

PROPERTIES OF AN L -CARDINAL

By

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When we study the set theory $ZF(aa)$, (Ref. [1] or [3]) it may be natural to consider a cardinal κ such that for every formula in the language of usual set theory,

$$R(\kappa) \models aa\alpha\phi \longleftrightarrow V \models aa\alpha\phi.$$

Let κ be measurable, M a transitive isomorph of V^*/U where U is a normal ultrafilter on κ , and j the canonical elementary embedding of V into M . If “ aa ” is interpreted by the closed unbounded filter of κ and $j(\kappa)$ respectively, in M ,

$$R(\kappa) \models aa\alpha\phi \longleftrightarrow R(j(\kappa)) \models aa\alpha\phi.$$

Therefore measurability is sufficient to show the consistency of the desired situation. But when we want κ to have this property in full V , a new cardinal axiom is needed.

1. Definitions of an L -cardinal and its basic properties.

DEFINITION. Let ϕ be a formula in set theory whose constants are all in $R(\kappa)$, and λ be an ordinal $\geq \kappa$.

a) A cardinal κ is a $(\phi-\lambda)$ -cardinal, if there exists an elementary embedding $j: V \rightarrow M$ such that

- (i) $j(\kappa) > \lambda$ and κ is the least ordinal moved by j ,
- (ii) for every x in $R(j(\kappa))^M$, $M \models \phi(x) \rightarrow V \models \phi(x)$.

b) κ is a ϕ -cardinal if for every $\lambda > \kappa$, κ is a $(\phi-\lambda)$ -cardinal.

c) κ is a Σ_n -cardinal if for every Σ_n formula ϕ , κ is a ϕ -cardinal.

d) Let A be a set of formulas, κ is a $(A-\lambda)$ -cardinal if for every formula in A , κ is a $(\phi-\lambda)$ -cardinal.

e) κ is an L -cardinal if for every formula ϕ , κ is a ϕ -cardinal.

The axiom of an L -cardinal definitely cannot be formulated in ZFC. However, all the arguments can be carried out in ZFC within some $R(\kappa)$ where κ is inaccessible.

The first lemma is trivial but basic in the development.

LEMMA 1.1. *Let ϕ be a formula such that $\{\alpha < \kappa \mid \phi(\alpha)\}$ is unbounded in κ and all constants are in $\mathbf{R}(\kappa)$. If κ is a $(\phi - \lambda)$ -cardinal, there exists an ordinal $\beta > \lambda$ such that $\phi(\beta)$ holds.*

PROOF. Let j be an elementary embedding from V into M as in the definition of $(\phi - \lambda)$ -cardinal. By assumption,

$$V \models \forall \alpha < \kappa \exists \beta < \kappa \{\alpha < \beta \wedge \phi(\beta)\}$$

Hence,

$$M \models \forall \alpha < j(\kappa) \exists \beta < j(\kappa) \{\alpha < \beta \wedge \phi(\beta)\}$$

Since $\lambda < j(\kappa)$, there is a $\beta < j(\kappa)$ such that $M \models \phi(\beta)$ and $\beta > \lambda$. Also $V \models \phi(\beta)$ because $\beta \in \mathbf{R}(j(\kappa))^M$.

COROLLARY 1.2. *Let κ be a $\{\phi, \neg\phi\}$ -cardinal. If $\{\alpha < \kappa \mid \phi(\alpha)\}$ is closed unbounded in κ , then $\{\alpha \mid \phi(\alpha)\}$ is closed unbounded in \mathbf{OR} .*

PROOF. Assume $\{\alpha < \kappa \mid \phi(\alpha)\} = C$ is closed unbounded in κ . By LEMMA 1.1. $\{\alpha \mid \phi(\alpha)\} = C'$ is an unbounded class. Let α be a limit point of C' and $j: V \rightarrow M$ be an elementary embedding that satisfies the definition of $(\{\phi, \neg\phi\} - \alpha)$ -cardinal.

$$V \models \forall \beta < \alpha \exists \gamma < \alpha \{\beta < \gamma \wedge \phi(\gamma)\} \quad \text{implies}$$

$$M \models \forall \beta < \alpha \exists \gamma < \alpha \{\beta < \gamma \wedge \phi(\gamma)\}.$$

As C is closed, $M \models \forall \alpha < j(\kappa) \{\forall \beta < \alpha \exists \gamma < \alpha \{\beta < \gamma \wedge \phi(\gamma)\} \rightarrow \phi(\alpha)\}$. Hence $M \models \phi(\alpha)$. Also $V \models \phi(\alpha)$.

For simplicity we consider a fixed formula $\Phi_0(x) \equiv \exists \alpha \{x = R(\alpha)\}$. If κ is a $(\Phi_0 - \lambda)$ -cardinal, $\mathbf{R}(j(\kappa))^M = \mathbf{R}(j(\kappa))$. Of course some results follow by the weaker assumption that κ is a $(\phi - \lambda)$ -cardinal which is reduced from the fact that κ is a $(\Phi_0 - \lambda)$ -cardinal.

LEMMA 1.3. *If κ is a Φ_0 -cardinal, $\{\alpha \mid \alpha \text{ is strongly inaccessible}\}$ is a proper class.*

PROOF. Since κ is measurable, $\{\alpha < \kappa \mid \alpha \text{ is strongly inaccessible}\}$ is unbounded in κ . Since $\mathbf{R}(j(\kappa))^M = \mathbf{R}(j(\kappa))$, the strongly inaccessible in M which are less than $j(\kappa)$ are also strongly inaccessible in V . Now the conclusion is clear by Corollary 1.2.

LEMMA 1.4. *If κ is a $(\Phi_0 - \kappa)$ -cardinal, κ is the κ -th measurable.*

PROOF. Let $j: V \rightarrow M$ be an associated embedding. $j(\kappa) > \kappa$ and $\text{lim}(\kappa)$ implies

$j(\kappa) > \kappa + \omega$. Hence $\mathbf{R}(\kappa+2)^M = \mathbf{R}(\kappa+2)$. Let U be a normal ultrafilter on κ . Since $U \in \mathbf{R}(\kappa+2)$, $M \models (U \text{ is a normal ultrafilter on } \kappa)$. Hence $M \models (\kappa \text{ is measurable})$. As usual, define U'

$$X \in U' \text{ iff } X \subset \kappa \wedge \kappa \in j(X)$$

U' is a normal ultrafilter on κ and $\{\alpha < \kappa \mid \alpha \text{ is measurable}\} \in U'$.

COROLLARY 1.5. *If κ is a Φ_0 -cardinal, $\{\alpha \mid \alpha \text{ is measurable}\}$ is a proper class.*

PROOF. By Corollary 1.2 and Lemma 1.4.

1.3 and 1.5 follow from the assumption of an extendible cardinal. 1.4 follows from 2^κ -supercompactness. As our definition of $(\phi - \lambda)$ -cardinal does not assume ${}^\lambda M \subset M$, it is not known whether κ carries a normal ultrafilter on $P_\kappa \lambda$. Even it is not clear whether κ is λ -compact (Ref. [2]). The next definition is also due to [2].

DEFINITION. Σ_n relativizes down to $\mathbf{R}(\kappa)$ iff for each formula $\phi \in \Sigma_n$,

$$\forall a \in \mathbf{R}(\kappa) \{ \phi(a) \rightarrow \mathbf{R}(\kappa) \models \phi(a) \}.$$

LEMMA 1.6. *If κ is a Σ_n -cardinal, Σ_n relativizes down to $\mathbf{R}(\kappa)$.*

PROOF. By induction on n . Π_2 relativizes down to $\mathbf{R}(\kappa)$ since κ is strongly inaccessible (Ref. [4]). Thus Σ_1 relativizes down to $\mathbf{R}(\kappa)$. Assume $\phi \in \Sigma_n$ and $a \in \mathbf{R}(\kappa)$, $\phi(a)$ holds. ($n \geq 2$) There is a Σ_{n-1} formula ψ such that $\phi(a) \equiv \exists x \psi(x, a)$. Since $\phi(a)$, for some b , $V \models \psi(b, a)$. Choose λ large enough to $b \in \mathbf{R}(\lambda)$. Let $j: V \rightarrow M$ be an associated embedding of $(\neg\phi - \lambda)$ -ness of κ . (Note that $\neg\phi \in \Sigma_n$.) Since $b \in \mathbf{R}(j(\kappa))^M$ and $V \models \psi(b, a)$, $M \models \psi(b, a)$. (As Φ_0 is a Σ_2 formula, we can assume $b \in M$.) But in V , Σ_{n-1} relativizes down to $\mathbf{R}(\kappa)$. Hence in M , Σ_{n-1} relativizes down to $\mathbf{R}(j(\kappa))$. Therefore $M \models (\mathbf{R}(j(\kappa)) \models \psi(b, a))$. Hence $M \models \mathbf{R}(j(\kappa)) \models \phi(a)$. By elementarity of j and $j(a) = a$, $\mathbf{R}(\kappa) \models \phi(a)$ in V .

Now the followings are all clear.

THEOREM 1.7. *If κ is an L -cardinal, Σ_ω relativizes down to $\mathbf{R}(\kappa)$.*

COROLLARY 1.8. *If κ is an L -cardinal, $\mathbf{R}(\kappa)$ is an elementary substructure of V .*

COROLLARY 1.9. *If κ is an L -cardinal and α is a definable cardinal, then $\alpha < \kappa$.*

Note that: If κ is supercompact, Σ_2 relativizes down to $R(\kappa)$. If κ is extendible, Σ_3 relativizes down to $R(\kappa)$. (Ref. [2])

Also recall the notion of "ghost cardinal" of M. Takahashi. It is the least cardinal not definable by \tilde{A}_1^2 -formula. Of course an L -cardinal is not definable in set theory, ZFC.

COROLLARY 1.10. *Let κ be an L -cardinal and ϕ be a formula whose constants are all in $R(\kappa)$. If there exists an ordinal $\gamma \geq \kappa$, such that $\phi(\gamma)$ holds, then $\{\alpha \mid \phi(\alpha)\}$ is a proper class and $\{\alpha < \kappa \mid \phi(\alpha)\}$ is unbounded in κ .*

PROOF. By Corollary 1.2, it suffices to show $\{\alpha < \kappa \mid \phi(\alpha)\}$ is unbounded in κ . If not, there is an $\alpha < \kappa$ such that $\forall \beta < \kappa (\alpha < \beta \rightarrow \neg \phi(\beta))$. By the above Theorem, $\forall \beta < \kappa (\alpha < \beta \rightarrow R(\kappa) \models \neg \phi(\beta))$. Then $R(\kappa) \models \forall \beta (\alpha < \beta \rightarrow \neg \phi(\beta))$. Using Corollary 1.9, we have $\forall \beta (\alpha < \beta \rightarrow \neg \phi(\beta))$. Contradicting $\phi(\gamma)$.

1.7-1.10 are too strong and give us suspicion about consistency.

2. Cohen extension and an L -cardinal.

Once L -cardinal is defined, many problems are raised. If V is a model of $ZFC + \exists \kappa : L\text{-cardinal}$, is there a Cohen extension of V where $ZFC + \exists \kappa : L\text{-cardinal} + G.C.H.$ hold? Is not $j(\kappa)$ necessarily measurable? (When κ is extendible, $j(\kappa)$ is always measurable.) Can an L -cardinal be strongly compact?

All these questions are unclear now. We need some technics to preserve an L -cardinal. We only get a quite easy fact that is not useful to solve the above problems.

LEMMA 2.1. *If $|P| < \kappa$ and κ is a $((\Vdash \phi) - \lambda)$ -cardinal, then κ is a $(\phi - \lambda)$ -cardinal in $V[G]$.*

PROOF. We can assume $P \in R(\kappa)$. Let $j : V \rightarrow M$ be an elementary embedding that witnesses κ is a $((\Vdash \phi) - \lambda)$ -cardinal. We extend j to \tilde{j} as usual.

$$\tilde{j}(K_G(x)) = K_G(j(x))$$

(We use the notations of [7].) \tilde{j} is an elementary embedding of $V[G]$ into $M[G]$. (Ref. [5]) If $x \in R(j(\kappa))^{M[G]}$, there is a name $\underline{x} \in R(j(\kappa))^M$ such that $K_G(\underline{x}) = x$. $M[G] \models \phi(x)$ iff $M \models (p \Vdash \phi(\underline{x}))$ for some $p \in G$. The latter implies $\exists p \in G (p \Vdash \phi(\underline{x}))$ in V . Therefore $V[G] \models \phi(x)$.

THEOREM 2.2. *If κ is an L -cardinal in V and $|P| < \kappa$, κ is an L -cardinal in $V[G]$.*

Using Theorem 2.2, we consider the relation between an L -cardinal and strongly compact cardinals.

THEOREM 2.3. *Let κ be an L -cardinal.*

- (i) *If κ is strongly compact, κ is the κ -th strongly compact.*
- (ii) *If κ is not strongly compact, there is no strongly compact cardinal greater than κ , and it is consistent that there is an L -cardinal and there is no strongly compact cardinal.*

PROOF. (i) By Corollary 1.10.

(ii) At first we assert that κ is not a limit of strongly compacts. For a measurable cardinal that is a limit of strongly compacts is strongly compact and κ is clearly measurable. (Ref. [6])

Thus there is no strongly compact above κ by Corollary 1.10. And there exists a regular cardinal $\alpha < \kappa$ such that there is no strongly compact cardinal between α and κ . We use the forcing condition that collapses α to ω_1 . In the extended universe there is no strongly compact cardinal and κ remains an L -cardinal by Theorem 2.2.

References

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