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NON-STANDARD REAL NUMBER SYSTEMS WITH REGULAR GAPS

By

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The purpose of this paper is to show that if an enlargement $^*\boldsymbol{M}$ of the universe $\boldsymbol{M}$ is saturated, then the non-standard real number system $^*\boldsymbol{R}$ has a regular gap and the uniform space $(^*\boldsymbol{R}, E[L(1)])$ is not complete.

Our notions and terminologies follow the usual use in the model theory. Let $\boldsymbol{G}=\langle G, +, <\rangle$ be a first order structure which satisfies

(a) the axioms of ordered abelian groups,

(b) the axioms of dense linear order.

(i. e. $\langle G, +, <\rangle$ is an ordered abelian group and $\langle G, <\rangle$ is a densely ordered set.) A Dedekind cut (X, Y) in $\boldsymbol{G}$ is said to be a *gap* if $sup(X)$ $(inf(Y))$ does not exist. A gap (X, Y) is said to be *regular* if, for all e in G_+ $(=\{g\in G\,;\ g>0\})$, $X+e\neq X$.

THEOREM. *Suppose that* $\boldsymbol{G}$ *is saturated. Then,* $\boldsymbol{G}$ *has a regular gap. Moreover,* $\boldsymbol{G}$ *has* 2^{κ}*-th regular gaps, where* κ *is the cardinality of* G.

PROOF. Since $\boldsymbol{G}$ is saturated, the coinitiality of G_+ is κ. Let $\langle g_\alpha | \alpha<\kappa\rangle$ be an enumeration of G and let $\langle e_\alpha | \alpha<\kappa\rangle$ be a strictly decreasing coinitial sequence in G_+. By the induction on $\alpha<\kappa$, we shall define a set $\{I(x_u, y_u)\,;\ u\in {}^{\alpha}2\}$ of open intervals in $\boldsymbol{G}$ such that

(1) $I(x_u, y_u)\neq\varnothing$ for all u in ${}^{\alpha}2$,

(2) $y_u-x_u<e_\alpha$ for all u in ${}^{\alpha}2$,

(3) $g_\alpha\notin I(x_u, y_u)$ for all u in ${}^{\alpha}2$,

(4) $I(x_u, y_u)\cap I(x_v, y_v)=\varnothing$ for all distinct elements u, v in ${}^{\alpha}2$,

(5) for all $\beta<\alpha$, for all $v\in{}^{\beta}2$ and for all $u\in{}^{\alpha}2$, if $v\subset u$, then $I(x_v, y_v)\supset I(x_u, y_u)$.

The construction is as follows:

(Case 1) $\alpha=0$.

This case is obvious.

(Case 2) $\alpha=\beta+1$ for some β.

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Suppose that $\{I(x_v, y_v); v\in {}^{\beta}2\}$ has been defined and satisfies (1)~(5). For each v in ${}^{\beta}2$, choose z_v, z'_v, w_v and w'_v in $I(x_v, y_v)$ such that

$$I(z_v, w_v)\neq\emptyset,\quad I(z'_v, w'_v)\neq\emptyset,$$

$$I(z_v, w_v)\cap I(z'_v, w'_v)=\emptyset,$$

$$w_v-z_v<e_\alpha,\quad w'_v-z'_v<e_\alpha,$$

$$g_\alpha\notin I(z_v, w_v)\cup I(z'_v, w'_v).$$

Set

$$x_{v\frown\langle 0\rangle}=z_v,$$

$$y_{v\frown\langle 0\rangle}=w_v,$$

$$x_{v\frown\langle 1\rangle}=z'_v,$$

$$y_{v\frown\langle 1\rangle}=w'_v.$$

Then,

$$\{I(x_{v\frown\langle i\rangle}, y_{v\frown\langle i\rangle});\ v\in {}^{\beta}2 \text{ and } i=0, 1\}$$

satisfies (1)~(5).

(Case 3) α is limit.

Suppose that, for all $\beta<\alpha$, $\{I(x_v, y_v); v\in {}^{\beta}2\}$ has been defined and satisfies (1)~(5). Let u be in ${}^{\alpha}2$. For each $\beta<\alpha$, put

$$x_\beta=x_{u\restriction\beta} \quad\text{and}\quad y_\beta=y_{u\restriction\beta}$$

(where $u\restriction\beta$ denotes the restriction of u to β).

The sequence $\langle I(x_\beta, y_\beta)\,|\,\beta<\alpha\rangle$ satisfies that

$$I(x_\beta, y_\beta)\neq\emptyset \quad\text{for all } \beta<\alpha,$$

$$I(x_\beta, y_\beta)\subset I(x_\gamma, y_\gamma) \quad\text{for all } \gamma<\beta<\alpha.$$

Since G is saturated, $\bigcap_{\beta<\alpha} I(x_\beta, y_\beta)$ contains elements x and y such that $x<y$. Since $I(x, y)\subset\bigcap_{\beta<\alpha} I(x_\beta, y_\beta)$, we can choose x_u, y_u in $I(x, y)$ such that

$$x_u<y_u<x_u+e_\alpha \quad\text{and}\quad g_\alpha\notin I(x_u, y_u).$$

Then, $\{I(x_u, y_u); u\in {}^{\alpha}2\}$ satisfies (1)~(5).

Now, $\{I(x_u, y_u); u\in\bigcup_{\alpha<\kappa} {}^{\alpha}2\}$ is a set which satisfies (1)~(5). For each f in ${}^{\kappa}2$, define subsets X_f and Y_f of G by

$$X_f=\{g\in G;\ \exists\alpha<\kappa(g<x_{f\restriction\alpha})\},$$

$$Y_f=\{g\in G;\ \exists\alpha<\kappa(y_{f\restriction\alpha}<g)\}.$$

By (3) and (5), (X_f, Y_f) is a cut in $\boldsymbol{G}$. By (4), if f, h are distinct elements in ${}^{\kappa}2$, then $(X_f, Y_f)\neq(X_h, Y_h)$. To complete the proof of our theorem, it suffices to show that (X_f, Y_f) is regular. Let e be any element in G_+. Since $\langle e_\alpha | \alpha<\kappa\rangle$ is coinitial in G_+, there exists some $\alpha<\kappa$ such that $e_\alpha\leqq e$. By (2),

$$y_{f1\alpha}<x_{f1\alpha}+e_\alpha\leqq x_{f1\alpha}+e\,.$$

Since $y_{f1\alpha}$ is in Y_f, $x_{f1\alpha}+e$ is in Y. Thus $X_f+e\neq X_f$. #

Let $\boldsymbol{R}$ be the set of real numbers, let $\boldsymbol{M}$ be a universe with $\boldsymbol{R}\in\boldsymbol{M}$, let ${}^*\boldsymbol{M}$ be an enlargement of $\boldsymbol{M}$ and let ${}^*\boldsymbol{R}$ be the scope of $\boldsymbol{R}$. We shall regard ${}^*\boldsymbol{R}$ as an ordered group $\langle {}^*R, +, <\rangle$. (${}^*\boldsymbol{R}$ may be of the form $\langle {}^*R, {}^*+, {}^*<\rangle$. But we shall omit asterisks in ${}^*+$ and ${}^*<$, because there is no danger of confusion.)

COROLLARY 1. *Suppose that ${}^*\boldsymbol{M}$ is saturated. Then, ${}^*\boldsymbol{R}$ has a regular gap.*

PROOF. Since ${}^*\boldsymbol{M}$ is saturated, ${}^*\boldsymbol{R}$ is saturated. So, this follows from Theorem. #

For each r in ${}^*R_+$, define $E(r)$ by

$$E(r)=\{(s, t)\in{}^*R\times{}^*R\,;\ |s-t|<r\}\,.$$

Define $L(1)$ and $E[L(1)]$ by

$$L(1)=\{r\in{}^*R\,;\ \forall r'\in R(r'<r)\}\,,$$

$$E[L(1)]=\{E(r)\,;\ r\in L(1)\}\,.$$

$E[L(1)]$ is the base of some uniform topology on ${}^*R$. This uniform space is denoted by $({}^*\boldsymbol{R}, E[L(1)])$ (see [6]). Define $\bar{R}$ by

$$\bar{R}=\{r\in{}^*R\,;\ \exists r'\in R(|r|<r')\}\,.$$

$\bar{R}$ is a convex subgroup of ${}^*\boldsymbol{R}$. So, the quotient group ${}^*\boldsymbol{R}/\bar{R}$ becomes an ordered group.

LEMMA. *$({}^*\boldsymbol{R}, E[L(1)])$ is complete if and only if ${}^*\boldsymbol{R}/\bar{R}$ does not have a regular gap.*

PROOF. It is easy from simple calculations. #

COROLLARY 2. *Suppose that ${}^*\boldsymbol{M}$ is saturated. Then, $({}^*\boldsymbol{R}, E[L(1)])$ is not complete.*

PROOF. From Theorem and Lemma. #

Assume GCH. There exists an enlargement $^*\boldsymbol{M}$ which is saturated (see [4, Proposition 5.1.5(ii)]). Therefore, from Corollaries 1 and 2, there exists an enlargement $^*\boldsymbol{M}$ such that

(1) $^*\boldsymbol{R}$ has a regular gap,

(2) $(^*\boldsymbol{R}, E[L(1)])$ is not complete.

This is another proof of Theorems 4.5 and 4.2 in my paper [6].

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