TSUKUBA J. MATH.
Vol. 6 No. 1 (1982). 9–19

ON THE ADJUNCTION SPACES OF FREE L-SPACES AND M_1-SPACES

By

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A class of free L-spaces is defined by Nagami [7]. This class contains all Lašnev spaces and is contained in the class of M_1-spaces in the sense of Ceder [3]. In this paper, we consider the sum theorem of free L-spaces and the property of being M_1-spaces and free L-spaces of the adjunction spaces. The main results are as follows:

1. Let $Z=X\cup Y$ be stratifiable, where X, Y are free L-spaces and X is a closed set of Z with a uniformly approaching anti-cover in Z. Then Z is a free L-space.
2. The adjunction space $X\cup_f Y$ is a free L-space if X is an L-space in the sense of Nagami [6] and Y is a free L-space.
3. Let $Z=X\cup Y$ be stratifiable, where X, Y are M_1-spaces and X is a closed set with a uniformly approaching anti-cover in Z. Then Z is an M_1-space.
4. The adjunction space $Z=X\cup_f Y$ is an M_1-space if X is a free L-space and Y is an M_1-space.
5. Every closed set of a free L-space has a closure-preserving open neighborhood base.
6. The closed irreducible image of an M_1-space with $\dim=0$ is also an M_1-space.

All spaces are assumed to be Hausdorff and mappings to be continuous and onto unless the contrary is stated explicitly. N always denotes the positive integers. As for undefined term, see Nagami [6] and [7], or [4].

A space X is called a *monotonically normal space* if the following (MN) is satisfied:

(MN) To each pair (H, K) of separated subsets of X, one can assign an open set $U(H, K)$ in such a way that

(i) $H\subset U(H, K)\subset\overline{U(H, K)}\subset X-K$ and

(ii) if (H', K') is a pair of separated sets having $H\subset H'$ and $K'\subset K$, then $U(H, K)\subset U(H', K')$.

LEMMA 1 ([4, Lemma 3.1]). *Let X be a monotonically normal space, F a*

Received May 13, 1981.

closed set of X and $\{W_\alpha : \alpha\in A\}$ an anti-closure-preserving family of open neighborhoods of F. Then there exists an anti-cover $\mathcal{U}$ of F that each W_α is a semi-canonical neighborhood of F with respect to $\mathcal{U}$.

THEOREM 1. *Let X, Y be a free L-spaces and $Z=X\cup Y$ be a stratifiable space, where X is a closed set which has a uniformly approaching anti-cover in Z. Then Z is a free L-space.*

PROOF. Part 1: Let $(\mathcal{F}, \{\mathcal{U}_F : F\in\mathcal{F}\})$ be a free L-structure of X. Let $\mathcal{V}_X=\{V_\beta : \beta\in B\}$ be a uniformly approaching anti-cover of X in Z. For each $F\in\mathcal{F}$, let $\mathcal{U}_F=\{U_\alpha : \alpha\in A_F\}$ be assumed to be locally finite in $X-F$. Set

$$\Delta(F)=\{\delta\subset A_F : W(\delta)=F\cup(\bigcup\{U_\alpha : \alpha\in\delta\}) \text{ is an open neighborhood of } F \text{ in } X\}.$$

Then $\{W(\delta) : \delta\in\Delta(F)\}$ is anti-closure-preserving in X. For each $x\in X-F$, set

$$V(x)=U(\{x\},\ F\cup(\bigcup\{X-W(\delta) : x\in W(\delta),\ \delta\in\Delta(F)\})),$$
$$\mathcal{V}_F=\mathcal{V}_X\cup\{V(x) : x\in X-F\},$$

where U is the monotonically normal operator assured by (MN). Then $\mathcal{V}_F$ is an anti-cover of F in Z. We shall show that $\mathcal{V}_F$ has the following property:

(*) If W_1 is a canonical neighborhood of F with respect to $\mathcal{U}_F$ in X, then there exists a semi-canonical neighborhood U_2 of F in Z with respect to $\mathcal{V}_F$ such that

$$F\subset U_2\cap X\subset W_1, \qquad \bar{U}_2\cap(X-W_1)=\phi.$$

To see (*), choose $\delta\in\Delta(F)$ such that

$$W_2=W(\delta), \qquad \bar{W}_2\subset W_1.$$

Set

$$U_1=U(X-W_2\ F), \qquad U_2=U(\bar{W}_2,\ X-W_1).$$

Then U_2 is an open neighborhood of F in Z such that

$$U_2\cap X\subset W_1, \qquad \bar{U}_2\cap(X-W_1)=\phi.$$

Since $\mathcal{V}_X$ is uniformly approaching in Z,

$$\overline{S(Z-U_2,\ \mathcal{V}_X)}\cap F=\phi.$$

Suppose

$$V(x)\cap(Z-U_2)\neq\phi, \qquad x\in X-F.$$

Note that if $x\in W_2$, then $V(x)\subset U_2$. Therefore $x\notin W_2$. This implies $V(x)\subset U_1$. Since $\bar{U}_1\cap F=\phi$, we have

$$\overline{S(Z-U_2, \mathcal{V}_F)} \cap F=\phi.$$

Part 2: Let $(\mathcal{H}, \{\mathcal{U}_H : H\in\mathcal{H}\})$ be a free L-structure of Y. Write

$$X=\bigcap_{n=1}^{\infty} G_n, \ G_{n+1}\subset G_n, \ n\in N,$$

where each G_n is open in Z. Let $\mathcal{H}=\bigcup_{i=1}^{\infty}\mathcal{H}_i$, where each $\mathcal{H}_i$ is discrete in Y. For each $i\in N$ and $H\in\mathcal{H}_i$, set

$$H_n=H\cap(Z-G_n),$$
$$\mathcal{H}_{in}=\{H_n : H\in\mathcal{H}_i\}, \ n\in N.$$

Then each $\mathcal{H}_{in}$ is a discrete closed collection of Z. Since Z is paracompact, there exists a discrete open collection $\mathcal{V}_{in}=\{V(H_n) : H_n\in\mathcal{H}_{in}\}$ of Z such that

$$H_n\subset V(H_n), \ H_n\in\mathcal{H}_{in}, \ n\in N.$$

Since Z is perfectly normal, there exists an anti-cover $\mathcal{V}_{H_n}$ of H_n in Z with respect to which $V(H_n)$ is a canonical neighborhood of H_n in Z. Choose canonical neighborhoods $V(H_n)_1$ and $V(H_n)_2$ of H_n with respect to $\mathcal{V}_{H_n}$ such that

$$H_n\subset V(H_n)_1\subset \overline{V(H_n)_1}\subset V(H_n)_2\subset \overline{V(H_n)_2}\subset V(H_n).$$

Let $\mathcal{U}_H=\{U_\alpha : \alpha\in A_H\}$ be assumed to be locally finite in $Y-H$. Set

$$\Delta(H)=\{\delta\subset A_H : W(\delta)=H\cup(\bigcup\{U_\alpha : \alpha\in\delta\}) \text{ is an open neighborhood of } H \text{ in } Y\}.$$

For each $\delta\in\Delta(H)$, set

$$W(\delta, n)=(W(\delta)\cap V(H_n)_2)\cup(V(H_n)_2-\overline{V(H_n)_1}).$$

Then $W(\delta, n)$ is an open neighborhood of $H'_n=\overline{V(_n)_1}\cap H$. Morever, it is easily seen that $\{W(\delta, n) : \delta\in\Delta(H)\}$ is anti-closure-preserving in Z. Therefore by Lemma 1, there exists an anti-cover $\mathcal{V}_{H'_n}$ of H'_n in Z such that each $W(\delta, n)$ is a semi-cannonical neighborhood of H'_n with respect to $\mathcal{V}_{H'_n}$. Obseve that for each $\delta\in\Delta(H)$

$$V(H_n)_1\cap W(\delta, n)=W(\delta)\cap V(H_n)_1$$

is an open neighborhood of H_n in Z, and that

$$\mathcal{H}'_{in}=\{H'_n : H_n\in\mathcal{H}_{in}\}$$

is a closed discrete collection of Z. Set

$$\mathcal{F}'=\mathcal{F}\cup\{X\}\cup(\bigcup\{\mathcal{H}_{in} : i, n\in N\}$$
$$\cup(\bigcup\{\mathcal{H}'_{in} : i, n\in N\}).$$

Then $\mathcal{F}'$ is a σ-discrte closed collection of Z. Set

$$\mathcal{P}=(\mathcal{F}',\ \{\mathcal{V}_F : F\in\mathcal{F}\}\cup\{\mathcal{V}_X\}\cup\{\mathcal{V}_{H_n} : H_n\in\mathcal{H}_{in},\ i,\ n\in N\}\cup\{\mathcal{V}_{H'_n} : H_n\in\mathcal{H}_{in},\ i,\ n\in N\}).$$

We shall show that $\mathcal{P}$ forms a free L-structure of Z. Suppose $p\in W$ for an arbitrary open set W of Z and an arbitrary point p of Z. Consider two cases. The first is the case $p\in X$. Since $(\mathcal{F}, \{\mathcal{V}_F : F\in\mathcal{F}\})$ is a free L-structure of X, there exist $F_1, \cdots, F_k\in\mathcal{F}$ and their canonical neighborhoods $V_1, \cdots, V_k$ such that

$$p\in\bigcap_{j=1}^{k} F_j\subset\bigcap_{j=1}^{k} V_j\subset W\cap X.$$

By (*) there exists for each j a semi-canonical neighborhood W_j of F_j with respect to $\mathcal{V}_{F_j}$ such that

$$F_j\subset W_j\cap X\subset V_j,\ \overline{W}_j\cap(X-V_j)=\phi.$$

Note that $Z-(\bigcap_{j=1}^{k}\overline{W}_j-W)$ is an open neighborhood of X in Z. Since $\mathcal{V}_X$ is approaching to X in Z, there exists a canonical neighborhood W_0 of X with respect to $\mathcal{V}_X$ such that

$$W_0\cap\left(\bigcap_{j=1}^{k}\overline{W}_j-W\right)=\phi.$$

Thus we have

$$p\in\bigcap_{j=1}^{k} F_j\cap X\subset\bigcap_{j=0}^{k} W_j\subset W.$$

The second case is $p\in Z-X$. Since $(\mathcal{H}, \{\mathcal{V}_H : H\in\mathcal{H}\})$ is a free L-structure of Y, there exist $H_1, \cdots, H_k\in\mathcal{H}$ and their canonical neighborhoods $W(\delta_1), \cdots, W(\delta_k)$ with $\delta_1\in\Delta(H_1), \cdots, \delta_k\in\Delta(H_k)$ such that

$$p\in\bigcap_{j=1}^{k} H_j\subset\bigcap_{j=1}^{k} W(\delta_j)\subset W\cap Y.$$

Choose $n\in N$ such that $p\in Z-G_n$. Then we have

$$\begin{aligned} p&\in\bigcap_{j=1}^{k}(H_j)_n\cap\bigcap_{j=1}^{k}(H_j)'_n\\ &\subset\bigcap_{j=1}^{k} W(\delta_j,\ n)\cap\bigcap_{j=1}^{k} V((H_j)_n)_1\subset W.\end{aligned}$$

As is shown in the above, each $W(\delta_j,\ n)$ and each $V((H_j)_n)_1$ are semi-canonical and canonical with respect to $\mathcal{V}_{(H_j)'_n}$ and $\mathcal{V}_{(H_j)_n}$, respectively. Therefore by the result of [4], Z is a free L-space.

Let f be a mapping of a closed set of a space X into a space Y. The adjunction space Z of X, Y is denoted as $Z=X\cup_f Y$. In the sequel, the mapping f in $Z=X\cup_f Y$ is assumed to be one of a closed set H into Y, and $p: X\vee Y\to Z$ denotes the quotient mapping. As the Ito's example in [4] shows, the adjunction space of free L-spaces need not be a free L-space. Miwa in [5] showed that the adjunction space of X and Y is a free L-space if X is a metric space and Y is a free L-space. The following corollary and the next theorem refine the result.

COROLLOARY 1. *Let X, Y be free L-spaces and H a closed set of X having a uniformly approaching anti-cover in X. Then $Z=X\cup_f Y$ is a free L-space.*

PROOF. As is well known, Z is a stratifiable space. Set

$$Z=X'\cup Y', \qquad X'=p(Y), \qquad Y'=Z-p(Y).$$

Then it is easily seen that $\{X', Y'\}$ satisfies the condition of the above theorem.

COROLLARY 2. *$X=\bigcup_{n=1}^{\infty} X_n$ be a stratifiable space, where each X_n is a closed free L-space, and has a uniformly approaching anti-cover in X. Then X is a free L-space.*

COROLLARY 3. *Let $X=\bigcup\{X_\alpha : \alpha\in A\}$ be a stratifiable space, where $\{X_\alpha : \alpha\in A\}$ is locally finite in X and each X_α is a closed free L-space and has a uniformly approaching anti-cover in X. Then X is a free L-space.*

THEOREM 2. *Let X be an L-space and Y a free L-space. Then $Z=X\cup_f Y$ is a free L-space.*

PROOF. Set

$$X'=p(Y), \qquad Y'=Z-p(Y).$$

Then $Z=X'\cup Y'$ and X', Y' are free L-spaces. Obviously Z is stratifiable and X' is a closed set of Z. We shall modify the part 1 of the proof of Theorem 1. Let $(\mathfrak{F}, \{\mathcal{U}_F : F\in\mathfrak{F}\})$ be a free L-structure of X' and let $\mathcal{U}_F$, $\Delta(F)$ and $W(\delta)$ be the same as in the part 1 with X replaced by X'. By the same way we define $V(x)$ for each $x\in X'-F, F\in\mathfrak{F}$. Since Z is hereditarily normal, there exists an open set U_F of $Z(F)=Z-F$ (and hence of Z) such that

$$X'-F\subset U_F\subset \mathrm{Cl}_{Z(F)}(U_F)\subset\bigcup\{V(x) : x\in X'-F\},$$

where $\mathrm{Cl}_{Z(F)}(U_F)$ denotes the closure in the subspace $Z(F)$. Since X is an L-space, $p_X^{-1}(F)$ has an approaching anti-cover $\mathcal{V}(p_X^{-1}(F))$ in X, where $p_X=p|X$ is the restriction of the quotient mapping. Set

$$\mathcal{V}_F=\{V(x) : x\in X'-F\}\cup p(\mathcal{V}(p_X^{-1}(F)))\,|\,((Z-\mathrm{Cl}_{Z(F)}(U_F)).$$

Then obviously $\mathcal{V}_F$ is an anti-cover of F in Z. We shall show that $\mathcal{V}_F$ has the property (*) stated there. Let W_1 be a cannonical neighborhood of F with respect to $\mathcal{U}_F$ in X'. Take $\delta\in\Delta(F)$ and open sets U_1, U_2 of Z such that

$$W_2=W(\delta),\ \bar{W}_2\subset W_1,$$
$$U_1=U(X'-W_2, F),\qquad U_2=U(\bar{W}_2, X'-W_1).$$

Then we have

$$S(Z-U_2, \{V(x): x\in X'-F\})\subset U_1,\ \bar{U}_1\cap F=\phi.$$

Since $\mathcal{V}(p_X^{-1}(F))$ is approaching to $p_X^{-1}(F)$ in X, there exists an open neighborhood V of $p_X^{-1}(F)$ in X such that

$$S(X-p_X^{-1}(U_2),\ \mathcal{V}(p_X^{-1}(F)))\cap V=\phi.$$

Set

$$N=p(V)\cup U_F.$$

Then N is an open neighborhood of F in Z such that

$$N\cap S(Z-U_2, p(\mathcal{V}(p_X^{-1}(F))\,|\,(Z-\mathrm{Cl}_{Z(F)}(U_F)))=\phi,$$

which implies that U_2 is semi-canonical with respect to $\mathcal{V}_F$. Since H has an approaching anti-cover in X, X' has an approaching anti-cover $\mathcal{V}_{X'}$ in Z. If we observe that in the part 2 of the proof of Theorem 1 we use merely the fact that $\mathcal{V}_X$ is approaching, then the proof is obviously completed.

THEOREM 3. *Let $Z=X\cup Y$ be a stratifiable space, where X, Y are M_1-spaces and X is a closed set which has a uniformly approaching anti-cover in Z. Then Z is an M_1-space.*

PROOF. Let $\mathcal{U}=\cup_{j=1}^{\infty}\mathcal{U}_j$ be a base for X, where each $\mathcal{U}_j=\{U_\alpha: \alpha\in A_j\}$ is closure-preserving in X. Write

$$U_\alpha=\bigcup_{j=1}^{\infty}F_{\alpha j},$$

where each $F_{\alpha j}$ is closed in X. Set

$$U'_\alpha=\bigcup_{j=1}^{\infty}U(F_{\alpha j}, X-U_\alpha).$$

Then U'_α satisfies the following conditions:

(i) U'_α is an open set of Z such that

$$U'_\alpha\cap X=U_\alpha,\ \alpha\in A_j,\ j\in N.$$

(ii) For an arbitrary subset B of $A_j, j\in N$, if $p\in X$ and $p\notin\overline{\bigcup\{U_\alpha: \alpha\in B\}}$, then

$p\notin\overline{\bigcup\{U'_\alpha : \alpha\in B\}}$.

(i) is obvious. To see (ii), suppose $p\notin\overline{\bigcup\{U_\alpha : \alpha\in B\}}$. Set

$$N(p)=Z-\overline{U(\overline{\bigcup\{U_\alpha : \alpha\in B\}}, \{p\})}.$$

Then $N(p)$ is an open neighborhood of p in Z such that

$$N(p)\cap U'_\alpha=\phi \text{ for every } \alpha\in B.$$

We shall construct collections $\mathcal{U}_\alpha=\{U_{\alpha\beta} : \beta\in B_\alpha\}$, $\alpha\in A_j$, $j\in N$, satisfying the following:

(1) Each $U_{\alpha\beta}$ is an open set of Z such that

$$U_{\alpha\beta}\cap X=U_\alpha \text{ and } U_{\alpha\beta}\subset U'_\alpha \text{ for every } \beta\in B_\alpha.$$

(2) $\bigcup\{\mathcal{U}_\alpha : \alpha\in A_j\}$ is closure-preserving in Z for every $j\in N$.

(3) If U is an open set of Z such that $U\cap X=U_\alpha$ for $\alpha\in A_j$, $j\in N$, then $U_{\alpha\beta}\subset U$ for some $\beta\in B_\alpha$.

Since Z is hereditarily paracompact, the uniformly approaching anti-cover $\mathcal{V}=\{V_\lambda : \lambda\in\Lambda\}$ of X can be assumed to be locally finite in $Z-X$. For each $\alpha\in A$, $j\in N$, set

$$B_\alpha=\{\beta\subset\Lambda : U_{\alpha\beta}=U_\alpha\cup(\bigcup\{V_\lambda : \lambda\in\beta\}) \text{ is an open neighborhood of } U_\alpha \text{ in Z such that } U_{\alpha\beta}\subset U'_\alpha\}, \mathcal{U}_\alpha=\{U_{\alpha\beta} : \beta\in B_\alpha\}.$$

Then (1) and (3) follow easily. (2) follows from (ii) and from the fact that $\mathcal{U}_j$ is closure-preserving in X. It is obvious from (3) that $\bigcup\{\mathcal{U}_\alpha : \alpha\in A_j, j\in N\}$ forms a local base of each point of X in Z. Since X is a closed set of a stratifiable space Z and Y is an M_1-space, there exists a σ-closure-preserving open collection $\mathcal{B}$ of Z such that $\mathcal{B}$ forms a local base of each point of $Z-X$ in Z. Set

$$\mathcal{W}=\bigcup\{\mathcal{U}_\alpha : \alpha\in A_j, j\in N\}\cup\mathcal{B}.$$

Then $\mathcal{W}$ is a σ-closure-preserving base of Z.

We define the property (P) as follows:

(P) Suppose that we are given a closure-preserving open collection $\mathcal{U}=\{U_\alpha : \alpha\in A\}$ of a closed set F of a space X. Then for each $\alpha\in A$, there exists an open collection $\mathcal{U}_\alpha=\{U_{\alpha\beta} : \beta\in B_\alpha\}$ of X satisfying the following:

(1) $U_{\alpha\beta}\cap F=U_\alpha$ for each $\beta\in B_\alpha$, $\alpha\in A$.

(2) $\mathcal{U}'=\bigcup\{\mathcal{U}_\alpha : \alpha\in A\}=\{U_{\alpha\beta} : \beta\in B_\alpha, \alpha\in A\}$

is closure-preserving in X.

(3) If V is an open set of X such that $V\cap F=U_\alpha$ for $\alpha\in A$, then there exists $\beta\in B_\alpha$ such that $U_{\alpha\beta}\subset V$.

LEMMA 2. *Every closed set F of a free L-space X has the property* (P).

PROOF. First we consider the case of dim $X=0$. Suppose that we are given a closure-preserving open collection $\mathcal{U}=\{U_\alpha:\alpha\in A\}$ of a closed set F of a free L-space X with dim $X=0$. Write

$$F=\bigcap_{n=1}^{\infty}H_n,\ H_{n+1}\subset H_n,\ n\in N,\ H_1=X,$$

where each H_n is closed and open in X. Since X is an M_1-space, there exists a base $\mathcal{B}=\bigcap_{i=1}^{\infty}\mathcal{B}_i$ for X, where each $\mathcal{B}_i$ is closure-preserving in X. For each $i\in N$ and $B\in\mathcal{B}_i$, set $B_i=B\cap H_i$. Let $\{\mathcal{S}_\lambda:\lambda\in\Gamma\}$ be the totality of subcollections of $\mathcal{B}$. For each $\lambda\in\Gamma$ set

$$V_{\lambda i}=\cup\{B_i:B\in\mathcal{S}_\lambda\cap\mathcal{B}_i\},$$
$$V_\lambda=\bigcup_{i=1}^{\infty}V_{\lambda i}.$$

For each $\alpha\in A$, set

$$B'_\alpha=\{\lambda\in\Gamma: V_\lambda \text{ is an open set of } X \text{ such that } V_\lambda\cap F=U_\alpha\}.$$

For each $\alpha\in A$, we expand U_α to an open set U'_α of X by the same method as in the proof of Theorem 3. Thus each U'_α satisfies (i) and (ii) stated there. Set

$$B_\alpha=\{\beta\in B'_\alpha: V_\beta\subset U'_\alpha\},$$
$$\mathcal{U}_\alpha=\{U_{\alpha\beta}=V_\beta:\beta\in B_\alpha\}.$$

Obviously each $\mathcal{U}_\alpha$ satisfies (1). To see (2), let B_0 be an arbitrary subset of $\cup\{\{\alpha\}\times B_\alpha:\alpha\in A\}$ and suppose

$$p\notin\bigcup\{\overline{U_{\alpha\beta}}:(\alpha,\beta)\in B_0\}.$$

Write

$$B_0=\bigcup\{\{\alpha\}\times B_\alpha^0:\alpha\in A_0\}.$$

If $p\in F$, then $p\notin\overline{\bigcup\{U_\alpha:\alpha\in A_0\}}$, because $\mathcal{U}$ is closure-preserving in F. Therefore by the property (ii) of U'_α, $p\notin\overline{\bigcup\{U'_\alpha:\alpha\in A_0\}}$. This implies

$$p\notin\overline{\bigcup\{U_{\alpha\beta}:(\alpha,\beta)\in B_0\}}.$$

If $p\in X-F$, then there exists $k\in N$ with $p\in H_k-H_{k+1}$. Write

$$U_{\alpha\beta}=\bigcup\{V_{\beta i}:i\in N\},\ \beta\in B_\alpha^0,\ \alpha\in A_0,$$
$$V_{\beta i}=\bigcup\{B_i:B\in\mathcal{S}_\beta\cap\mathcal{B}_i\},\ \beta\in B_\alpha^0,\ \alpha\in A_0.$$

Since $X-H_{k+1}$ is an open neighborhood of p such that

$$(X-H_{k+1})\cap V_{\lambda n}=\phi,\ n\geqq k+1,\ \lambda\in\Lambda,$$
$$p\notin\overline{\bigcup\{V_{\beta n}:n\geqq k+1,\ \beta\in\bigcup\{B_\alpha^0:\alpha\in A_0\}\}}.$$

Therefore if we assume

$$p\in\overline{\bigcup\{U_{\alpha\beta}:(\alpha,\beta)\in B_0\}},$$

then

$$p\in\overline{\bigcup\{V_{\beta m}:m\leqq k,\ \beta\in\bigcup\{B_\alpha^0:\alpha\in A_0\}\}}.$$

This implies for some $m\leqq k$

$$p\in\overline{\bigcup\{V_{\beta m}:\beta\in\bigcup\{B_\alpha^0:\alpha\in A_0\}\}}.$$

Since $\mathcal{B}_m$ is closure-presering in $X, p\in\bar{B}$ for some $B\in\mathcal{S}_\beta\cap\mathcal{B}_m$, $\beta\in\bigcup\{B_\alpha^0:\alpha\in A_0\}$. Since $p\in H_m$ and H_m is open, it follows that

$$p\in\overline{B\cap H_m}=\overline{B_m}\subset\overline{V_{\beta m}}.$$

Hence $p\in\overline{U_{\alpha\beta}}$ for $(\alpha,\beta)\in B_0$, a contradiction. Thus (2) is satisfied. To see (3), let V be an arbitrary open set of X such that $V\cap F=U_\alpha$. For each $p\in U_\alpha$, there exist $n(p)\in N$ and $B_p\in\mathcal{B}_{n(p)}$ such that

$$p\in B_p\subset V\cap U'_\alpha.$$

Obviously $p\in(B_p)_{n(p)}\subset V$. If we put

$$\mathcal{S}_\beta=\{B_p:p\in U_\alpha\},$$

then $U_{\alpha\beta}\subset V$.

Next, we consider the general case. Let X be a free L-space. Then by [7, Theorem 2.10] there exists a perfect mapping f of a free L-space Z with dim $Z\leqq 0$ onto X. By [2, Lemma 3.2 (a)] we can assume that f is irreducible. Suppose that we are given a closure-preserving open collection $\mathcal{U}=\{U_\alpha:\alpha\in A\}$ of a closed set F of X. In the preceding manner, we construct for each $\alpha\in A$ an open collection $\{(f^{-}(U_\alpha))_\beta:\beta\in B'_\alpha\}$ of Z satisfying the following:

(1)′ $\quad (f^{-1}(U_\alpha))_\beta\cap f^{-1}(F)=f^{-1}(U_\alpha),\ \beta\in B'_\alpha,\ \alpha\in A.$

(2)′ $\{(f^{-1}(U_\alpha))_\beta:\beta\in\bigcup\{B'_\alpha:\alpha\in A\}$ is closure-preserving in $Z-f^{-1}(F)$.

(3)′ If V is an open set of Z such that $V\cap f^{-1}(F)=f^{-1}(U_\alpha)$, then $(f^{-1}(U_\alpha))_\beta\subset V$ for some $\beta\in B'_\alpha$.

For each $\alpha\in A$, $\beta\in B'_\alpha$, put

$$U_{\alpha\beta}=X-f(Z-(f^{-1}(U_\alpha))_\beta).$$

We expand each U_α to an open set U'_α of X by the same method as in the proof of Theorem 3. Construct

$$\mathcal{U}_\alpha=\{U_{\alpha\beta}:\beta\in B_\alpha\},\ \alpha\in A,$$

$$B_\alpha=\{\beta\in B'_\alpha : U_{\alpha\beta}\subset U'_\alpha\}.$$

(1) follows easily from (1)$'$. To see (2), let B_0 be an arbitrary subset of $\bigcup\{\{\alpha\}\times B_\alpha : \alpha\in A\}$ and suppose

$$p\notin\bigcup\{\overline{U_{\alpha\beta}} : (\alpha, \beta)\in B_0\}.$$

Write

$$B_0=\bigcup\{\{\alpha\}\times B^0_\alpha : \alpha\in A_0\}.$$

If $p\in F$, then $p\notin\overline{\bigcup\{U'_\alpha : \alpha\in A_0\}}$ by the property (ii) of U'_α. Consequently we have $p\notin\overline{\bigcup\{U_{\alpha\beta} : (\alpha, \beta)\in B_0\}}$. Let $p\in X-F$ and assume $p\in\overline{\bigcup\{U_{\alpha\beta} : (\alpha, \beta)\in B_0\}}$. Then we have

$$f^{-1}(p)\subset Z-f^{-1}(F),$$
$$f^{-1}(p)\cap\overline{\bigcup\{(f^{-1}(U_\alpha))_\beta : (\alpha, \beta)\in B_0\}}\neq\phi.$$

By (2)$'$, there exist $\beta\in B^0_\alpha$, $\alpha\in A_0$ such that

$$f^{-1}(p)\cap\overline{(f^{-1}(U_\alpha))_\beta}=\phi.$$

Since f is irreducible, $p\in\overline{U_{\alpha\beta}}$ follows from the argument of [2, Lemma 3.3]. Therefore (2) is proved. (3) follows easily from (3)$'$. This completes the proof.

So far as I know, it is not known whether each closed set of an M_1-space admits a σ-closure-preserving open neighborhood base. It is also an open question whether $X|A$ is an M_1-space for each closed set A of an M_1-space. But as far as we are concerned with the class of free L-spaces, these hold positively.

COROLLARY 1. *Every closed set of a free L-space has a closure-preserving open neighborhood base.*

COROLLARY 2. *$X|A$ is an M_1-space for each closed set A of a free L-space X.*

COROLLARY 3. *Let f be a closed irreducible mapping of a free L-space X onto Y. Then Y is an M_1-space.*

PROOF. The closed image of a paracompact σ-space is also paracompact σ. It is similarly shown to [2, Lemma 3.2] that every closed set of Y has a closure-preserving open neighborhood base.

Note that we use only the fact that X is an M_1-space in the proof of the case of dim $X=0$ of Lemma 2. Thus we have the following:

COROLLARY 3$'$. *Let f be a closed irreducible mapping of an M_1-space X with* dim $X\leqq 0$ *onto Y. Then Y is an M_1-space.*

It is unknown whether the adjunction space of M_1-spaces is M_1. From the result of Borges [1], it is known that the adjunction space is at least stratifiable.

THEOREM 4. *Let X be a free L-space and Y an M_1-space. Then $Z=X\cup_f Y$ is an M_1-space.*

PROOF. Let $\mathcal{U}=\bigcup_{j=1}^{\infty}\mathcal{U}_j$ be a base for $p(Y)$, where each $\mathcal{U}_j=\{U_\alpha : \alpha\in A_j\}$ is closure-preserving in $p(Y)$. By the same method of the proof of Theorem 3, we expand each U_α to an open set U'_α of Z. By the same method as in the proof of Lemma 2, we can show that there exists for each $\alpha\in A_j$ an open collection $\mathcal{U}_\alpha=\{U_{\alpha\beta} : \beta\in B_\alpha\}$ of X satisfying the following:

(1) $U_{\alpha\beta}\cap H=p_X^{-1}(U_\alpha)$, $U_{\alpha\beta}\subset p_X^{-1}(U'_\alpha)$ for each $\beta\in B_\alpha$, $\alpha\in A_j$.

(2) $\bigcup\{\mathcal{U}_\alpha : \alpha\in A_j\}$ is closure-preserving in $X-H$.

(3) If U is an open set of X such that $U\cap H=p_X^{-1}(U_\alpha)$ for $\alpha\in A_j$, then $U_{\alpha\beta}\subset U$ for some $\beta\in B_\alpha$.

Set

$$\mathcal{V}_\alpha=\{V_{\alpha\beta}=U_\alpha\cup p(U_{\alpha\beta}) : \beta\in B_\alpha\},\ \alpha\in A_j,$$
$$\mathcal{V}_j=\bigcup\{\mathcal{V}_\alpha : \alpha\in A_j\},$$
$$\mathcal{V}=\bigcup_{j=1}^{\infty}\mathcal{V}_j.$$

Then $\mathcal{V}$ is a σ-closure-preserving open collection of Z, which forms a local base of each point of $p(Y)$ in Z. Since Z is perfectly normal and X is an M_1-space, there exists a σ-closure-preserving open collection $\mathcal{W}$ of Z, which forms a local base of each point of $Z-p(Y)$ in Z. Then $\mathcal{V}\cup\mathcal{W}$ is a σ-closure-preserving base for Z. This completes the proof.

COROLLARY 1. *Let X be the perfect irreducible image of an M_1-space with* $\dim X\leqq 0$ *and Y an M_1-space. Then $X\cup_f Y$ is an M_1-space.*

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