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PROPERTIES OF NORMAL EMBEDDINGS CONCERNING STRONG SHAPE THEORY, I

By

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Abstract. Normal embeddings are characterized in terms of an approximate extension property, whence a generalization of certain cofibration properties to normal embeddings in the context of strong shape is deduced. Statements of Mayer-Vietoris type and a description of inclusion maps, which are invertible in the strong shape category, are presented as examples.

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54C20, 54C55, 54C56, 54D20, 55N40

1. Introduction

Normal embeddings play a particular role in strong shape theory, in fact: they are shape analogues of cofibrations in ordinary homotopy theory. In this paper we are going to demonstrate this in four examples, the first three of which are "Mayer-Vietoris"-arguments. Let us suppose a space X is covered by two closed subspaces A and B, such that the triad $(X; A, B)$ is "excisive" in an appropriate sense. If we have some knowledge about A, B, and $A\cap B$ we can draw conclusions concerning X. Usually excisiveness is related to the cofibration property of the inclusion maps, but in our context the normality of the embeddings is sufficient.

1.1) THEOREM. *A topological triad $(X; A, B)$ with $X=A\cup B$, such that A and B are closed and $A\cap B$ is normally embedded in X, is excisive with respect to any homology or cohomology functor factoring over the strong shape category.*

1.2) THEOREM. *Let $(X; A, B)$ be a topological triad satisfying the assumptions of theorem* 1.1. *Then the shape dimension of X has the following upper bound:*

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$$sd\,X \leqq \max(sd\,A,\ sd\,B,\ 1+sd\,A\cap B)$$

We recall that a space X has shape dimension less than or equal to a positive integer n if and only if every map $f: X\to P$ into a polyhedron P is homotopic to a map, whose full image is contained in the n-skeleton of P (cf. [11] p. 96). Theorem 1.2 has been proved for strongly paracompact spaces by Nowak and Spież (cf. [12] theorem 1.7).

1.3) THEOREM (Lisica, Mardešić): *If a triad* $(X;\, A,\, B)$ *satisfies the assumptions of theorem* 1.1, *and if two strong shape morphisms* $\alpha: A\to Y$ *and* $\beta: B\to Y$ *to an arbitrary space* Y *satisfy* $\alpha_{|A\cap B}=\beta_{|A\cap B}$, *then there exists a strong shape morphism* $\gamma: X\to Y$ *with* $\gamma_{|A}=\alpha$ *and* $\gamma_{|B}=\beta$.

This is the pasting theorem of Lisica and Mardešić, which has been proved in [8]. In our framework we are able to give a new, simplified proof.

In classical homotopy theory a map of pairs $f:(X, A)\to(Y, B)$ is a homotopy equivalence if and only if the total map $f: X\to Y$ and the relative map $f_{|A}$: $A\to B$ are homotopy equivalences of spaces, provided the inclusion maps $A\hookrightarrow X$ and $B\hookrightarrow Y$ are cofibrations. This carries over to the strong shape setting for normal inclusions, but in the present context we can treat only a special case which is interesting, because for pairs we can get by with ordinary shape:

1.4) THEOREM. *Let us suppose that* A *is normally embedded in the space* X. *Then the inclusion map* $i: A\hookrightarrow X$ *is a strong shape equivalence if and only if the inclusion map* $j:(A, A)\hookrightarrow(X, A)$ *is an ordinary shape equivalence of pairs.*

In the compact metric case theorem 1.4 has been proved by Dydak and Segal (cf. [4] theorem 6.2).

To deal with these subjects we need to know only a few basic facts of strong shape theory, which are recalled here. For a detailed treatment we refer to [1], [2], [6] and [7].

There is a strong shape category *ssh* (constructed somehow), whose objects are all topological spaces and which is related to the homotopy category by the strong shape functor $\eta: HTop\to ssh$. This functor has a right adjoint T: $ssh\to HTop$, which means the existence of spaces $T(Y)$ and bijections $ssh(X, Y)\approx HTop(X, T(Y))$, natural with respect to continuous maps in the first variable. A continuous map $f: X\to Y$ induces an isomorphism $\eta(f)$ if and only if it has the following two properties:

a) Every map $\varphi: X\to P$ into an ANR-space P admits a factorization $\varphi\cong\psi f$

with a suitable map $\psi: Y \to P$.

b) If two maps $\varphi, \psi: Y \to P$, P an ANR, and a homotopy $H: \varphi f \simeq \psi f$ are prescribed, then we can find a homotopy $G: \varphi \simeq \psi$ with $G(f \times id_I) \simeq H$.

(cf. [3] def. 1.1.) Homotopies between homotopies are understood relative to the boundary. A map of pairs $f: (X, A) \to (Y, B)$ is an ordinary shape equivalence if and only if for every ANR-pair (P, Q) the induced map $f^*: HTop(Y, B; P, Q) \to HTop(X, A; P, Q)$ is bijective. If a space X, an inverse system of spaces $Y = \{g_\mu^\lambda: Y_\lambda \to Y_\mu;\ \lambda \geqq \mu \in \Lambda\}$ and an inverse family of maps $\boldsymbol{f} = \{f_\lambda: X \to Y_\lambda\}$ are given, then $\boldsymbol{f}$ is called strong expansion, if conditions (c) and (d) are satisfied (cf. [9] def. 1, [6] def. 1.10; [3] def. 4.1 contains a more general notion):

c) For every map $\varphi: X \to P$ into an ANR-space P there exist an index $\lambda \in \Lambda$ and a map $\psi: Y_\lambda :\to P$ with $\psi f_\lambda \simeq \varphi$.

d) If two maps $\varphi, \psi: Y_\lambda \to P$ into an ANR-space P and a homotopy H: $\varphi f_\lambda \simeq \psi f_\lambda$ are given, then we can find an index $\mu \geqq \lambda$ and a homotopy $G: \varphi g_\lambda^\mu \simeq \psi g_\lambda^\mu$ with $G(f_\mu \times id_I) \simeq H$.

Every space admits a strong expansion in an inverse system of ANR-spaces or polyhedra; any resolution will do (cf. [9] and [10]).

2. Normal embeddings

We recall that an open covering of a topological space is said to be normal, if it admits a subordinated, locally finite partition of unity (cf. [11] p. 324). A subspace A of a space X is normally embedded, if for every normal covering $\mathcal{U}$ of A there is a normal covering $\mathcal{V}$ of X, whose trace on A refines $\mathcal{U}$ (cf. [11] p. 89). For our purpose it is necessary to characterize these embeddings in terms of an approximate extension property of maps instead of coverings. We introduce the following convention:

2.1) Definition. A neighborhood U of a subspace A of X is called normal, if A can be separated from $X \setminus U$ by an Urysohn function, i.e. by a map f: $X \to I$ vanishing outside of U and taking the constant value 1 on A.

This notion is natural in the following sense: If U is a normal neighborhood of A in X with corresponding Urysohn function f, $g: Y \to X$ a map and B a subset of Y with $g(B) \subseteqq A$, then $g^{-1}(U)$ is a normal neighborhood of B in Y, because $fg: Y \to I$ is an Urysohn function separating B and $Y \setminus g^{-1}(U)$. To give the reader an idea of the techniques applicable to normal neighborhoods we prove:

2.2) PROPOSITION. *Let us consider subsets* $A \subseteqq U \subseteqq X$. *The conditions* (a)-(c) *are equivalent:*

a) *U is a normal neighborhood of A in X.*

b) $\{\mathrm{Int}\, U, X \setminus \bar{A}\}$ *is a normal covering of X.*

c) *There is a normal covering $\mathcal{U}$ of X with $St(A, \mathcal{U}) \subseteqq U$.*

$\mathrm{St}(A, \mathcal{U})$ is the star of A with respect to $\mathcal{U}$, i.e. the union of all elements of $\mathcal{U}$ meeting A (cf. [5] p. 376).

Proof of proposition 2.2: If (a) holds and f is a corresponding Urysohn function, then $\{f, 1-f\}$ is a locally finite partition of unity subordinate to the covering defined in (b). This covering in turn fulfills (c). If $\mathcal{U}$ is an arbitrary covering satisfying (c) and $\{\varphi_\iota | \iota \in M\}$ a subordinate locally finite partition of unity, then we define $N := \{\iota \in M | A \cap \varphi_\iota^{-1}]0, 1] \neq \emptyset\}$ and $f := \sum_{\iota \in N} \varphi_\iota$. f is an Urysohn function separating A and $X \setminus U$, hence (a). q.e.d.

2.3) THEOREM. *Let A be a subspace of a topological space X.*

a) *A is normally embedded in X if and only if for every map $f: A \to P$, where P is an ANR, and every open covering $\mathcal{U}$ of P there are a normal neighborhood W of A in X and a map $g: W \to P$, whose restriction to A is $\mathcal{U}$-near to f.*

b) *If A is normally embedded in X, then so is $X \times \dot{I} \cup A \times I$ in $X \times I$.*

Proof. At first we want to show that condition (a) is necessary, and we suppose that a normally embedded subspace A of X, a map f from A into an ANR P and an open covering $\mathcal{U}$ of P are given. By the Kuratowski-Wojdislawski embedding theorem (cf. [11] p. 35) we may suppose that P is a subspace of a normed vector space and that P is closed in its convex hull K. We choose open neighborhoods U and V of P in K with $\bar{V} \subseteqq U$ such that there is a retraction $r: U \to P$. Let $\mathcal{V}_1$ be an open covering of V consisting of convex sets and refining $r^{-1}(\mathcal{U})$, and let $\mathcal{V}_2$ be a star refinement of $\mathcal{V}_1$ (cf. [5] p. 377). By assumption there is a normal covering $\mathcal{W}$ of X, whose trace on A is finer than $f^{-1}(\mathcal{V}_2)$, and to $\mathcal{W}$ corresponds a subordinate, locally finite partion of unity $\{\varphi_\iota | \iota \in M\}$. For each index $\iota \in M$ we choose elements $y_\iota \in V_\iota \in \mathcal{V}_2$ with $A \cap \varphi_\iota^{-1}(]0, 1]) \subseteqq f^{-1}(V_\iota)$. Now we can define a continuous map $g: X \to K$ by $g(x) := \sum \varphi_\iota(x) y_\iota$, and I claim:

$$g(A) \subseteqq V \tag{1}$$

$$f \text{ and } rg_{|A} \text{ are } \mathcal{U}\text{-near}. \tag{2}$$

$$W := g^{-1}(U) \text{ is a normal neighborhood of } A \text{ in } X. \tag{3}$$

For any point $a \in A$ and each index $\iota \in M$ with $\varphi_\iota(a)>0$ the elements y_ι and $f(a)$ belong to V_ι, especially: $y_\iota, f(a) \in \mathrm{St}(f(a), \mathcal{V}_2)$. If $V' \in \mathcal{V}_1$ contains $\mathrm{St}(f(a), \mathcal{V}_1)$, then it also contains $g(a)$ and $f(a)$, because V' does not depend on the index ι and it is convex. (1) follows immediately, and to prove (2) it suffices to choose $U' \in \mathcal{U}$ with $V' \subseteqq r^{-1}(U')$; then $rg(a)$ and $f(a)=rf(a)$ both lie in U'. The neighborhood W of A in X is normal, because the covering $\{g^{-1}(U), g^{-1}(K \setminus \overline{V})\}$ of X refines $\{W, X \setminus \overline{A}\}$ and is the continuous inverse image of the normal covering $\{U, K \setminus \overline{V}\}$ of K.

To prove that (a) is sufficient we consider a normal covering $\mathcal{U}$ of A. There exist an ANR P, an open covering $\mathcal{W}'$ of P, and a map $f: A \to P$, such that $f^{-1}(\mathcal{W}')$ refines $\mathcal{U}$, and in addition we choose a star refinement $\mathcal{W}$ of $\mathcal{W}'$. Condition (a) provides us with a normal neighborhood V of A in X and a map $g: V \to P$, whose restriction to A is $\mathcal{W}$-near to f. $\mathcal{V} := \{X \setminus \overline{A}\} \cup g^{-1}(\mathcal{W})$ is a normal covering of X, and I claim that its trace on A refines $\mathcal{U}$. To an element $W \in \mathcal{W}$ there are $W' \in \mathcal{W}'$ and $U \in \mathcal{U}'$ with $\mathrm{St}(W, \mathcal{W}) \subseteqq W'$ and $f^{-1}(W') \subseteqq U$, and for each point $a \in A \cap g^{-1}(W)$ we can find $W_1 \in \mathcal{W}$ containing $f(a)$ as well as $g(a)$. Since W_1 intersects W in $g(a)$ it belongs to $\mathrm{St}(W, \mathcal{W})$, and we have: $f(a) \in W_1 \subseteqq \mathrm{St}(W, \mathcal{W}) \subseteqq W'$, hence $g^{-1}(W) \cap A \subseteqq f^{-1}(W') \subseteqq U$.

To prove (b) we use the following lemma as a convenient substitute for the concept of stacked coverings:

2.4) LEMMA. *For every ANR-space P and each open covering $\mathcal{U}$ of P there exist an open covering $\mathcal{V}$ of the path space P^I and a map $\varphi: P^I \to]0, 1]$ with the following property: If two paths $\omega, \nu \in P^I$ are $\mathcal{V}$-near and two real numbers $s, t \in I$ satisfy $|s-t| < \varphi(\omega)$, then the points $\omega(s)$ and $\nu(t)$ are $\mathcal{U}$-near.*

Proof of lemma 2.4: Let $\mathcal{W}$ be a star refinement of $\mathcal{U}$. For each finite sequence $\alpha=(W_0, \cdots W_n)$ in $\mathcal{W}$ we denote by $\tilde{\alpha} \subseteqq P^I$ the open set of all paths ω with $\omega(I \cap [(k-1)/n, (k+1)/n]) \subseteqq W_k$ for $0 \leqq k \leqq n$. By Lebesgue's theorem the sets of the form $\tilde{\alpha}$ cover P^I; let $\{f_\iota | \iota \in M\}$ be a subordinate, locally finite partition of unity. For each index $\iota \in M$ we choose a sequence α_ι in $\mathcal{W}$ of length $n_\iota+1$ with $f_\iota^{-1}(]0, 1]) \subseteqq \tilde{\alpha}_\iota$, and we define:

$$\varphi := \sum_{\iota \in M} \frac{1}{n_\iota} f_\iota \tag{4}$$

The map φ and the open covering consisting of the sets $\tilde{\alpha}$ fulfill the requirements of lemma 2.4. q.e.d.

Proof of theorem 2.3.b: We want to apply 2.3.a and consider a map f:

$X\times\dot{I}\cup A\times I\to P$ into an ANR-space P and an open covering $\mathcal{U}$ of P. Let $\mathcal{U}_1$ denote a star refinement of $\mathcal{U}$ and $\mathcal{U}_2$ a refinement of $\mathcal{U}_1$, such that any two $\mathcal{U}_2$-near maps of an arbitrary space into P are $\mathcal{U}_1$-homotopic (cf. [11] p. 39), and let $\mathcal{U}_3$ be a locally finite open covering of P consisting of sets, whose closures refine $\mathcal{U}_2$. Lemma 2.4 provides us with an open covering $\mathcal{V}$ of P^I and with a map $\varphi: P^I\to]0, 1]$, such that $\omega(s)$ and $\nu(t)$ are $\mathcal{U}_3$-near if the assumptions of our lemma are satisfied. Since P^I is an ANR-space too there exist a normal neighborhood V' of A in X and a map $\bar{h}: V'\to P^I$, whose restriction to A is $\mathcal{V}$-near to the adjoint map $\bar{f}: A\to P^I$. Now I claim the existence of a normal neighborhood V of A in X contained in V', such that there are $\mathcal{U}_1$-homotopies $F: f_{|V\times\{0\}}\cong h_{|V\times\{0\}}$ and $G: f_{|V\times\{1\}}\cong h_{|V\times\{1\}}$. To see this we consider the maps $\Phi_t: V'\to P\times P$, defined by $\Phi_t(x):=(f(x, t), h(x, t))$ for $t=0, 1$. By construction we have:

$$\Phi_t(A)\subseteq\cup\{\bar{U}\times\bar{U}\,|\,U\in\mathcal{U}_3\}=:M\subseteq\cup\{U\times U\,|\,U\in\mathcal{U}_2\}=:N \tag{5}$$

The set M is closed in $P\times P$, because $\mathcal{U}_3$ is locally finite, and therefore the closure of $\Phi_t(A)$ is contained in the open set N. Since $P\times P$ is metrizable $V:=\Phi_0^{-1}(N)\cap\Phi_1^{-1}(N)$ is a normal neighborhood of A in V' and a fortiori in X, such that $f_{|V\times\{t\}}$ and $h_{|V\times\{t\}}$ are $\mathcal{U}_2$ near and therefore $\mathcal{U}_1$-homotopic. We take any map $\psi: X\to I$, $0<\psi<1/2$, such that:

$$\frac{\psi(a)}{1-2\psi(a)}<\varphi\bar{h}(a) \qquad \text{for} \quad a\in A. \tag{6}$$

Then $U:=V\times I\cup\{(x, t)\in X\times I\,|\,t<(1/2)\psi(x) \text{ or } t>1-(1/2)\psi(x)\}$ is a normal neighborhood of $X\times\dot{I}\cup A\times I$ in $X\times I$, and we can define a map $g: U\to P$ with restriction $\mathcal{U}$-near to f as follows:

$$g(x, t)=\begin{cases} f(x, 0) & t\in\left[0, \frac{1}{2}\psi(x)\right] \\ F\left(x, \frac{2t-\psi(x)}{\psi(x)}\right) & t\in\left[\frac{1}{2}\psi(x), \psi(x)\right] \\ h\left(x, \frac{t-\psi(x)}{1-2\psi(x)}\right) & t\in[\psi(x), 1-\psi(x)] \\ G\left(x, \frac{2(1-t)-\psi(x)}{\psi(x)}\right) & t\in\left[1-\psi(x), 1-\frac{1}{2}\psi(x)\right] \\ f(x, 1) & t\in\left[1-\frac{1}{2}\psi(x), 1\right] \end{cases} \tag{7}$$

q.e.d.

2.5) COROLLARY. *If A is normally embedded in X, then the normal neighborhoods of A in X form a strong expansion of A.*

Remark. It follows from theorem 2.3 that the normal neighborhoods of A form a resolution, and therefore corollary 2.5 follows from a general theorem of Mardešić (cf. [10]). But in the present, special situation a direct proof is simpler:

Proof. i) We want to show that condition (c) of the introduction holds and consider a map $\varphi: A\to P$ into an ANR-space P. Taking $\mathcal{U}$ as an open covering of P, such that any two $\mathcal{U}$-near maps are homotopic, we apply theorem 2.3.a and get a normal neighborhood W of A in X and a map $\psi: W\to P$ with $\psi_{|A}\cong\varphi$.

ii) Now condition (d) is to be proved. We consider a normal neighborhood U of A in X, two maps $\varphi, \psi: U\to P$ into an ANR-space P and a homotopy H: $\varphi_{|A}\cong\psi_{|A}$. Again we take $\mathcal{U}$ as an open covering of P, such that $\mathcal{U}$-near maps are homotopic, and we define a map $g: U\times\dot{I}\cup A\times I\to P$ as follows:

$$g(x, t):=\begin{cases}\varphi(x) & t\leqq\frac{1}{3}\\ H(x, 3t-1) & \frac{1}{3}\leqq t\leqq\frac{2}{3}\\ \psi(x) & t\geqq\frac{2}{3}\end{cases}\tag{8}$$

By theorem 2.3.b $U\times\dot{I}\cup A\times I$ is normally embedded in $U\times I$, and 2.3.a provides us with a normal neighborhood $V\subseteqq U$ of A in X and with a map $h: V\times I\to P$, whose restriction to $V\times\dot{I}\cup A\times I$ is $\mathcal{U}$-near to g. Let Φ: $g_{V\times\dot{I}\cup A\times I}\cong h_{V\times\dot{I}\cup A\times I}$ be a connecting homotopy. The homotopy G: $\varphi_{|V}\cong\psi_{|V}$ we are looking for is defined as follows:

$$G(x, t):=\begin{cases}\Phi(x, 0, 3t) & t\leqq\frac{1}{3}\\ h(x, 3t-1) & \frac{1}{3}\leqq t\leqq\frac{2}{3}\\ \Phi(x, 1, 3(1-t)) & t\geqq\frac{2}{3}\end{cases}\tag{9}$$

A connecting homotopy Ψ: $G_{|A\times I}\cong H$ is given by $\Psi(a, s, t):=\Phi(a, \omega(s, t))$, where $\omega: I^2\to I^2$ is subject to the following boundary conditions:

$$\omega(s,0):=\begin{cases}(0,3s) & s\leqq\frac{1}{3}\\ (3s-1,1) & \frac{1}{3}\leqq s\leqq\frac{2}{3}\\ (1,3(1-s)) & s\geqq\frac{2}{3}\end{cases}\tag{10}$$

$$\omega(s,1)=\left(\frac{1+s}{3},0\right)\quad\omega(0,t)=\left(\frac{t}{3},0\right)\quad\omega(1,t)=\left(1-\frac{t}{3},0\right)\tag{11}$$

q. e. d.

The next theorem displays that for the application of "Mayer-Vietoris" arguments normal embeddings are as good as cofibrations:

2.6) THEOREM. *We suppose that a space X is given as the union of two closed subspaces A and B, whose intersection is normally embedded in X. Then the inclusion map*

$$A\times\{0\}\cup(A\cap B)\times I\cup B\times\{1\}\hookrightarrow X\times I$$

is a strong shape equivalence.

The proof requires the following lemma:

2.7) LEMMA. *If the triad $(X; A, B)$ satisfies the assumptions of theorem* 2.6, *then*

a) *A and B are normally embedded in X, and*

b) *$\Gamma:=A\times\{0\}\cup(A\cap B)\times I\cup B\times\{1\}$ is normally embedded in $X\times I$.*

Proof of theorem 2.6, *assuming lemma* 2.7. We have to show that conditions (a) and (b) of the introduction are satisfied, and we start by considering a map $\varphi:\Gamma\to P$ into an ANR P. Corollary 2.5 provides us with a normal neighborhood U of Γ in $X\times I$ and with a map $\phi':U\to P$, whose restriction to Γ is homotopic to φ. Let $f: X\times I\to I$ be an Urysohn function separating Γ from the complement of U and define a continuous map $g: X\to I$ by $g(x):=\inf\{f(x,t)\mid t\in I\}$. On $A\cap B$ g equals 1, and if $g(x)$ does not vanish, then (x, t) belongs to U for all $t\in I$. Therefore we can define a map $\phi: X\times I\to P$ as follows:

$$\phi(x,t):=\begin{cases}\phi'(x,tg(x)) & x\in A\\ \phi'(x,1-(1-t)g(x)) & x\in B\end{cases}\tag{12}$$

On Γ the map ϕ coincides with ϕ', and condition (a) is proved. The proof of (b) follows exactly the same pattern. q. e. d.

Proof of 2.7.a: Let a map $f: A\to P$ into an ANR-space P and an open covering $\mathcal{U}$ of P be given. We choose a refinement $\mathcal{U}_1$ of $\mathcal{U}$, such that any two $\mathcal{U}_1$-near maps are $\mathcal{U}$-homotopic, and a locally finite covering $\mathcal{U}_2$, such that the closures of the sets of $\mathcal{U}_2$ form a refinement of $\mathcal{U}_1$. By assumption there exist a normal neighborhood W of $A\cap B$ in X and a map $g: W\to P$, whose restriction to $A\cap B$ is $\mathcal{U}_2$-near to $f_{|A\cap B}$. Now we consider the map $\varphi: A\cap W\to P\times P$ given by $\varphi(x):=(f(x), g(x))$; since $\varphi(A\cap B)$ is contained in $\cup\{\bar{U}\times\bar{U}\,|\,U\in\mathcal{U}_2\}\subseteqq\cup\{U\times U\,|\,U\in\mathcal{U}_1\}$ there is a normal neighborhood W' of $A\cap B$ in $A\cap W$, such that f and g are $\mathcal{U}_1$-near and consequently $\mathcal{U}$-homotopic on W'. Let us take such a $\mathcal{U}$-homotopy $H: f_{|W'}\cong g_{|W'}$ and Urysohn functions $\chi: X\to I$ and $\psi: A\cap W\to I$ separating $A\cap B$ from $X\setminus W$ respectively $A\cap W\setminus W'$. We define $r: A\to I$ and $h: A\cup W\to P$ as follows:

$$r(a):=\begin{cases} 0 & \chi(a)\leqq\frac{1}{2} \\ (2\chi(a)-1)\psi(a) & \chi(a)\geqq\frac{1}{2} \end{cases} \tag{13}$$

$$h(x):=\begin{cases} f(x) & x\in A,\ r(x)\leqq\frac{1}{2} \\ H(x, 2r(x)-1) & x\in A,\ r(x)\geqq\frac{1}{2} \\ g(x) & x\in B\cap W \end{cases} \tag{14}$$

Then $h_{|A}$ and f are $\mathcal{U}$-near, and $A\cup W$ is a normal neighborhood of A in X, because we can set up the following Urysohn function:

$$\rho(x):=\begin{cases} 1 & x\in A \\ \chi(x) & x\in B \end{cases} \tag{15}$$

Proof of 2.7.b: Again we consider a prescribed map $f: \Gamma\to P$ into an ANR-space P and an open covering $\mathcal{U}$ of P, but now we need four consecutive refinements $\mathcal{U}_1, \cdots \mathcal{U}_4$ of $\mathcal{U}$: $\mathcal{U}_1$ shall be a star refinement of $\mathcal{U}$, any two $\mathcal{U}_2$-near maps shall be $\mathcal{U}_1$-homotopic, the closures of the sets of $\mathcal{U}_3$ are required to be a locally finite refinement of $\mathcal{U}_2$ and $\mathcal{U}_4$ is taken as a star refinement of $\mathcal{U}_3$. At last we choose an open covering $\mathcal{V}$ of P^I and a map $\varphi: P^I\to]0, 1]$ by applying lemma 2.4 to $\mathcal{U}_4$.

Since A, B and $A\cap B$ are normally embedded in X we can find normal neighborhoods U_0 of A, U_1 of B and U_2 of $A\cap B$ in X and maps $g_0: U_0\to P$, $g_1: U_1\to P$ and $\bar{g}_2: U_2\to P^I$, such that $g_{0|A}$ and $f_{|A\times\{0\}}$ respectively $g_{1|B}$ and $f_{|B\times\{1\}}$ are $\mathcal{U}_4$-near and $\bar{g}_{2|A\cap B}$ is $\mathcal{V}$-near to the adjoint map $\bar{f}: A\cap B\to P^I$.

Then $g_{0|A\cap B}$ and $g_{2|A\cap B\times\{0\}}$ respectively $g_{1|A\cap B}$ and $g_{2|A\cap B\times\{1\}}$ are $\mathcal{U}_3$-near, and this implies the existence of a normal neighborhood V of $A\cap B$ in X contained in $U_0\cap U_1\cap U_2$, such that $g_{0|V}$ and $g_{2|V\times\{0\}}$ respectively $g_{1|V}$ and $g_{2|V\times\{1\}}$ are $\mathcal{U}_2$-near. Therefore we can find $\mathcal{U}_1$-homotopies Φ: $g_{0|V}\cong g_{2|V\times\{0\}}$ and Ψ: $g_{1|V}\cong g_{2|V\times\{1\}}$, and surely we can find a map ϕ: $X\to I$ with $0<\phi<1/2$ and:

$$\frac{\phi(x)}{1-2\phi(x)}<\varphi\bar{g}_2(x) \qquad \text{for} \quad x\in A\cap B \tag{16}$$

$W:=V\times I\cup\{(x, t)\in A\times I\mid t<(1/2)\phi(x)\}\cup\{(x, t)\in B\times I\mid t>1-(1/2)\phi(x)\}$ is a normal neighborhood of Γ in $X\times I$, and the restriction of the following map $h: W\to P$ to Γ is $\mathcal{U}$-near to f:

$$h(x, t):=\begin{cases} g_0(x) & t\in\left[0, \frac{1}{2}\phi(x)\right] \\ \Phi\left(x, \dfrac{2t-\phi(x)}{\phi(x)}\right) & t\in\left[\frac{1}{2}\phi(x), \phi(x)\right] \\ g_2\left(x, \dfrac{t-\phi(x)}{1-2\phi(x)}\right) & t\in[\phi(x), 1-\phi(x)] \\ \Psi\left(x, \dfrac{2(1-t)-\phi(x)}{\phi(x)}\right) & t\in\left[1-\phi(x), 1-\frac{1}{2}\phi(x)\right] \\ g_1(x) & t\in\left[1-\frac{1}{2}\phi(x), 1\right] \end{cases} \tag{17}$$

q. e. d.

Proof of theorem 1.1: Instead of $(X; A, B)$ we may consider $(X'; A', B')$ with $X':=\Gamma$, $A':=A\times\{0\}\cup(A\cap B)\times I$ and $B':=B\times\{1\}\cup(A\cap B)\times I$. This triad is excisive because the interiors of A' and B' cover X'. q. e. d.

Proof of theorem 1.3: Theorem 2.6 permits us to replace the triad $(X; A, B)$ by $(X'; A', B')$ with $X':=\Gamma$, $A':=\Gamma\cap X\times[0, 1/2]$ and $B':=\Gamma\cap X\times[1/2, 1]$, so that the inclusion map $A'\cap B'\hookrightarrow A'$ is a cofibration. Using the right adjoint T of the strong shape functor the statement of 1.3 may be rephrased as follows: If two maps α: $A'\to T(Y)$ and β: $B'\to T(Y)$ with $\alpha_{|A'\cap B'}\cong\beta_{|A'\cap B'}$ are given, then there is γ: $X'\to T(Y)$ with $\gamma_{|A'}\cong\alpha$ and $\gamma_{|B'}\cong\beta$. But this follows from the cofibration property. q. e. d.

For the proof of theorem 1.2 we need the following lemma, which corresponds to the well known fact that homotopic cellular maps between CW-complexes can be connected by cellular homotopies.

2.8) LEMMA.

a) *Let* $f, g: X \to P$ *be two maps from a space* X, *whose shape dimension* $sd\, X$ *does not exceed a given number* n, *to a polyhedron* P, *and suppose that the full images of* f *and* g *are contained in the n-skeleton* $P_{(n)}$ *of* P. *Then every homotopy* $G: f \simeq g$ *in* P *is homotopic (relative boundary!) to a homotopy* $H: f \simeq g$ *in* $P_{(n+1)}$.

b) *Let* (X, A) *be a topological pair with closed cofibre subspace* A *and let* $f: X \to P$ *be a map into a polyhedron with* $f(A) \subseteq P_{(n-1)}$, $\max(sd\, X, 1+sd\, A) \leqq n$. *Then there exist a map* $g: X \to P$ *with* $g(X) \subseteq P_{(n)}$ *and a homotopy* $g \simeq f$ *relative* A.

Proof. a) We take a strong expansion $\{\rho_\lambda\}: X \to \{\pi^\mu_\lambda: Q_\mu \to Q_\lambda\}$ of X in an inverse system of polyhedra. There exist an index λ, maps $f', g': Q_\lambda \to P_{(n)}$ and homotopies $\Phi: f \simeq f'\rho_\lambda$ and $\Psi: g \simeq g'\rho_\lambda$ in $P_{(n)}$, and for a suitable index $\mu \geqq \lambda$ we can find a homotopy $G': f'\pi^\mu_\lambda \simeq g'\pi^\mu_\lambda$ in P with $G'(\rho_\mu \times id_I) \simeq \Phi^{-1} \circ G \circ \Psi$. The condition $sd\, X \leqq n$ implies the existence of a polyhedron Q' with $\dim Q' \leqq n$, an index $\nu \geqq \mu$, of maps $a: Q' \to Q_\mu$ and $b: Q_\nu \to Q'$ and of a homotopy A: $ab \simeq \pi^\nu_\mu$. Since the maps $f'\pi^\mu_\lambda a$ and $g'\pi^\mu_\lambda a: Q' \to P_{(n)}$ are homotopic to cellular maps and since the composition of these homotopies with $G'(a \times id_I)$ is homotopic to a cellular homotopy these maps can be connected by a homotopy $G'': f'\pi^\mu_\lambda a \simeq g'\pi^\mu_\lambda a$ in $P_{(n+1)}$ with $G'' \simeq G'(a \times id_I)$. This gives rise to a homotopy H': $f'\pi^\nu_\lambda \simeq g'\pi^\nu_\lambda$ in $P_{(n+1)}$ defined by $H' := f'\pi^\mu_\lambda A^{-1} \circ G''(b \times id_I) \circ g'\pi^\mu_\lambda A$, and this in turn leads to $H := \Phi \circ H'(\rho_\nu \times id_I) \circ \Psi^{-1}: f \simeq g$. H connects f and g in $P_{(n+1)}$ and is homotopic to G in P, because A and G' commute (cf. [13] lemma 1, the "Godement interchange law").

b) Since the inclusion map $A \hookrightarrow X$ is a cofibration and because of the relations $sd\, X \leqq n$ and $sd\, A \leqq n-1$ we can surely find a homotopy $H: f \simeq g'$, where g' satisfies $g'(X) \subseteq P_{(n)}$ and $g'(A) \subseteq P_{(n-1)}$, but a priori H need not be stationary on A. By part (a) $H_{|A \times I}$ is homotopic to $\Phi: f_{|A} \simeq g'_{|A}$ with $\Phi(A \times I) \subseteq P_{(n)}$. A second application of the cofibration property provides us with a map $g: X \to P_{(n)}$ and a homotopy $\Psi: g \simeq g'$ in $P_{(n)}$ with $\Psi_{|A \times I} = \Phi$. Then the restriction of $G := H \circ \Psi^{-1}: f \simeq g$ to $A \times I$ is homotopic to a stationary homotopy, and by a third cofibration argument (b) is proved. q.e.d.

Proof of theorem 1.2: By the same trick as in the proof of 1.3 we can reduce the general case to the special situation, where the inclusion maps are cofibrations. We set $n := \max(sd\, A, sd\, B, 1+sd\, A \cap B)$ and consider a map f: $X \to P$ into a polyhedron P. Because of the cofibration property and the relation $sd\, A \cap B \leqq n-1$ we are able to replace f by a homotopic map f' with $f'(A \cap B) \subseteq P_{(n-1)}$. Lemma 2.8.b ensures the existence of maps and homotopies $f'_{|A} \simeq \varphi$

and $f'_{|B} \cong \phi$ relative $A \cap B$ with $im\,\varphi$, $im\,\phi \subseteq P_{(n)}$. Then the map $g: X \to P$ with $g_{|A} = \varphi$ and $g_{|B} = \phi$ is homotopic to f and its full image is contained in $P_{(n)}$.

q. e. d.

Proof of theorem 1.4: At first we deal with the special case, in which the subspace A is closed and the inclusion map i is a cofibration. Let us suppose that i is a strong shape equivalence; we have to show that $j^*: HTop(X, A; P, Q) \to HTop(A, A; P, Q)$ is bijective for every ANR-pair (P, Q). For a map $\varphi: (A, A) \to (P, Q)$ condition (a) of the introduction ensures the existence of a map $\phi: X \to Q$ with $\phi_{|A} \cong \varphi$, hence j^* is surjective. If two maps $\varphi, \phi: (X, A) \to (P, Q)$ and a homotopy $H: \varphi_{|A} \cong \phi_{|A}$ in Q are given, then (b) provides us with a homotopy $G: \varphi \cong \phi$ in P with $G_{|A \times I} \cong H$. Since $X \times \dot{I} \cup A \times I$ is a cofibre subspace of $X \times I$ we can now construct $G': \varphi \cong \phi$ with $G'_{|A \times I} = H$, so φ and ϕ are homotopic as maps of pairs and j^* is injective.

If on the other hand j^* is bijective, then (a) follows immediatly when we consider pairs of the form (P, P). To prove (b) we assume that maps $\varphi, \phi: X \to P$ and a homotopy $H: \varphi_{|A} \cong \phi_{|A}$ are given; using the cofibration property we can find a map $\varphi': X \to P$ with $\varphi'_{|A} = \varphi_{|A}$ and a homotopy $G': \varphi \cong \varphi'$ with $G'_{|A \times I} = H$. Now it suffices to construct a homotopy $G: \varphi' \cong \varphi$ relative A. The maps $\alpha, \beta: X \to P \times P$ with coordinates $\alpha = (\varphi', \varphi)$ and $\beta = (\varphi, \varphi)$ map A to the diagonal $\Delta \subseteq P \times P$; therefore they may be considered as elements $\alpha, \beta \in HTop(X, A; P \times P, \Delta)$ with $j^*\alpha = j^*\beta$. Consequently α and β are homotopic as maps of pairs, and the projections Φ, Ψ of such a connecting homotopy are homotopies $\Phi: \varphi' \cong \varphi$ and $\Psi: \varphi \cong \varphi$ with $\Phi_{|A \times I} = \Psi_{|A \times I}$. Then the restriction of $G'' := \Phi \circ \Psi^{-1}: \varphi' \cong \varphi$ to $A \times I$ is homotopic to a stationary homotopy, and we can construct G from G'' by an application of the cofibration property.

Now the general case follows from lemma 2.9, which says that in the ordinary shape category every pair with normally embedded subspace is equivalent to a pair with closed cofibre subspace.

2.9) LEMMA. *We assume that A is normally embedded in a space X and set $X' := X \times \{0\} \cup A \times I \subseteq X \times I$, $A' := A \times \{1\} \subseteq X'$ and denote by $p: (X', A') \to (X, A)$ the natural projection. Then p is an ordinary shape equivalence of pairs.*

Proof. We consider an ANR-pair (P, Q) and the induced map $p^*: HTop(X, A; P, Q) \to HTop(X', A'; P, Q)$.

a) p^* is surjective: Let $f: (X', A') \to (P, Q)$ be given; we want to construct a normal neighborhood U of A in X and a map $f': (X \times \{0\} \cup U \times I, U \times \{1\}) \to (P, Q)$, whose restriction to (X', A') is homotopic to f. Once f' is constructed we can choose an Urysohn function $\varphi: X \to I$ separating A from $X \setminus U$ and define a map $g: (X, A) \to (P, Q)$ and a homotopy $H: gp \cong f'_{|(X', A')}$ by:

$$g(x) := f'(x, \varphi(x)) \tag{18}$$

$$H(x, s; t) := f'(x, ts + (1-t)\varphi(x)) \tag{19}$$

Corollary 2.5 implies the existence of a normal neighborhood V of A in X, of a map $f'': V \to Q$ and of a homotopy $G'': f_{|A'} \cong f''_{|A}$ in Q. Applying corollary 2.5 again we can find a closed normal neighborhood U of A in X contained in V and a homotopy $G: f_{|U\times\{0\}} \cong f''_{|U}$ with $G_{|A\times I} \cong f_{|A\times I} \circ G''$ in P. Then our map f' is defined as follows:

$$f'(x, s) := \begin{cases} f(x, 0) & s = 0 \\ G(x, s) & x \in U \end{cases} \tag{20}$$

b) p^* is injective: Let two maps $f, g: (X, A) \to (P, Q)$ and a homotopy $H: fp \cong gp$ as maps of pairs be given. We may assume the existence of a normal neighborhood U of A in X with $f(U), g(U) \subseteqq Q$, because otherwise we could replace f by ρf and g by ρg, where $\rho: P \to P$ is a map with $\rho \cong id$ relative Q and $\rho(U') \subseteqq Q$ for a suitable neighborhood U' of Q in P. Corollary 2.5 implies the existence of a normal neighborhood V of A in X contained in U and of homotopies $G: f_{|V} \cong g_{|V}$ and $\Psi: H_{|A\times\{1\}\times I} \cong G_{|A\times I}$ in Q relative $A \times \dot{I}$. Now we define a map $\Phi: X \times \{0\} \times I \cup V \times \dot{I}^2 \cup A \times I^2 \to P$ as follows:

$$\Phi(x, s, t) := \begin{cases} f(x) & t \leqq \frac{1}{3} \\ H(x, 3s, 3t-1) & \frac{1}{3} \leqq t \leqq \frac{2}{3},\ s \leqq \frac{1}{3} \\ \Psi(x, 3t-1, 3s-1) & \frac{1}{3} \leqq t \leqq \frac{2}{3},\ \frac{1}{3} \leqq s \leqq \frac{2}{3} \\ G(x, 3t-1) & \frac{1}{3} \leqq t \leqq \frac{2}{3},\ s \geqq \frac{2}{3} \\ g(x) & t \geqq \frac{2}{3} \end{cases} \tag{21}$$

The division of I^2 into so many parts is necessary to ensure the continuity of Φ if A is not closed. We observe: $\Phi(V \times \{1\} \times I) \subseteqq Q$. Since $V \times \dot{I}^2 \cup A \times I^2$ is normally embedded in $V \times I^2$ there are a normal neighborhood W of A in X contained in V and a map $\Phi': V \times \dot{I}^2 \cup W \times I^2 \to P$, whose restriction to $V \times \dot{I}^2 \cup A \times I^2$ is homotopic to Φ, and since the inclusion map $V \times \dot{I}^2 \hookrightarrow V \times \dot{I}^2 \cup W \times I^2$ is a cofibration, we may assume $\Phi'_{|V\times\dot{I}^2} = \Phi_{|V\times\dot{I}^2}$. We choose an Urysohn function $\varphi: X \to I$ separating A from $X \setminus W$ and define a homotopy $\Gamma: f \cong g$ as follows:

$$\Gamma(x, t) := \begin{cases} \Phi'(x, 2\varphi(x)-1, t) & \varphi(x) \geqq \frac{1}{2} \\ \Phi(x, 0, t) & \varphi(x) \leqq \frac{1}{2} \end{cases} \tag{22}$$

q.e.d.

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