TSUKUBA J. MATH.
Vol. 10 No. 2 (1986). 255-262

SOME ALMOST-HOMOGENEOUS COMPLEX STRUCTURES ON $P^2 \times P^2$

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1. Introduction

It is well known that on $P^1(C) \times P^1(C)$ there exists an infinite sequence of different complex structures, namely the Hirzebruch surfaces Σ_{2m}, $m \in N$. These surfaces are of the form $P(\mathcal{O}_{P^1}(m) \oplus \mathcal{O}_{P^1}(-m))$ and are all almost-homogeneous (see [H]). In generalization of this, Brieskorn has studied P^n-bundles over P^1 and has proved that all complex structures on $P^1 \times P^n$ satisfying some supplementary conditions (see [Br], (5.3)) are such P^n-bundles. All these structures are almost-homogeneous.

Motivated by these results, it is natural to consider complex structures on $P^2 \times P^2$ of the form $P(E)$, where E is a topologically trivial holomorphic vector bundle of rank 3 on P^2. In contrast with the situation on P^1, a complete classification of such bundles is not known, however Bănică has classified all topologically trivial rank 2 vector bundles on P^2 (see [B], §2). In particular these bundles do not depend only on discrete parameters, but also on "continuous" moduli. Using rank 3 vector bundles on P^2 of the form $E := F \oplus \mathcal{O}_{P^2}$, with F topologically trivial of rank 2, one can easily construct complex structures on $P^2 \times P^2$, depending on "continuous" moduli, which are not almost-homogeneous.

Here we study some examples of almost-homogeneous complex structures on $P^2 \times P^2$ of the form $P(E)$, for homogeneous and almost-homogeneous E. In §2 are studied the cases when E is $\mathcal{T}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-1)$ or its dual (together with $\mathcal{O}_{P^2}^{\oplus 3}$, these are the only topologically trivial homogeneous rank 3 vector bundles on P^2, see for example [M]). It turns out that the automorphism group of $X_1 := P(\mathcal{T}_{P^2}(-1) \oplus \mathcal{O}_{P^2}(-1))$ has an open orbit, whose complement is an irreducible homogeneous hypersurface (hence X_1 gives an example of the manifolds classified by Ahiezer [Ah]), while the automorphism group of $X_2 := P(\mathcal{T}_{P^2}(-2) \oplus \mathcal{O}_{P^2}(1))$ has an open orbit, whose complement is irreducible and homogeneous of codimension 2. In §3 we consider the complex manifold $X := P(F \oplus \mathcal{O}_{P^2})$ with F a topologically trivial rank 2 vector bundle on P^2 of generic splitting type $(-1, 1)$ and we prove that the automorphism group of X has an open orbit, whose complement is an irreducible hypersurface, which contains a whole fiber of $P(E)$.

This paper was written with the financial support of M.P.I. (Italian Ministry of Education).
The author is a member of G.N.S.A.G.A. of the C.N.R.
Received September 2, 1985.

Section 1.

In this section we introduce some notations and preliminary material.

Here a vector bundle is always a holomorphic vector bundle and we often identify vector bundles and locally free sheaves. Following the notations of [OSS] for a vector bundle $E \underset{\pi}{\to} S$ we denote by $E(x)$ the fiber over a point $x \in S$.

Let S be a complex manifold, $E \underset{\pi}{\to} S$ a rank m vector bundle on S, and let $\boldsymbol{P}(E) \underset{p}{\to} S$ be the corresponding projective bundle. Set $X := \boldsymbol{P}(E)$. We denote by

—Aut (E) the group of all biholomorphic maps $E \to E$, which tranform fibers in fibers and are linear on fibers;

—Aut $(\boldsymbol{P}(E))$ the group of all biholomorphic maps of $\boldsymbol{P}(E)$ into itself, which transform fibers in fibers;

—$\mathrm{Aut}_S(E)$, $\mathrm{Aut}_S(\boldsymbol{P}(E))$ the subgroups of all elements of Aut (E) and Aut $(\boldsymbol{P}(E))$ respectively, which induce the identity on S;

—PGL (E) the subgroup of all elements of $\mathrm{Aut}_S(\boldsymbol{P}(E))$, which are induced by elements of $\mathrm{Aut}_S(E)$;

—Aut (X) the group of all biholomorphic maps of X onto itself.

(1.1) PROPOSITION.

With the same notations as above, let us suppose S simply connected. Then

$$\mathrm{Aut}_S(\boldsymbol{P}(E)) = \mathrm{PGL}(E).$$

PROOF: Let U be a simply connected open subset of S such that $E|U$ is trivial. An element $\phi \in \mathrm{Aut}_U(\boldsymbol{P}(E|U))$ can be regarded as a holomorphic map $\bar{\phi}: U \to \mathrm{PGL}(m)$. Since U is simply connected and SL (m) is a coverning space of PGL (m), the map $\bar{\phi}$ can be lifted to a holomorphic map $\bar{\Phi}: U \to \mathrm{SL}(m)$, and this gives an element $\Phi \in \mathrm{Aut}_S(E|U)$, which induces ϕ.

Now let $\phi \in \mathrm{Aut}_S(\boldsymbol{P}(E))$ and let $\mathcal{U} := \{U_i\}_{i \in I}$ be an open covering of S such that every U_i is simply connected and $E|U_i$ is trivial. Then for each U_i there exists $\Phi_i \in \mathrm{Aut}_{U_i}(E|U_i)$ with det $\Phi_i = 1$, which induces ϕ. On $U_i \cap U_j$ the two matrices Φ_i and Φ_j induce the same projective automorphism ϕ, therefore they coincide up to the multiplication by an m-th root of unity. Thus we obtain an element of $H^1(S, \mu_m)$, where μ_m is the locally constant sheaf of m-th roots of unity. Since S is simply connected $H^1(S, \mu_m) = 0$, hence $\Phi_i \cdot \Phi_j^{-1} = \lambda_i \cdot \lambda_j^{-1}$ where λ_i, λ_j are m-th roots of unity. Thus the $\lambda_i^{-1} \cdot \Phi_i$ can be glued together to give an element $\Phi \in \mathrm{Aut}_S(E)$ which induces ϕ.

We recall the following

(1.2) DEFINITION.

Let S be a homogeneous complex manifold and let $E \underset{\pi}{\to} S$ be a vector bundle on S.

We say that E is *homogeneous* if for all $g \in \mathrm{Aut}(S)$ one has $g^*E \simeq E$.

We say that E is *almost-homogeneous* if there exists a subgroup G of Aut (S), one of whose orbits is a Zariski open dense subset of S, such that $g^*E \simeq E$ for all $g \in G$.

(1.3) COROLLARY.

Let S be a homogeneous simply connected complex manifold, $E \underset{\pi}{\to} S$ a homogeneous vector bundle on S and let $\boldsymbol{P}(E) \underset{p}{\to} S$ be the corresponding projective bundle.
Then there is the exact sequence

$$0 \to \mathrm{PGL}(E) \to \mathrm{Aut}(\boldsymbol{P}(E)) \to \mathrm{Aut}(S) \to 0.$$

PROOF: This is an obvious consequence of (1.1).

Section 2.

In this section we show that the complex manifolds $X_1 := \boldsymbol{P}(T_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2})$ and $X_2 := \boldsymbol{P}(T_{\boldsymbol{P}^2}(-3) \oplus \mathcal{O}_{\boldsymbol{P}^2})$ are almost-homogeneous and we determine the orbits with respect to the action of the group of automorphisms.

It is well known (see [A], th. 3) that in the decomposition $E := E_1 \oplus \cdots \oplus E_n$ of a vector bundle over a compact variety into direct sum of indecomposable bundles, the bundles E_i are uniquely determined up to order and isomorphy; however in general the bundles E_i are not uniquely determined as subbundles of E.

(2.1) LEMMA.

Let $E_1 := T_{\boldsymbol{P}^2} \oplus \mathcal{O}_{\boldsymbol{P}^2}$. In this decomposition only the vector bundle $F_1 := T_{\boldsymbol{P}^2} \oplus 0$ is uniquely determined as subbundle of E_1.

PROOF: We first observe that the vector bundle $G_1 := 0 \oplus \mathcal{O}_{\boldsymbol{P}^2}$ is not uniquely determined as subbundle of E, since it is not invariant under an automorphism $\phi \in \mathrm{Aut}_S(E_1)$ of the form

$$\left(\begin{array}{c|c} id_{T_{\boldsymbol{P}^2}} & \psi \\ \hline 0 & id_{\mathcal{O}_{\boldsymbol{P}^2}} \end{array}\right)$$

with $\psi \in \mathrm{Hom}(\mathcal{O}_{\boldsymbol{P}^2}, T_{\boldsymbol{P}^2})$, $\psi \neq 0$.

On the other hand, from the Euler sequence

$$0 \to \mathcal{O}_{\boldsymbol{P}^2}(-1) \to \mathcal{O}_{\boldsymbol{P}^2}^{\oplus 3} \to T_{\boldsymbol{P}^2}(-1) \to 0,$$

if follows that the vector bundle $T_{\boldsymbol{P}^2}(-1)$ is generated by global sections, therefore $F_1(-1)$ is the subbundle of $E_1(-1)$ generated by $\Gamma(\boldsymbol{P}^2, E_1(-1))$ and this characterizes $F_1(-1)$ as subbundle of $E_1(-1)$.

(2.2) COROLLARY.

Let $E_2 := T_{P^2}(-3) \oplus \mathcal{O}_{P^2}$. *In this decomposition only the vector bundle* $G_2 := 0 \oplus \mathcal{O}_{P^2}$ *is uniquely determined as subbundle of* E_2.

PROOF: Since E_2 is the dual bundle of E_1 and G_2 consists of the linear forms on E_1, which are zero on $F_1 := T_{P^2} \oplus 0$, the assertion follows from (2.1).

(2.3) THEOREM.

Let $E_1 := T_{P^2} \oplus \mathcal{O}_{P^2}$ *and let* $X_1 := \boldsymbol{P}(E_1)$. *The group* Aut (X_1) *has exactly two orbits*: $A_1 := \boldsymbol{P}(T_{P^2} \oplus 0)$ *and* $A_2 := X_1 - A_1$.

PROOF: We first prove that A_1 is transformed into itself by all $\phi \in \mathrm{Aut}\,(X_1)$. By [S], th. A, Aut $(X_1) = $ Aut $(\boldsymbol{P}(E_1))$, hence ϕ determines an automorphism $\bar{\phi} \in \mathrm{Aut}\,(S)$ and an isomorphism $\boldsymbol{P}(E_1) \simeq \bar{\phi}^* \boldsymbol{P}(E_1)$, which induces the identity on $\boldsymbol{P}^2$ and which can be identified with ϕ. Therefore E_1 is isomorphic to $\bar{\phi}^* E_1 \otimes \mathcal{O}_{P^2}(k)$ and calculating the first Chern classes one sees that $k=0$. Thus ϕ induces an isomorphism $\Phi: E_1 \simeq \bar{\phi}^* E_1$, which must transform $(T_{P^2} \oplus 0)(x)$ into $(\bar{\phi}^*(T_{P^2} \oplus 0))(x) = (T_{P^2} \oplus 0)(\bar{\phi}(x))$ for all $x \in \boldsymbol{P}^2$. Therefore $\phi(\boldsymbol{P}(T_{P^2} \oplus 0)) = \boldsymbol{P}(T_{P^2} \oplus 0)$.

Now we prove that the action of Aut (X_1) is transitive on both A_1 and A_2, by showing that for all $x \in \boldsymbol{P}^2$ the subgroup of Aut (X_1), which fixes the fiber $\boldsymbol{P}(E_1)_x$, acts transitively on $A_1 \cap \boldsymbol{P}(E_1)_x$ and on $A_2 \cap \boldsymbol{P}(E_1)_x$. Let $\xi, \xi' \in A_1 \cap \boldsymbol{P}(E_1)_x$. They correspond to lines r, r' of $\boldsymbol{P}^2$ through the point x. Let $\alpha \in \mathrm{Aut}\,(\boldsymbol{P}^2)$ be such that $\alpha(x) = x$ and $\alpha(r) = r'$ and take an element $\phi \in \mathrm{Aut}\,(\boldsymbol{P}(E_1))$ such that $\bar{\phi} = \alpha$. It is easy to show that $\phi(\xi) = \xi'$.

$$\text{Since End } E_1 = \left(\begin{array}{c|c} \mathrm{End}\, T_{P^2} & \mathrm{Hom}\,(\mathcal{O}_{P^2}, T_{P^2}) \\ \hline \mathrm{Hom}\,(T_{P^2}, \mathcal{O}_{P^2}) & \mathrm{End}\, \mathcal{O}_{P^2} \end{array}\right)$$

and since $\mathrm{Aut}_{P^2}(T_{P^2}) \simeq \mathrm{Aut}_{P^2}(\mathcal{O}_{P^2}) \simeq \boldsymbol{C}^*$ the action on $\boldsymbol{P}(E_1)_x$ of an element of PGL (E_1) can be thought as the action on $\boldsymbol{P}^2$, with projective coordinates $(x_1{:}x_2{:}x_3)$, of a matrix like

$$\begin{pmatrix} \lambda & 0 & a_1 \\ 0 & \lambda & a_2 \\ 0 & 0 & \mu \end{pmatrix} \quad \text{with} \quad \lambda, \mu \in \boldsymbol{C}^*, a_1, a_2 \in \boldsymbol{C}.$$

In this $\boldsymbol{P}^2$, $A_1 \cap \boldsymbol{P}(E_1)_x$ can be identified with the line $x_3 = 0$ and $A_2 \cap \boldsymbol{P}(E_1)_x$ with the complement of such a line; therefore it is clear that PGL (E) acts transitively on $A_2 \cap \boldsymbol{P}(E_1)$.

(2.4) THEOREM.

Let $F_2 := T_{P^2}(-3) \oplus \mathcal{O}_{P^2}$ *and let* $X_2 := \boldsymbol{P}(E_2)$. *The orbits of* X_2 *with respect to the action of* Aut (X_2) *are exacty* $B_1 := \boldsymbol{P}(0 \oplus \mathcal{O}_{P^2})$ *and* $B_2 := X_2 - B_1$.

PROOF: With an argument similar to the one used in prop. (2.3) and using (2.2), one has that all $\phi \in \mathrm{Aut}\,(X_2) = \mathrm{Aut}\,(\boldsymbol{P}(E_2))$ transform B_1 into itself.

Since for all $x \in \boldsymbol{P}^2$, $B_1 \cap \boldsymbol{P}(E_2)_x$ consists exactly of one point, we have only to prove that for all $x \in \boldsymbol{P}^2$ the action of the subgroup Σ of Aut (X_2), containing all automorphisms, which fix the fiber $\boldsymbol{P}(E_2)_x$, is transitive on $B_2 \cap \boldsymbol{P}(E_2)_x$.

Let $(x_1{:}x_2{:}x_3)$ and $(y_1{:}y_2{:}y_3)$ be two points in $B_2 \cap \boldsymbol{P}(E_2)_x$. With an argument similar to the one used in proving prop. (2.3), there exists $\phi \in \Sigma$, which transforms $(x_1{:}x_2{:}x_3)$ into $(y_1{:}y_2{:}y_3')$. Now there exists an element ψ in PGL (E_2) (whose action on $\boldsymbol{P}(E_2)_x$ can be thought as the action on $\boldsymbol{P}^2$ of a matrix like

$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ a_1 & a_2 & \mu \end{pmatrix}$ with $\lambda, \mu \in \boldsymbol{C}^*$, $a_1, a_2 \in \boldsymbol{C}$), which transforms $(y_1{:}\, y_2{:}\, y_3')$ in $(y_1{:}\, y_2{:}\, y_3)$.

Section 3.

In this section we show that the complex manifold $\boldsymbol{P}(F \oplus \mathcal{O}_{\boldsymbol{P}^2})$, where F is a rank 2 topologically trivial vector bundle on $\boldsymbol{P}^2$ of generic splitting type $(-1, 1)$, is almost-homogeneous.

(3.1) PROPOSITION.

Let F be a rank 2 topologically trivial vector bundle on $\boldsymbol{P}^2$ of generic splitting type $(-1, 1)$. *Then*

i) *there is an exact sequence*:

$$(*) \qquad 0 \to \mathcal{O}_{\boldsymbol{P}^2}(1) \underset{\alpha}{\to} F \underset{\beta}{\to} \mathcal{I}_Z(-1) \to 0,$$

where Z is a simple point of $\boldsymbol{P}^2$, which determines the bundle F *up to isomorphy*;

ii) $F \simeq F^{\vee}$ (that is F is self-dual);

iii) *F is almost-homogeneous.*

PROOF: i) The existence of the exact sequence (*) has been proved by Bănică (see [B], lemma 4).

Now let F' be a vector bundle on $\boldsymbol{P}^2$, which makes exact the sequence

$$0 \to \mathcal{O}_{\boldsymbol{P}^2}(1) \to F' \to \mathcal{I}_Z(-1) \to 0.$$

Both F and F' correspond to elements $\eta, \eta' \in \mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_{\boldsymbol{P}^2}(1))$, which are not zero, since the trivial extension $\mathcal{O}_{\boldsymbol{P}^2}(1) \oplus \mathcal{I}_Z(-1)$ is not a vector bundle. But, from [B], §2, dim $\mathrm{Ext}^1(\mathcal{I}_Z(-1), \mathcal{O}_{\boldsymbol{P}^2}(1)) = 1$, therefore $\eta = a\eta'$ with $a \in \boldsymbol{C}^*$, hence $F \simeq F'$.

ii) Since F has rank 2, we have $F^{\vee} \simeq F \otimes \det F^{\vee} \simeq F$.

iii) Let $G := \{g \in \mathrm{Aut}\,(\boldsymbol{P}^2) \mid g(Z) = Z\}$, and let $g \in G$.

The vector bundle g^*F makes exact the sequence

$$0 \to \mathcal{O}_{\boldsymbol{P}^2} \to g^*F \to \mathcal{I}_Z(-1) \to 0$$

and with the same argument used in (i), $g^*F \simeq F$.

(3.2) LEMMA.
Let $E := F \oplus \mathcal{O}_{P^2}$, *with* F *as in* (3.1), *and let* $\mathcal{V}_1 := \alpha(\mathcal{O}_{P^2}(1)) \oplus 0$, $\mathcal{V} := \alpha(\mathcal{O}_{P^2}(1)) \oplus \mathcal{O}_{P^2}$. *The filtration* $\mathcal{V}_1 \subset \mathcal{V} \subset E$ *is invariant with respect to* Aut (E).

PROOF: From the exact sequence $(*)$, one has $\Gamma(\boldsymbol{P}^2, F) = \Gamma(\boldsymbol{P}^2, \alpha(\mathcal{O}_{P^2}(1)))$. It follows $\mathcal{V} = \mathcal{O}_{P^2} \cdot \Gamma(\boldsymbol{P}^2, E)$. In the same way, from the exact sequence

$$0 \to \mathcal{O}_{P^2} \underset{\alpha'}{\to} F(-1) \underset{\beta'}{\to} \mathcal{I}_Z(-2) \to 0$$

one has $\Gamma(\boldsymbol{P}^2, F(-1)) = \Gamma(\boldsymbol{P}^2, \alpha'(\mathcal{O}_{P^2}))$, hence $\mathcal{V}_1(-1) = \mathcal{O}_{P^2} \cdot \Gamma(\boldsymbol{P}^2, E(-1))$.

(3.3) THEOREM.
Let $E := F \oplus \mathcal{O}_{P^2}$, *with* F *as in* (3.1), *and let* $X := \boldsymbol{P}(E)$. *The action of* Aut (X) *on* X *has an open orbit, whose complement is an irreducible hypersurface* $H \subset X$, *which can be described as follows: let* V *be the subbundle of* $E \mid \boldsymbol{P}^2 - Z$ *defined by* $V := (\alpha(\mathcal{O}_{P^2}(1)) \oplus \mathcal{O}_{P^2}) \mid \boldsymbol{P}^2 - Z$. *Then* $H = \boldsymbol{P}(V) \cup \boldsymbol{P}(E)_Z$.

PROOF: We first observe that, from the fact that Z is characterized by the property that every non-zero section of $E(-1)$ vanishes exactly on Z (see the proof of (3.2)), it follows that every $\phi \in \mathrm{Aut}\,(X)$ ($= \mathrm{Aut}\,(\boldsymbol{P}(E))$ by [S], th. A) transforms $\boldsymbol{P}(E)_Z$ into itself. By lemma (3.2), with an argument similar to the one used in proving prop. (2.3), one has that every $\phi \in \mathrm{Aut}\,(X)$ transforms also $\boldsymbol{P}(V)$ into itself.

Now we show that Aut (X) acts transitively on $A := X - (\boldsymbol{P}(V) \cup \boldsymbol{P}(E)_Z)$, by proving that for all $x \in \boldsymbol{P}^2 - Z$ the subgroup PGL (E) of Aut (X) acts transitively on $A \cap \boldsymbol{P}(E)_x$.

We observe that $\mathrm{End}\ E = \left(\begin{array}{c|c} \mathrm{End}\ F & \mathrm{Hom}\,(\mathcal{O}_{P^2}, F) \\ \hline \mathrm{Hom}\,(F, \mathcal{O}_{P^2}) & \mathrm{End}\ \mathcal{O}_{P^2} \end{array}\right)$.

From the exact sequence $(*)$ we get the exact sequence

$$(**) \qquad 0 \to \mathrm{Hom}\,(F, \mathcal{O}_{P^2}(1)) \underset{\sigma}{\to} \mathrm{End}\ F \underset{\tau}{\to} \mathrm{Hom}\,(F, \mathcal{I}_Z(-1)) \to \cdots$$

Since the endomorphisms of F, which are in $\mathrm{Im}\ \sigma = \mathrm{Ker}\ \tau$, cannot be surjective, $id_F \notin \mathrm{Ker}\ \tau$, hence dim $(\mathrm{Hom}\,(F, \mathcal{I}_Z(-1))) \geqq 1$. On the other hand, from $(*)$ we have also the exact sequence

$$0 \to \mathrm{Hom}\,(\mathcal{I}_Z(-1), \mathcal{I}_Z(-1)) \to \mathrm{Hom}\,(F, \mathcal{I}_Z(-1)) \to \mathrm{Hom}\,(\mathcal{O}_{P^2}(1), \mathcal{I}_Z(-1)) \to \cdots$$

where $\mathrm{Hom}\,(\mathcal{O}_{P^2}(1), \mathcal{I}_Z(-1))$ $\mathrm{Hom}\,(\mathcal{O}_{P^2}(1), \mathcal{O}_{P^2}(-1)) = 0$. Therefore $\mathrm{Hom}\,(F, \mathcal{I}_Z(-1)) \simeq \mathrm{Hom}\,(\mathcal{I}_Z(-1), \mathcal{I}_Z(-1))$ $\mathrm{Hom}\,(\mathcal{I}_Z(-1), \mathcal{O}_{P^2}(-1))$ and the last, by Riemann's extension theorem, is isomorphic to $\mathrm{Hom}\,(\mathcal{O}_{P^2}(-1), \mathcal{O}_{P^2}(-1)) \simeq \boldsymbol{C}$. Therefore, $\mathrm{Hom}\,(F, \mathcal{I}_Z(-1)) \simeq \boldsymbol{C}$, and, since $\tau(id_F) \neq 0$, the morphism τ of $(**)$ is surjective and $(**)$ becomes

$$(**)' \qquad 0 \to \mathrm{Hom}\,(F, \mathcal{O}_{P^2}(1)) \underset{\sigma}{\to} \mathrm{End}\ F \underset{\tau}{\to} \mathrm{Hom}\,(F, \mathcal{I}_Z(-1)) \to 0.$$

Again from $(*)$, we have the exact sequence

$$0 \to \mathrm{Hom}\,(\mathcal{I}_Z(-1),\, \mathcal{O}_{P^2}(1)) \to \mathrm{Hom}\,(F,\, \mathcal{O}_{P^2}(1)) \to \mathrm{End}\,(\mathcal{O}_{P^2}(1)) \to 0,$$

since $H^1(\mathcal{H}om\,(\mathcal{I}_Z(-1),\, \mathcal{O}_{P^2}(1))\simeq H^1(\mathcal{O}_{P^2}(2))=0$

as codim $Z=2$. Moreover, since $\mathcal{H}om\,(\mathcal{I}_Z(-1),\, \mathcal{O}_{P^2}(1))\simeq \mathcal{H}om\,(\mathcal{O}_{P^2}(-1),\, \mathcal{O}_{P^2}(1))\simeq \mathcal{O}_{P^2}(2)$ and $\mathcal{E}nd\,(\mathcal{O}_{P^2}(1))\simeq \mathcal{O}_{P^2}$ are globally generated, for any point x in $\boldsymbol{P}^2-Z$ every homomorphism of $F(x)$ into $(\mathcal{O}_{P^2}(1))(x)$ is induced by an element of Hom $(F,\, \mathcal{O}_{P^2}(1))$. Now we fix a point x in $\boldsymbol{P}^2-Z$ and a base v_1, v_2 of $F(x)$ such that $v_1\in(\alpha(\mathcal{O}_{P^2}(1))(x)$. With respect to such a base the automorphisms of $F(x)$ induced by global automorphisms of F can be represented by matrices of the type $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with a, $c\in\boldsymbol{C}^*$, $b\in\boldsymbol{C}$. From $(*)$ we have the exact sequence

$$0 \to \mathrm{Hom}\,(\mathcal{O}_{P^2},\, \mathcal{O}_{P^2}(1)) \to \mathrm{Hom}\,(\mathcal{O}_{P^2},\, F) \to \mathrm{Hom}\,(\mathcal{O}_{P^2},\, \mathcal{I}_Z(-1)=0,$$

hence Hom $(\mathcal{O}_{P^2}, F)\simeq$Hom $(\mathcal{O}_{P^2},\, \mathcal{O}_{P^2}(1))$, that is every homomorphism of $\mathcal{O}_{P^2}$ into F has values in $\alpha(\mathcal{O}_{P^2}(1))$. From $(*)$ we have also

$$0 \to \mathrm{Hom}\,(\mathcal{I}_Z(-1),\, \mathcal{O}_{P^2}) \to \mathrm{Hom}\,(F,\, \mathcal{O}_{P^2}) \to \mathrm{Hom}\,(\mathcal{O}_{P^2}(1),\, \mathcal{O}_{P^2})=\Gamma(\boldsymbol{P}^2,\, \mathcal{O}_{P^2}(-1))=0,$$

hence Hom $(F,\, \mathcal{O}_{P^2})\simeq$Hom $(\mathcal{I}_Z(-1),\, \mathcal{O}_{P^2})$, that is every homomorphism of F into $\mathcal{O}_{P^2}$ is zero on $\alpha(\mathcal{O}_{P^2}(1))$. Now we complete the base v_1, v_2 of $F(x)\simeq(F\oplus 0)(x)$ to a base v_1, v_2, $v_3\in E(x)$ by adding a vector $v_3\in(0\oplus\mathcal{O}_{P^2})(x)$. With respect to such a base the automorphisms of $E(x)$ induced by global automorphisms of E can be represented by a matrix of the type

$$\begin{pmatrix} a & b & d \\ 0 & c & 0 \\ 0 & e & f \end{pmatrix} \quad \text{with} \quad a,\, c,\, f\in\boldsymbol{C}^*,\, b,\, d,\, e\in\boldsymbol{C}.$$

Since $A\cap\boldsymbol{P}(E)_x$ can be identified with the complement of the line $x_2=0$, an easy computation shows that the action of PGL (E) on $A\cap\boldsymbol{P}(E)_x$ is transitive.

Now we prove that $H:=\boldsymbol{P}(V)\cup\boldsymbol{P}(E)_Z$ is an irreducible hypersurface in X, by showing that $H=\overline{\boldsymbol{P}(V)}$. Let U be an open neighbourhood of Z, over which the bundles F and $\mathcal{O}_{P^2}(1)$ are trivial and let (e_1, e_2) and e bases of $F\,|\,U$ and $\mathcal{O}_{P^2}(1)\,|\,U$ respectively. With respect to these bases, the morphism $\alpha{:}\mathcal{O}_{P^2}(1)\to F$ can be described as $\alpha(e)=f_1e_1+f_2e_2$, where f_1, f_2 are holomorphic functions on U, which have exactly one common zero in the point Z. In $\boldsymbol{P}(E)\,|\,U\simeq U\times\boldsymbol{P}^2$ we have

$$\boldsymbol{P}(V)\,|\,U-Z=\{(p;\, t_1{:}t_2{:}t_3)\in(U-Z)\times\boldsymbol{P}^2\,|\,t_1f_2(p)-t_2f_1(p)=0\},$$

hence

$$\begin{aligned}\overline{\boldsymbol{P}(V)\,|\,U-Z}&=\{(p;\, t_1{:}t_2{:}t_3)\in U\times\boldsymbol{P}^2\,|\,t_1f_2(p)-t_2f_1(p)=0\}\\ &=\boldsymbol{P}(V)\,|\,(U-Z)\cup\boldsymbol{P}(E)_Z.\end{aligned}$$

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