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FRAMES AND RIESZ BASES FOR BANACH SPACES, AND BANACH SPACES OF VECTOR-VALUED SEQUENCES

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Abstract. This paper is devoted to an investigation of frames and Riesz bases for general Banach sequence spaces. We establish various relationships between Bessel (respectively, frames) and Riesz sequences (respectively, Riesz bases), and then some of their applications are presented. Some recent results for Banach frames and atomic decompositions are sharpened with simple proofs. Banach spaces consisting of Bessel or Riesz sequences are introduced and it is shown that they are isometrically isomorphic to some Banach spaces of bounded linear operators, and that some subspaces of those Banach spaces are isometrically isomorphic to some Banach spaces of compact operators.

1. Introduction

A sequence (f_n) in a Hilbert space H is called a *frame* if there exist constants $A, B > 0$ such that

$$A\|f\| \le \Big(\sum_n |\langle f, f_n\rangle|^2\Big)^{\frac{1}{2}} \le B\|f\|$$

for every $f \in H$. The concept is well known and the theory for it has been much studied. The introductory text of Christensen [9] and the survey article of Casazza [6] contain many results and references for the frame theory for Hilbert spaces. We are now naturally led to a Banach space version of the frame. For

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$1 \le p \le \infty$, a sequence (x_n^*) in the dual space X^* of a Banach space X is called a *p-frame* for X if there exist constants $A, B > 0$ such that

$$A\|x\| \le \Big(\sum_n |x_n^*(x)|^p\Big)^{\frac{1}{p}} \le B\|x\|$$

for every $x \in X$. For the case $p = \infty$, $(\sum_n |x_n^*(x)|^p)^{\frac{1}{p}}$ is replaced by $\sup_n |x_n^*(x)|$. If there exists a 2-frame for a Banach space, then the Banach space is isomorphic to a Hilbert space. The concept was introduced by Aldroubi, Sun and Tang [2] and some abstract theories for it were studied by Christensen and Stoeva [11, 19].

Casazza, Christensen and Stoeva [7] introduced and studied a more general notion. They defined that a sequence (x_n^*) in X^* is a B_s*-frame* for X, where B_s is a scalar-valued Banach sequence space that is a linear space of sequences with a norm which makes it a Banach space and for which the coordinate functionals are continuous, if

(i) $(x_n^*(x)) \in B_s$ for every $x \in X$,

(ii) there exist constants $A, B > 0$ such that

$$A\|x\| \le \|(x_n^*(x))\|_{B_s} \le B\|x\|$$

for every $x \in X$. An l_p-frame for a Banach space is exactly a p-frame. We say that (x_n^*) is a B_s-*Bessel sequence* for X if (i) and the upper B_s-frame condition are satisfied. Since a Banach space X can be identified with a subspace of the bidual space X^{**} of X, for a given sequence in X, the B_s-Bessel sequence (resp. frame) for X^* can be analogously defined.

For a sequence (x_n^*) in X^* (resp. (x_n) in X), if the map

$$F_{(x_n^*)} : X \to B_s,\ x \mapsto (x_n^*(x))$$

$$(\text{resp. } F_{(x_n)} : X^* \to B_s,\ x^* \mapsto (x^*(x_n)))$$

is well defined, then from the closed graph theorem it is automatically bounded. This means that the condition of B_s-Bessel sequence is only (i) in the frame conditions. The operator is called the *analysis operator*. A sequence is a B_s-frame if and only if the analysis operator is an isomorphism.

Considering the classical Banach sequence spaces l_p $(1 \le p \le \infty)$ and c_0, then an l_p $(1 \le p < \infty)$ (resp. c_0)-Bessel sequence (x_n) for X^* is a weakly p-summable (resp. null) sequence, an l_∞-Bessel sequence (x_n) for X^* is a bounded sequence, and for a sequence (x_n^*) in X^*, a c_0-Bessel sequence (x_n^*) for X is a weak* null sequence.

We say that a sequence (x_n) in X is a B_s-*Riesz basic sequence* for X if

(i) $\sum_n \alpha_n x_n$ converges for every $(\alpha_n) \in B_s$,

(ii) there exist constants $A, B > 0$ such that

$$A\|(\alpha_n)\|_{B_s} \le \Big\|\sum_n \alpha_n x_n\Big\| \le B\|(\alpha_n)\|_{B_s}$$

for every $(\alpha_n) \in B_s$.

In particular, a B_s-Riesz basic sequence (x_n) for X is a B_s-*Riesz basis* if $X = \overline{\operatorname{span}}\{x_n\}$ and in this paper we call (x_n) a B_s-*Riesz sequence* for X if (i) is satisfied.

Note that (x_n) is an l_∞-Riesz sequence for X if and only if (x_n) is unconditionally summable (cf. [17, Theorem 4.2.8]).

For a B_s-Riesz sequence (x_n) for X, the *synthesis operator* is defined by

$$R_{(x_n)} : B_s \to X,\ (\alpha_n) \mapsto \sum_n \alpha_n x_n.$$

From the Banach-Steinhaus theorem, a sequence is a B_s-Riesz sequence if and only if the synthesis operator is well defined and bounded. Also a sequence is a B_s-Riesz basic sequence if and only if the synthesis operator is an isomorphism. The Riesz basis for a Hilbert space is well known (cf. [6, 9]), and in [2, 11], l_p-Riesz bases for Banach spaces were introduced and studied. This paper is organized as follows.

In Section 2 we establish some relationships between Bessel and Riesz sequences. It is well known that a sequence (x_n) in X is an l_1-Bessel sequence for X^* if and only if (x_n) is a c_0-Riesz sequence for X (cf. [17, Proposition 4.3.9]). Christensen and Stoeva [11, Proposition 2.2] showed that a sequence (x_n^*) in X^* is an l_p $(1 < p < \infty)$-Bessel sequence for X if and only if (x_n^*) is an l_{p^*}-Riesz sequence for X^*, where $p^* = p/(p-1)$. We extend those results, more precisely, for the dual Banach sequence space Y_s of B_s, it is shown that a sequence (x_n) in X (resp. (x_n^*) in X^*) is a Y_s-Bessel sequence for X^* (resp. X) if and only if (x_n) (resp. (x_n^*)) is a B_s-Riesz sequence for X (resp. X^*). Moreover, we establish some relationships between B_s-Bessel and Y_s-Riesz sequences.

In Section 3 we study some relationships between B_s (resp. Y_s)-Riesz bases and Y_s (resp. B_s)-frames, necessary and sufficient conditions for Y_s (resp. B_s)-frames to be B_s (resp. Y_s)-Riesz bases.

In Section 4 we study Banach frames and atomic decompositions. Some recent results [3, 4, 7] for them are sharpened with simple proofs.

We denote the collection of B_s-Bessel sequences in X for X^* (resp. X^* for X) by $B_s^w(X)$ (resp. $B_s^{w^*}(X^*)$). If $B_s = l_p$ $(1 \le p < \infty)$ (resp. $B_s = l_\infty$), then $B_s^w(X)$ is the collection of weakly p-summable (resp. bounded) sequences in X, and $c_0^w(X)$ (resp. $c_0^{w^*}(X^*)$) is the collection of weakly (resp. weak*) null sequences in X (resp. X^*). We denote the collection of B_s-Riesz sequences in X by $B_sR(X)$. These collections are vector spaces under the standard operation of scalar multiplication and addition for sequences. In Section 5 we show that these vector spaces are Banach spaces endowed with some norms and that they are isometrically isomorphic to some Banach spaces of bounded linear operators.

In Section 6 we introduce the $\check{B}_s$-Bessel and Riesz sequences which are special Bessel and Riesz sequences. We show that the Banach spaces consisting of them are isometrically isomorphic to some Banach spaces of compact operators. Also it is shown that a sequence is a $\check{Y}_s$-Bessel (resp. $\check{B}_s$-Bessel) sequence if and only if it is a $\check{B}_s$-Riesz (resp. $\check{Y}_s$-Riesz) sequence.

2. Relationships between Bessel and Riesz sequences

The purpose of this section is to establish some relationships between Bessel and Riesz sequences. In order to do this, we need the well known representation of the dual space B_s^* of B_s; cf. [7, Lemma 3.1]. Let (e_n) be the sequence of the canonical unit vectors and suppose that (e_n) is a Schauder basis for B_s. Let

$$Y_s = \{(x_s^*(e_n)) | x_s^* \in B_s^*\}$$

and $\|(x_s^*(e_n))\|_{Y_s} = \|x_s^*\|$. Then we see that $(Y_s, \|\cdot\|_{Y_s})$ is a normed space and the coordinate functionals for Y_s are continuous. Consider the map $j_s : Y_s \to B_s^*$ defined by $j_s[(x_s^*(e_n))] = x_s^*$. Then j_s is a surjective linear isometry and so Y_s is a Banach sequence space. For example, if $B_s = l_p$ $(1 < p < \infty)$ (resp. l_1), then $Y_s = l_{p^*}$ (resp. l_∞), and if $B_s = c_0$, then $Y_s = l_1$.

Now for every $x_s^* \in B_s^*$ and $(\alpha_n) \in B_s$

$$x_s^*((\alpha_n)) = x_s^*\Big(\sum_n \alpha_n e_n\Big) = \sum_n \alpha_n x_s^* e_n = \sum_n \alpha_n (j_s^{-1} x_s^*)_n,$$

where $(j_s^{-1} x_d^*)_n$ is the n-th element of $j_s^{-1} x_d^*$. Let (f_n) be the sequence of the canonical unit vectors in Y_s. Fix $k \in \mathbb{N}$. Then for every $(\alpha_n) \in B_s$

$$j_s f_k((\alpha_n)) = \sum_n \alpha_n (j_s^{-1} j_s f_k)_n = \sum_n \alpha_n (f_k)_n = \alpha_k.$$

This shows that $(j_s f_n)$ is the sequence of the coordinate functionals for B_s. If (f_n) is a Schauder basis for Y_s, then for every $x_s^* \in B_s^*$

$$x_s^* = j_s[(x_s^*(e_n))] = j_s\Big(\sum_n x_s^*(e_n) f_n\Big) = \sum_n x_s^*(e_n) j_s f_n.$$

Throughout this paper we use the objects Y_s, j_s, (e_n), (f_n), the analysis and synthesis operators in the introduction. Recall that an operator S from Y^* to X^* is weak* to weak* continuous if and only if there exists an operator T from X to Y such that S is the adjoint operator T^* of T; cf. see [17, Theorem 3.1.11]. We now have

Theorem 2.1. *Suppose that (e_n) is a Schauder basis for B_s and let (x_n) be a sequence in X. Then the following are equivalent.*

(a) *The analysis operator $F_{(x_n)} : X^* \to Y_s$ is well defined.*

(b) *The synthesis operator $R_{(x_n)} : B_s \to X$ is well defined.*

(c) *The analysis operator $F_{(x_n)} : X^* \to Y_s$ is well defined and the operator $j_s F_{(x_n)} : X^* \to B_s^*$ is weak* to weak* continuous.*

Hence $(x_n) \in Y_s^w(X)$ if and only if $(x_n) \in B_s R(X)$.

Proof. (c)$\Longrightarrow$(a) is trivial.

(a)$\Longrightarrow$(b) Recall that $F_{(x_n)}$ is bounded. Let $(\alpha_n) \in B_s$. Then

$$\begin{aligned}
\Big\| \sum_{n=m}^{l} \alpha_n x_n \Big\| &= \sup_{x^* \in B_{X^*}} \Big| \sum_{n=m}^{l} \alpha_n x^*(x_n) \Big| \\
&= \sup_{x^* \in B_{X^*}} \Big| \sum_{n=m}^{l} \alpha_n j_s[(x^*(x_k))](e_n) \Big| \\
&= \sup_{x^* \in B_{X^*}} \Big| j_s[(x^*(x_k))] \Big(\sum_{n=m}^{l} \alpha_n e_n \Big) \Big| \\
&\leq \sup_{x^* \in B_{X^*}} \| j_s[(x^*(x_k))] \|_{B_s^*} \Big\| \sum_{n=m}^{l} \alpha_n e_n \Big\|_{B_s} \\
&= \sup_{x^* \in B_{X^*}} \| (x^*(x_k)) \|_{Y_s} \Big\| \sum_{n=m}^{l} \alpha_n e_n \Big\|_{B_s} \\
&= \| F_{(x_n)} \| \Big\| \sum_{n=m}^{l} \alpha_n e_n \Big\|_{B_s} \longrightarrow 0 \quad \text{as } l, m \to \infty.
\end{aligned}$$

Hence $\sum_n \alpha_n x_n$ converges.

(b)$\Longrightarrow$(c) Recall that $R_{(x_n)}$ is bounded. Consider the operator $j_s^{-1} R_{(x_n)}^* : X^* \to Y_s$. Then for every $x^* \in X^*$

$$(x^*(x_n)) = (x^*(R_{(x_n)} e_n)) = ((R_{(x_n)}^* x^*) e_n) = j_s^{-1} R_{(x_n)}^* x^* \in Y_s.$$

Hence $F_{(x_n)}$ is well defined, $F_{(x_n)} = j_s^{-1} R_{(x_n)}^*$ and so $j_s F_{(x_n)} = R_{(x_n)}^*$ is weak* to weak* continuous. □

For example, for $1 < p < \infty$, $(x_n) \in l_p R(X)$ if and only if $(x_n) \in l_{p^*}^w(X)$, $(x_n) \in l_1 R(X)$ if and only if $(x_n) \in l_\infty(X)$, and $(x_n) \in c_0 R(X)$ if and only if $(x_n) \in l_1^w(X)$. For every Banach space X, let (x_n) be a bounded sequence in X, which does not weakly converge to 0. Then $(x_n) \in l_1 R(X)$ but $(x_n) \notin c_0^w(X)$.

Remark 2.2. For every Banach space X containing c_0 there exists a $(x_n) \in l_1^w(X)$ such that $(x_n) \notin l_\infty R(X)$ because a Banach space X does not contain c_0 if and only if every weakly summable sequence in X is unconditionally summable; see [14, (I.4.5)] or [17, Theorem 4.3.12].

We have the following duality result of Theorem 2.1.

Corollary 2.3. *Suppose that (e_n) is a Schauder basis for B_s and let (x_n^*) be a sequence in X^*. Then the following are equivalent.*

(a) *The analysis operator $F_{(x_n^*)} : X \to Y_s$ is well defined.*

(b) *The synthesis operator $R_{(x_n^*)} : B_s \to X^*$ is well defined.*

(c) *The analysis operator $F_{(x_n^*)} : X^{**} \to Y_s$ is well defined and the operator $j_s F_{(x_n^*)} : X^{**} \to B_s^*$ is weak* to weak* continuous.*

Hence $(x_n^*) \in Y_s^{w^*}(X^*)$ if and only if $(x_n^*) \in B_s R(X^*)$ if and only if $(x_n^*) \in Y_s^w(X^*)$.

Proof. (c)$\Longrightarrow$(a) is clear and (b)$\Longrightarrow$(c) follows from Theorem 2.1(b)$\Longrightarrow$(c).

(a)$\Longrightarrow$(b) Since for every $(\alpha_n) \in B_s$

$$\Big\| \sum_{n=m}^{l} \alpha_n x_n^* \Big\| = \sup_{x \in B_X} \Big| \sum_{n=m}^{l} \alpha_n x_n^*(x) \Big| = \sup_{x \in B_X} \Big| \sum_{n=m}^{l} \alpha_n j_s[(x_k^*(x))](e_n) \Big|,$$

from the proof of Theorem 2.1(a)$\Longrightarrow$(b) the conclusion follows. □

Interchanging the dual Banach sequence space Y_s with B_s, we have the following symmetric version of Theorem 2.1.

Theorem 2.4. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s and let (x_n) be a sequence in X. Then the following are equivalent.*

(a) *The analysis operator $F_{(x_n)} : X^* \to B_s$ is well defined.*

(b) *The synthesis operator $R_{(x_n)} : Y_s \to X^{**}$ is well defined and the operator $R_{(x_n)} j_s^{-1} : B_s^* \to X^{**}$ is weak* to weak continuous.*

(c) *The analysis operator $F_{(x_n)} : X^* \to B_s$ is well defined and weak* to weak continuous.*

Proof. (c)$\Longrightarrow$(a) is trivial.

(a)$\Longrightarrow$(b) Consider the operator $F_{(x_n)}^* j_s : Y_s \to X^{**}$. Then for every n and $x^* \in X^*$

$$(F_{(x_n)}^* j_s f_n) x^* = j_s f_n F_{(x_n)} x^* = j_s f_n [(x^*(x_k))] = x^*(x_n)$$

and so $F_{(x_n)}^* j_s f_n = x_n$ for every n. Now for every $(\beta_n) \in Y_s$

$$F_{(x_n)}^* j_s((\beta_n)) = F_{(x_n)}^* j_s\Big(\sum_n \beta_n f_n\Big) = \sum_n \beta_n F_{(x_n)}^* j_s f_n = \sum_n \beta_n x_n.$$

Hence $R_{(x_n)} = F_{(x_n)}^* j_s$ is well defined and $R_{(x_n)} j_s^{-1} = F_{(x_n)}^*$ is weak* to weak continuous because $R_{(x_n)}(Y_s) \subset X$.

(b)$\Longrightarrow$(c) By the assumption there exists an operator $T : X^* \to B_s$ such that $T^* = R_{(x_n)} j_s^{-1}$. But for every $x^* \in X^*$

$$Tx^* = (j_s f_n T x^*) = ((T^* j_s f_n) x^*) = (R_{(x_n)} f_n x^*) = (x^*(x_n)).$$

Hence $F_{(x_n)} = T$ is well defined and weak* to weak continuous because $F_{(x_n)}^*(B_s^*) = R_{(x_n)} j_s^{-1}(B_s^*) \subset X$. □

From Theorem 2.4, if $(x_n) \in B_s^w(X)$, then $(x_n) \in Y_s R(X)$, and the converse does not hold in general by Remark 2.2 above, but if B_s is reflexive, then the converse is true.

Remark 2.5. The assumption that (f_n) is a Schauder basis for Y_s is not used in the proof of Theorem 2.4(b)$\Longrightarrow$(c), but (a)$\Longrightarrow$(b) need the assumption (see Remark 2.2).

From the same argument as in the proof of Theorem 2.4, we have the following duality result.

Corollary 2.6. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s and let (x_n^*) be a sequence in X^*. Then the following are equivalent.*

(a) *The analysis operator* $F_{(x_n^*)} : X \to B_s$ *is well defined.*
(b) *The synthesis operator* $R_{(x_n^*)} : Y_s \to X^*$ *is well defined and the operator* $R_{(x_n^*)} j_s^{-1} : B_s^* \to X^*$ *is weak** *to weak** *continuous.*

3. B_s-Frames and Riesz bases

In this section we use the results in Section 2 to establish some relationships between frames and Riesz bases, necessary and sufficient conditions for frames to be Riesz bases.

Recall that an operator T is surjective if and only if T^* is an isomorphism, and T^* is surjective if and only if T is an isomorphism; cf. [17, Theorem 3.1.22]. Then we have

Theorem 3.1. *Suppose that* (e_n) *is a Schauder basis for* B_s *and let* (x_n) *be a sequence in* X*. Then the following are equivalent.*
(a) *The analysis operator* $F_{(x_n)} : X^* \to Y_s$ *is an isomorphism.*
(b) *The synthesis operator* $R_{(x_n)} : B_s \to X$ *is surjective.*
(c) *The analysis operator* $F_{(x_n)} : X^* \to Y_s$ *is an isomorphism and the operator* $j_s F_{(x_n)} : X^* \to B_s^*$ *is weak** *to weak** *continuous.*

Proof. (c)$\Longrightarrow$(a) is trivial.

(a)$\Longrightarrow$(b) By Theorem 2.1 $R_{(x_n)}$ is well defined. Then $R_{(x_n)}^* = j_s F_{(x_n)}$ in the proof of Theorem 2.1(b)$\Longrightarrow$(c). Hence by the assumption (a) $R_{(x_n)}$ is surjective.

(b)$\Longrightarrow$(c) Since $R_{(x_n)}^* = j_s F_{(x_n)}$, by the assumption (b) $F_{(x_n)}$ is an isomorphism. □

From Theorem 3.1, if (x_n) is a B_s-Riesz basis for X, then (x_n) is a Y_s-frame for X^*. For example, if (x_n) is an l_p-Riesz basis for X $(1 \le p < \infty)$, then (x_n) is an l_{p^*}-frame for X^*, and if (x_n) is a c_0-Riesz basis for X, then (x_n) is an l_1-frame for X^*. But a Y_s-frame for X^* does not imply a B_s-Riesz basis for X in general. Indeed, consider the sequence $(x_n) = (e_1, 0, e_2, 0, \cdots, 0, e_n, 0, \cdots)$ in c_0. Then for every $(\alpha_k) \in l_1$ $\|(\alpha_k)\|_1 = \|((\alpha_k)x_n)\|_1$. Thus (x_n) is an l_1-frame for l_1, but (x_n) fails the condition (ii) of a c_0-Riesz basic sequence for c_0.

In the proof of Theorem 2.4 the synthesis operator $R_{(x_n)} : Y_s \to X^{**}$ is the operator $F_{(x_n)}^* j_s$, hence we have the following.

Theorem 3.2. *Suppose that* (e_n) *is a Schauder basis for* B_s *and* (f_n) *is a Schauder basis for* Y_s *and let* (x_n) *be a sequence in* X*. Then the following are equivalent.*
(a) *The analysis operator* $F_{(x_n)} : X^* \to B_s$ *is an isomorphism.*
(b) *The synthesis operator* $R_{(x_n)} : Y_s \to X^{**}$ *is surjective, and the operator* $R_{(x_n)} j_s^{-1} : B_s^* \to X^{**}$ *is weak** *to weak continuous.*
(c) *The analysis operator* $F_{(x_n)} : X^* \to B_s$ *is weak** *to weak continuous and an isomorphism.*

Remark 3.3. In Theorem 3.2, if (a) holds, then X is reflexive because $R_{(x_n)}(Y_s) \subset X$ in the proof of Theorem 2.4 . Consequently, under the assumption in Theorem 3.2, a nonreflexive Banach space X cannot contain a B_s-frame for X^*.

We now consider sequences in dual spaces. The following result extends [11, Theorem 2.4].

Theorem 3.4. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s and let (x_n^*) be a sequence in X^*. Then the following are equivalent.*
(a) *The analysis operator $F_{(x_n^*)} : X \to B_s$ is an isomorphism.*
(b) *The synthesis operator $R_{(x_n^*)} : Y_s \to X^*$ is surjective and the operator $R_{(x_n^*)} j_s^{-1} : B_s^* \to X^*$ is weak* to weak* continuous.*

Proof. From the same argument as in the proof of Theorem 2.4, we see that the synthesis operator $R_{(x_n^*)} : Y_s \to X^*$ is the operator $F^*_{(x_n^*)} j_s$, hence the proof is established. □

In view of Corollary 2.6 and Theorem 3.4, if $(x_n^*) \in B_s^{w^*}(X^*)$ and (x_n^*) is a Y_s-Riesz basis for X^*, then (x_n^*) is a B_s-frame for X, and if B_s is reflexive and (x_n^*) is a Y_s-Riesz basis for X^*, then (x_n^*) is a B_s-frame for X. Consider the sequence $(x_n^*) = (e_1, 0, e_2, 0, \cdots, 0, e_n, 0, \cdots)$ in l_1. Then (x_n^*) is a c_0-frame for c_0, but (x_n^*) is not even an l_1-Riesz basic sequence for l_1.

In Theorem 2.1, we have shown that a sequence (x_n) in X is a B_s-Riesz sequence for X if and only if (x_n) is a Y_s-Bessel sequence for X^*. But even for a Y_s- Riesz basis in a dual space, it may not be a B_s-Bessel sequence.

Example 3.5. Let $x_1^* = e_1$ and for every $n \geq 2$ let $x_n^* = (1, 0, \cdots, 0, 1, 0, \cdots)$, where the second 1 is the n-th element. Consider the sequence (x_n^*) in l_1. Then for every $(\alpha_n) \in l_1$ $\sum_n \|\alpha_n x_n^*\|_1 = \sum_n 2|\alpha_n| < \infty$ and so $\sum_n \alpha_n x_n^*$ converges in l_1,

$$\begin{aligned}\|(\alpha_n)\|_1 &\leq \Big\|\Big(\sum_k \alpha_k, \alpha_2, \cdots, \alpha_n, \cdots\Big)\Big\|_1 + \Big\|\Big(-\sum_{k=2}^{\infty} \alpha_k, 0, \cdots\Big)\Big\|_1 \\ &\leq \Big\|\sum_n \alpha_n x_n^*\Big\|_1 + \sum_{k=2}^{\infty} |\alpha_k| \leq 2\Big\|\sum_n \alpha_n x_n^*\Big\|_1,\end{aligned}$$

and

$$\begin{aligned}(\alpha_n) &= \Big(\alpha_1 - \sum_{k=2}^{\infty} \alpha_k, 0, \cdots\Big) + (\alpha_2, \alpha_2, 0, \cdots) + \cdots + (\alpha_n, 0, \cdots, \alpha_n, 0, \cdots) + \cdots \\ &= \Big(\alpha_1 - \sum_{k=2}^{\infty} \alpha_k\Big) x_1^* + \sum_{n=2}^{\infty} \alpha_n x_n^*\end{aligned}$$

which shows $l_1 = \overline{\text{span}}\{x_n^*\}$. Hence (x_n^*) is an l_1-Riesz basis for l_1. But $x_n^*(e_1) = 1$ for every n and so (x_n^*) does not weak* converge to 0 in l_1. Hence (x_n^*) is not a c_0-Bessel sequence for c_0.

We next establish necessary and sufficient conditions for Bessel sequences (resp. frames) to be Riesz basic sequences (resp. Riesz bases).

Theorem 3.6. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s and let $(x_n) \in Y_s^w(X)$. Then the following are equivalent.*
(a) *(x_n) is a B_s-Riesz basic sequence for X.*
(b) *For $(\beta_n) \in B_s$, if $\sum_n \beta_n x_n = 0$, then $\beta_n = 0$ for all n, and $R_{(x_n)}(B_s)$ is closed in X.*
(c) *$F_{(x_n)}(X^*) = Y_s$.*

(d) (x_n) *has a biorthogonal sequence and* $F_{(x_n)}(X^*)$ *is closed in* Y_s.
(e) *For every* n $x_n \notin \overline{span}\{x_i\}_{i\neq n}$ *and* $R_{(x_n)}(B_s)$ *is closed in* X.

Proof. (a)$\Longrightarrow$(b) By definition of B_s-Riesz basic sequence this is clear.

(b)$\Longrightarrow$(a) Since $(x_n) \in Y_s^w(X)$, by Theorem 2.1 the synthesis operator $R_{(x_n)} : B_s \to X$ is bounded. By the assumption (b) $R_{(x_n)}$ is injective. Since $R_{(x_n)}(B_s)$ is closed in X, by the open mapping theorem $R_{(x_n)}$ is an isomorphism. Hence (a) follows.

(a)$\Longleftrightarrow$(c) In the proof of Theorem 2.1 $R^*_{(x_n)} = j_s F_{(x_n)}$. Hence the conclusion follows.

(c)$\Longrightarrow$(d) Since $F_{(x_n)}$ is surjective, for every n there exists an $x_n^* \in F^{-1}_{(x_n)}\{f_n\}$. Consider the sequence (x_n^*) in X^* and fix k. Then

$$x_k^*(x_n) = \varphi_n[F_{(x_n)}(x_k^*)] = \varphi_n(f_k),$$

where φ_n is the n-th coordinate functional for Y_s. Hence (x_n^*) is a biorthogonal sequence for (x_n) and so (d) follows.

(d)$\Longrightarrow$(c) Let (x_n^*) be a biorthogonal sequence for (x_n). Then for every n $F_{(x_n)}x_n^* = (x_n^*(x_n)) = f_n$. Consequently $\{f_n\} \subset F_{(x_n)}(X^*)$. Since (f_n) is a Schauder basis for Y_s and $F_{(x_n)}(X^*)$ is closed in Y_s, $Y_s = \overline{\text{span}}\{f_n\} = \overline{F_{(x_n)}(X^*)} = F_{(x_n)}(X^*)$.

(a)$\Longrightarrow$(e) It is easy to check that if a sequence (x_n) in X is a B_s-Riesz basic sequence under the assumption that (e_n) is a Schauder basis for B_s, then (x_n) is a Schauder basis for $\overline{\text{span}}\{x_n\}$. Hence (e) follows.

(e)$\Longrightarrow$(b) By the assumption $x_n \neq 0$ for all n. Suppose that there exists a $(\beta_n) \in B_s$ such that $\sum_n \beta_n x_n = 0$ but $\beta_m \neq 0$ for some m. Then $\sum_{n\neq m} \beta_n x_n = -\beta_m x_m$ and so $x_m = \sum_{n\neq m} -\frac{\beta_n}{\beta_m} x_n$. Consequently, $x_m \in \overline{\text{span}}\{x_i\}_{i\neq m}$. This contradicts the assumption (e). Hence (b) follows. □

In Theorem 3.6, the condition that (f_n) is a Schauder basis for Y_s is only used in the proof of (d)$\Longrightarrow$(c). From Theorem 3.6 we have

Corollary 3.7. *Suppose that* (e_n) *is a Schauder basis for* B_s *and* (f_n) *is a Schauder basis for* Y_s *and let* (x_n) *be a* Y_s*-frame for* X^*. *Then the following are equivalent.*
(a) (x_n) *is a* B_s*-Riesz basis for* X.
(b) *For* $(\beta_n) \in B_s$, *if* $\sum_n \beta_n x_n = 0$, *then* $\beta_n = 0$ *for all* n.
(c) $F_{(x_n)}(X^*) = Y_s$.
(d) (x_n) *has a biorthogonal sequence.*
(e) *For every* n $x_n \notin \overline{span}\{x_i\}_{i\neq n}$.

Using the operators in the proof of Theorem 2.4, then from the same argument as in the proof of Theorem 3.6 we have the following.

Theorem 3.8. *Suppose that* (e_n) *is a Schauder basis for* B_s *and* (f_n) *is a Schauder basis for* Y_s *and let* $(x_n) \in B_s^w(X)$. *Then the following are equivalent.*
(a) (x_n) *is a* Y_s*-Riesz basic sequence for* X.
(b) *For* $(\beta_n) \in Y_s$, *if* $\sum_n \beta_n x_n = 0$, *then* $\beta_n = 0$ *for all* n, *and* $R_{(x_n)}(Y_s)$ *is closed in* X.

(c) $F_{(x_n)}(X^*) = B_s$.
(d) (x_n) *has a biorthogonal sequence and* $F_{(x_n)}(X^*)$ *is closed in* B_s.
(e) *For every* n $x_n \notin \overline{span}\{x_i\}_{i\neq n}$ *and* $R_{(x_n)}(Y_s)$ *is closed in* X.

Use the operators in Corollary 2.6 to show Corollary 3.9.

Corollary 3.9. *Suppose that* (e_n) *is a Schauder basis for* B_s *and* (f_n) *is a Schauder basis for* Y_s *and let* $(x_n^*) \in B_s^{w^*}(X^*)$. *Then the following are equivalent.*
(a) (x_n^*) *is a* Y_s*-Riesz basic sequence for* X^*.
(b) $F_{(x_n^*)}(X) = B_s$.
(c) (x_n^*) *has a biorthogonal sequence in* X *and* $F_{(x_n^*)}(X)$ *is closed in* B_s.

Corollaries 3.10 and 3.11 extend [11, Proposition 2.7].

Corollary 3.10. *Suppose that* (e_n) *is a Schauder basis for* B_s *and* (f_n) *is a Schauder basis for* Y_s *and let* (x_n) *be a* B_s*-frame for* X^*. *Then the following are equivalent.*
(a) (x_n) *is a* Y_s*-Riesz basis for* X.
(b) *For* $(\beta_n) \in Y_s$, *if* $\sum_n \beta_n x_n = 0$, *then* $\beta_n = 0$ *for all* n.
(c) $F_{(x_n)}(X^*) = B_s$.
(d) (x_n) *has a biorthogonal sequence.*
(e) *For every* n $x_n \notin \overline{span}\{x_i\}_{i\neq n}$.

Corollary 3.11. *Suppose that* (e_n) *is a Schauder basis for* B_s *and* (f_n) *is a Schauder basis for* Y_s *and let* (x_n^*) *be a* B_s*-frame for* X. *Then the following are equivalent.*
(a) (x_n^*) *is a* Y_s*-Riesz basis for* X^*.
(b) $F_{(x_n^*)}(X) = B_s$.
(c) (x_n^*) *has a biorthogonal sequence in* X.

Now we obtain some applications for Riesz bases.

Theorem 3.12. *Suppose that* (e_n) *is a Schauder basis for* B_s *and* (f_n) *is a Schauder basis for* Y_s. *If* (x_n) *is a* B_s*-Riesz basis for* X, *then there exists a* Y_s*-Riesz basis* (x_n^*) *for* X^*, *which is a biorthogonal sequence for* (x_n), *so that*

$$x = \sum_n x_n^*(x)x_n, \quad x^* = \sum_n x^*(x_n)x_n^*$$

for every $x \in X$ *and* $x^* \in X^*$.

Proof. We have shown that if (x_n) is a B_s-Riesz basis for X, then (x_n) is a Y_s-frame for X^* and $F_{(x_n)}(X^*) = Y_s$. Consider the sequence $(F_{(x_n)}^{-1} f_n)$ in X^*. Then $F_{(x_n)}^{-1} f_n(x_k) = \varphi_k(F_{(x_n)} F_{(x_n)}^{-1} f_n) = \varphi_k(f_n)$, where φ_k is the k-th coordinate functional for Y_s. Therefore $(F_{(x_n)}^{-1} f_n)$ is a biorthogonal sequence for (x_n) and for every $x^* \in X^*$

$$x^* = F_{(x_n)}^{-1} F_{(x_n)} x^* = F_{(x_n)}^{-1}[(x^*(x_n))]$$
$$= F_{(x_n)}^{-1}\Big(\sum_n x^*(x_n) f_n\Big) = \sum_n x^*(x_n) F_{(x_n)}^{-1} f_n.$$

Let $x \in X$. Then $x = \sum_n \beta_n x_n$ for some sequence (β_n) of scalars and for every k $F^{-1}_{(x_n)} f_k(x) = \sum_n \beta_n F^{-1}_{(x_n)} f_k(x_n) = \beta_k$. Hence $x = \sum_n F^{-1}_{(x_n)} f_n(x) x_n$. It is immediate that $(F^{-1}_{(x_n)} f_n)$ is equivalent to (f_n). □

If B_s is reflexive and (x_n) is a Y_s-Riesz basis for X, then X is reflexive and so (x_n) is a B_s-frame for X^* and $F_{(x_n)}(X^*) = B_s$ by Theorem 3.2 and Corollary 3.10. Then by the proof of Theorem 3.12 we have

Theorem 3.13. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s. Let B_s be reflexive. If (x_n) is a Y_s-Riesz basis for X, then there exists a B_s-Riesz basis (x^*_n) for X^*, which is a biorthogonal sequence for (x_n), so that*

$$x = \sum_n x^*_n(x) x_n, \quad x^* = \sum_n x^*(x_n) x^*_n$$

for every $x \in X$ and $x^* \in X^*$.

If $(x^*_n) \in B^{w^*}_s(X^*)$ (or B_s is reflexive) and (x^*_n) is a Y_s-Riesz basis for X^*, then (x^*_n) is a B_s-frame for X and $F_{(x^*_n)}(X) = B_s$ by Theorem 3.4 and Corollary 3.11. Then by the proof of Theorem 3.12 we have the following which extends [11, Theorem 2.8].

Theorem 3.14. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s. Let $(x^*_n) \in B^{w^*}_s(X^*)$ (or B_s be reflexive). If (x^*_n) is a Y_s-Riesz basis for X^*, then there exists a B_s-Riesz basis (x_n) for X, which is a biorthogonal sequence for (x^*_n), so that*

$$x = \sum_n x^*_n(x) x_n, \quad x^* = \sum_n x^*(x_n) x^*_n$$

for every $x \in X$ and $x^* \in X^*$.

4. Banach frames and atomic decompositions

For an operator $S : B_s \to X$, a sequence (x_n) in X and a B_s-frame (x^*_n) in X^* for X, we say that $((x^*_n), S)$ (resp. $((x^*_n), (x_n))$) is a *Banach frame* (BF) (resp. an *atomic decomposition* (AD)) for X with respect to B_s if

$$S[(x^*_n(x))] = x \left(\text{resp. } \sum_n x^*_n(x) x_n = x\right)$$

for every $x \in X$. The BF and AD for Banach spaces were introduced and studied by Gröchenig [15], and Casazza, Han, Larson, Christensen and Heil [8, 10], respectively. Recently, some abstract theories for them were studied by Carando, Lassalle and Schmidberg [3, 4], and Casazza, Christensen, Stoeva [7]. The purpose of this section is to sharpen some recent results [3, 4, 7] for the BF and AD.

First we consider the existence problem of the BF and AD for Banach spaces. In [8, Theorem 2.10], it was shown that there exists an AD for a Banach space X if and only if X is separable and has the *bounded approximation property* (BAP). Thus there exist a separable reflexive Banach space Z, which does not have the BAP (see [12]), such that there is no AD for Z. Also we will see that there is no BF for every nonseparable reflexive Banach space from the following proposition.

Recall that a sequence (x_n^*) in X^* is called *total* on X if $x_n^*(x) = 0$ for every n implies $x = 0$.

Proposition 4.1. *The following are equivalent.*
(a) *There exists a BF for X.*
(b) *X^* is weak* separable.*
(c) *X^* has a total sequence.*

Proof. In [7, Lemma 2.6], (c)$\Longrightarrow$(a) was shown and since a weak* countable dense set in X^* is a total sequence, (b)$\Longrightarrow$(c) follows.

(a)$\Longrightarrow$(b) Let $((x_n^*), S)$ be a Banach frame for X with respect to B_s. Consider the countable subset $\{e_n^* F_{(x_n^*)}\}_{n=1}^{\infty}$ of X^*. If there would exist an $x^* \in X^*$ such that $x^* \notin \overline{\text{span}}^{weak^*}\{e_n^* F_{(x_n^*)}\}_{n=1}^{\infty}$, then by the separation theorem there exists an $x \in X$ such that $\overline{\text{span}}^{weak^*}\{e_n^* F_{(x_n^*)}\}_{n=1}^{\infty} \subset \ker(x)$ and $x^*(x) = 1$. But

$$x = SF_{(x_n^*)}x = S[(e_n^* F_{(x_n^*)}x)] = 0,$$

which is a contradiction. Hence $X^* = \overline{\text{span}}^{weak^*}\{e_n^* F_{(x_n^*)}\}_{n=1}^{\infty}$ and so X^* is weak* separable. $\square$

Corollary 4.2. *Let X be a reflexive Banach space. Then there exists a BF for X if and only if X is separable.*

If X is separable, then there exist Banach frames for X and X^*. Indeed, if X is separable, then there exist sequences (x_n) in X and (x_n^*) in X^* such that (x_n^*) is total on X and $(j_X(x_n))$ is total on X^* (see [16, Proposition 1.f.3]), where $j_X : X \to X^{**}$ is the natural isometry. Hence the assertion follows from Proposition 4.1(c)$\Longrightarrow$(a)[7, Lemma 2.6].

We now extend some results in [3, 4, 7] for the BF and AD. Our proofs are simple using some results in the previous sections. First, we note that a B_s-Bessel sequence (x_n^*) in X^* for X is a B_s-frame if and only if (x_n^*) is total and $F_{(x_n^*)}(X)$ is closed in B_s. Indeed, if (x_n^*) is total, then the frame operator $F_{(x_n^*)}$ is injective. Thus if $F_{(x_n^*)}(X)$ is closed, then by the open mapping theorem $F_{(x_n^*)}$ is an isomorphism. We also remark that if there exist operators $U : X \to B_s$ and $V : B_s \to X$ such that $VUx = x$ for every $x \in X$, then $((U^* e_n^*), V)$ is a BF for X with respect to B_s (see [4]), where each e_n^* is the coordinate functional for B_s.

We now establish necessary and sufficient conditions for B_s-Bessel sequences to be Banach frames with respect to B_s, which extend [7, Proposition 3.4].

Theorem 4.3. *Let (x_n^*) be a B_s-Bessel sequence in X^* for X. The following are equivalent.*
(a) *$F_{(x_n^*)}(X)$ is complemented in B_s and (x_n^*) is total.*
(b) *There exists an operator $V : B_s \to X$ such that $V[(x_n^*(x))] = x$ for every $x \in X$.*
(c) *There exists an operator $S : B_s \to X$ such that $((x_n^*), S)$ is a BF for X with respect to B_s.*

If (e_n) is a Schauder basis for B_s, then (a), (b) and (c) are equivalent to
(d) *There exists a sequence (x_n) in X such that $\sum_n \alpha_n x_n$ converges for every $(\alpha_n) \in B_s$ and $x = \sum_n x_n^*(x) x_n$ for every $x \in X$.*

(e) *There exists a Y_s-frame (x_n) in X for X^* such that $x = \sum_n x_n^*(x)x_n$ for every $x \in X$.*

If (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s, then the above statements are equivalent to

(f) *There exists a Y_s-frame (x_n) in X for X^* such that $x^* = \sum_n x^*(x_n)x_n^*$ for every $x^* \in X^*$.*

Proof. (a)$\Longrightarrow$(b) Since $F_{(x_n^*)}(X)$ is closed in B_s, by the note above $F_{(x_n^*)}$ is an isomorphism. Let $P : B_s \to F_{(x_n^*)}(X)$ be a projection. Then $F_{(x_n^*)}^{-1}P$ is the desired operator.

(b)$\Longrightarrow$(c) By the assumption $VF_{(x_n^*)}x = x$ for every $x \in X$. Then by the note above $((F_{(x_n^*)}^* e_n^*), V)$ is a BF for X with respect to B_s. But $F_{(x_n^*)}^* e_n^* = x_n^*$ for every n.

(c)$\Longrightarrow$(a) By the assumption (x_n^*) is total and $F_{(x_n^*)}S$ is a desired projection.

(b)$\Longrightarrow$(d) We see that (Ve_n) is the desired sequence.

(d)$\Longrightarrow$(b) By the assumption we can define the map $V : B_s \to X$ by $V((\alpha_n)) = \sum_n \alpha_n x_n$. Then by the Banach-Steinhaus theorem V is bounded and then it is the desired operator.

(d)$\Longrightarrow$(e) follows from Theorem 3.1 and (e)$\Longrightarrow$(d) follows from Theorem 2.1.

Assume (e). Then $(x^*(x_n)) \in Y_s$ for every $x^* \in X^*$. Hence by Corollary 2.6, (f) follows, and by Theorem 2.1 (f)$\Longrightarrow$(e) follows. □

Next, we consider the AD. Remark that an AD for X with respect to a Banach sequence space can be an AD with respect to another Banach sequence space in which the sequence of the canonical unit vectors is a Schauder basis for it (see [4]).

We say that an AD $((x_n^*), (x_n))$ for X is *shrinking* if for every $x^* \in X^*$

$$\sup_{x \in B_X} \Big| \sum_{n \geq N} x_n^*(x)x^*(x_n) \Big| \longrightarrow 0$$

as $N \to \infty$, and that it is *boundedly complete* if $\sum_n x^{**}(x_n^*)x_n$ converges for every $x^{**} \in X^{**}$. Also it is called *unconditional* if $\sum_n x_n^*(x)x_n$ converges unconditionally for every $x \in X$. The concepts were introduced in [3]. We now have the following which extends [4, Remark 3.2].

Theorem 4.4. *Let $((x_n^*), (x_n))$ be an AD for X such that $\sum_n |x_n^*(x)x^*(x_n)| < \infty$ for every $x \in X$ and $x^* \in X^*$. Then the following are equivalent.*

(a) *X is reflexive.*

(b) *X does not contain an isomorphic copy c_0 and l_1.*

(c) *X and X^* do not contain an isomorphic copy c_0.*

(d) *$((x_n), (x_n^*))$ is an unconditional AD for X^* and $\sum_n x^{**}(x_n^*)x_n$ unconditionally converges for every $x^{**} \in X^{**}$.*

(e) *$((x_n^*), (x_n))$ is shrinking and boundedly complete.*

Proof. (a)$\Longrightarrow$(b) and (d)$\Longrightarrow$(e) are clear, and (b)$\Longrightarrow$(c) and (e)$\Longrightarrow$(a) follows from [16, Proposition 2.e.8] and [3, Proposition 2.4], respectively.

(c)$\Longrightarrow$(d) Let $x^* \in X^*$. Then by the hypothesis and Corollary 2.3 $(x^*(x_n)x_n^*) \in l_1^w(X^*)$ and so $\sum_n x^*(x_n)x_n^*$ is weakly unconditionally Cauchy in X^*. By the assumption (c) and [17, Theorem 4.3.12] $\sum_n x^*(x_n)x_n^*$ unconditionally converges. Since for every $x \in X$

$$\sum_n x^*(x_n)x_n^*(x) = x^*\Big(\sum_n x_n^*(x)x_n\Big) = x^*(x),$$

$\sum_n x^*(x_n)x_n^* = x^*$ unconditionally converges. This shows the first part. Since for every $x^* \in X^*$ $\sum_n x^*(x_n)x_n^*$ is weakly unconditionally Cauchy, for every $x^{**} \in X^{**}$ $\sum_n x^{**}(x_n^*)x_n$ is weakly unconditionally Cauchy in X. By the assumption (c) for every $x^{**} \in X^{**}$ $\sum_n x^{**}(x_n^*)x_n$ unconditionally converges. □

From the same proof of Theorem 4.4 we have the following which extends [4, Theorem 3.3].

Corollary 4.5. *Let $((x_n^*), (x_n))$ be an AD for X such that $\sum_n |x_n^*(x)x^*(x_n)| < \infty$ for every $x \in X$ and $x^* \in X^*$. Then the following are equivalent.*
(a) *X^* is separable.*
(b) *X does not contain an isomorphic copy l_1.*
(c) *X^* does not contain an isomorphic copy c_0.*
(d) *$((x_n), (x_n^*))$ is an unconditional AD for X^*.*
(e) *$((x_n^*), (x_n))$ is shrinking.*

The following extends [4, Theorem 3.4 and Corollary 3.5].

Theorem 4.6. *Let $((x_n^*), (x_n))$ be an AD for X such that $\sum_n |x_n^*(x)x^*(x_n)| < \infty$ for every $x \in X$ and $x^* \in X^*$. Then the following are equivalent.*
(a) *X is complemented in X^{**}.*
(b) *X does not contain an isomorphic copy c_0.*
(c) *$\sum_n x^{**}(x_n^*)x_n$ unconditionally converges for every $x^{**} \in X^{**}$.*
(d) *$((x_n^*), (x_n))$ is boundedly complete.*

Proof. (c)$\Longrightarrow$(d) is clear and (b)$\Longrightarrow$(c) follows from the proof of Theorem 4.4(c)$\Longrightarrow$(d). (d)$\Longrightarrow$(a) is [4, Remark 2.5] and (a)$\Longrightarrow$(b) is an application of [1, Corollary 2.5.9]. □

We now apply the AD to the approximation property. We say that X has the the *approximation property* (AP) if for every compact subset of X and $\varepsilon > 0$ there exists a finite rank operator T on X such that $\sup_{x\in K} \|Tx - x\| \le \varepsilon$, and if we take the operator T such as $\|T\| \le \lambda$ for some $\lambda \ge 1$, then X is said to have the *bounded approximation property* (BAP).

An AD $((x_n^*), (x_n))$ for X with respect to B_s, in which the sequence of the canonical unit vectors is a Schauder basis for it, is called *strongly shrinking* [3] if for every $x^* \in X^*$

$$\sup\Big\{\Big|\sum_{n\ge N} \alpha_n x^*(x_n)\Big| : \|(\alpha_n)\|_{B_s} \le 1\Big\} \longrightarrow 0$$

as $N \to \infty$. The strongly shrinking property is strictly stronger than shrinking [3, Examples 1.12 and 1.13]. The following is a simple observation of known results but an interesting relation between the AP and AD.

Theorem 4.7. *The following are equivalent.*

(a) *There exists an AD for X with respect to a Banach sequence space in which the sequence of the canonical unit vectors is a shrinking Schauder basis for it.*
(b) *There exists a strongly shrinking AD for X.*
(c) *There exists a shrinking AD for X.*
(d) *X^* is separable and has the BAP.*
(e) *X^* is separable and has the AP.*

Proof. (b)$\Longrightarrow$(c) is clear and (a)$\Longrightarrow$(b) follows from [3, Proposition 1.9]. (c)$\Longrightarrow$(d) is [3, Corollary 1.5] and (d)$\Longleftrightarrow$(e) is well known; cf. [5, Theorem 3.6].

(d)$\Longrightarrow$(a) From [5, Theorem 4.9] there exists a Banach space Z with a shrinking basis (z_n) such that X embeds complementably into Z. Put

$$B_s = \Big\{(\alpha_n)| \sum_n \alpha_n z_n \text{ converges in } Z\Big\}$$

with $\|(\alpha_n)\|_{B_s} = \|\sum_n \alpha_n z_n\|_Z$. Then we see that B_s is a Banach sequence space in which the sequence of the canonical unit vectors is a shrinking Schauder basis for it. Since X embeds complementably into Z, we can find an AD for X with respect to B_s. $\square$

5. Banach spaces consisting of Bessel or Riesz sequences

Recall the vector spaces

$$B_s^w(X) = \{(x_n) \text{ in } X : (x^*(x_n)) \in B_s \text{ for every } x^* \in X^*\},$$

$$B_s^{w^*}(X^*) = \{(x_n^*) \text{ in } X^* : (x_n^*(x)) \in B_s \text{ for every } x \in X\},$$

$$B_sR(X) = \{(x_n) \text{ in } X : \sum_n \alpha_n x_n \text{ converges for every } (\alpha_n) \in B_s\}.$$

Then by boundedness of the analysis and synthesis operators, for every $(x_n) \in B_s^w(X), (x_n^*) \in B_s^{w^*}(X^*), (x_n) \in B_sR(X)$, respectively,

$$\|(x_n)\|_{B_s^w(X)} = \sup_{x^* \in B_{X^*}} \|(x^*(x_n))\|_{B_s}, \|(x_n^*)\|_{B_s^{w^*}(X^*)} = \sup_{x \in B_X} \|(x_n^*(x))\|_{B_s},$$

and

$$\|(x_n)\|_{B_sR(X)} = \sup_{(\alpha_n) \in B_{B_s}} \Big\|\sum_n \alpha_n x_n\Big\|$$

are all finite.

We now have

Proposition 5.1. *$(B_s^w(X), \|\cdot\|_{B_s^w(X)})$ and $(B_s^{w^*}(X^*), \|\cdot\|_{B_s^{w^*}(X^*)})$ are Banach spaces.*

Proof. The proofs of the two cases are the same and so we only prove the first case. It is easy to check that $\|\cdot\|_{B_s^w(X)}$ is a norm on $B_s^w(X)$. Let $((x_n^{(k)}))_k$ be a Cauchy sequence in $B_s^w(X)$ and let $m \in \mathbb{N}$ be fixed. Let $\varepsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ so that $k, l \geq N$ implies

$$\|(x_n^{(k)}) - (x_n^{(l)})\|_{B_s^w(X)} \leq \frac{\varepsilon}{\|e_m^*\|},$$

where e_m^* is the m-th coordinate functional for B_s. Then $k, l \geq N$ implies

$$\|x_m^{(k)} - x_m^{(l)}\| = \sup_{x^* \in B_{X^*}} |x^*(x_m^{(k)} - x_m^{(l)})|$$

$$\leq \sup_{x^* \in B_{X^*}} \|e_m^*\| \|(x^*(x_n^{(k)} - x_n^{(l)}))\|_{B_s} \leq \varepsilon.$$

Thus $(x_m^{(k)})_k$ is a Cauchy sequence in X and so there exists an $x_m \in X$ so that $x_m^{(k)} \longrightarrow x_m$. We have shown that there exists a sequence (x_n) in X so that for every n $x_n^{(k)} \longrightarrow x_n$ as $k \to \infty$. Since every $x^* \in X^*$ $((x^*(x_n^{(k)})))_k$ is a Cauchy sequence in B_s, for every $x^* \in X^*$ there exists a $(\alpha_n^{x^*}) \in B_s$ so that $\|(x^*(x_n^{(k)})) - (\alpha_n^{x^*})\|_{B_s} \longrightarrow 0$ as $k \to \infty$ and so for every n $x^*(x_n^{(k)}) \longrightarrow \alpha_n^{x^*}$. Consequently $(x^*(x_n)) = (\alpha_n^{x^*}) \in B_s$ for every $x^* \in X^*$ and so $(x_n) \in B_s^w(X)$. Now let $\varepsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ so that $k, l \geq N$ implies

$$\sup_{x^* \in B_{X^*}} \|(x^*(x_n^{(k)})) - (x^*(x_n^{(l)}))\|_{B_s} \leq \varepsilon.$$

Then $k \geq N$ implies that for every $x^* \in B_{X^*}$

$$\|(x^*(x_n^{(k)})) - (x^*(x_n))\|_{B_s} = \lim_l \|(x^*(x_n^{(k)})) - (x^*(x_n^{(l)}))\|_{B_s} \leq \varepsilon.$$

Thus $k \geq N$ implies $\|(x_n^{(k)}) - (x_n)\|_{B_s^w(X)} \leq \varepsilon$. Hence $(x_n^{(k)}) \longrightarrow (x_n)$ in $B_s^w(X)$ as $k \to \infty$. This completes the proof. □

The following theorem shows that they are actually some Banach spaces of bounded linear operators. Here $\mathcal{L}$ is the Banach space of all bounded linear operators between Banach spaces.

Theorem 5.2. *For every Banach space X, the following statements hold.*

(a) $B_s^w(X)$ *(resp. $B_s^{w^*}(X^*)$) is isometrically isomorphic to $\{T \in \mathcal{L}(X^*, B_s) : T^*(\{e_n^*\}_{n=1}^\infty) \subset X\}$ (resp. $\mathcal{L}(X, B_s)$).*

(b) *If (e_n) is a Schauder basis for B_s, then $B_sR(X)$ and $Y_s^w(X)$ are isometrically isomorphic to $\mathcal{L}(B_s, X)$.*

Proof. (a) Recall the analysis operator $F_{(x_n)}$ (resp. $F_{(x_n^*)}$) : X^* (resp. X) $\to B_s$ and then we define the map from $B_s^w(X)$ (resp. $B_s^{w*}(X^*)$) to $\{T \in \mathcal{L}(X^*, B_s) : T^*(\{e_n^*\}_{n=1}^\infty) \subset X\}$ (resp. $\mathcal{L}(X, B_s)$) via

$$(x_n) \text{ (resp. } (x_n^*)) \mapsto F_{(x_n)} \text{ (resp. } F_{(x_n^*)}).$$

Then the maps will be the desired isometries. Let us only check the first case. Since for every n and $x^* \in X^*$ $(F_{(x_n)}^* e_n^*)(x^*) = x^*(x_n)$, $F_{(x_n)}^*(\{e_n^*\}_{n=1}^\infty) \subset X$ and so the map is well defined and clearly a linear isometry. Now let T be an element in the codomain space and consider the sequence $(T^* e_n^*)$ in X. Then we see that $(T^* e_n^*) \in B_s^w(X)$ and $F_{(T^* e_n^*)} = T$. Hence the map is surjective.

(b) Define the map from $B_sR(X)$ to $\mathcal{L}(B_s, X)$ via $(x_n) \mapsto R_{(x_n)}$, where $R_{(x_n)}$ is the synthesis operator. Then it is easy to check that the map is a surjective linear isometry.

We have shown that for a sequence (x_n) in X $(x_n) \in Y_s^w(X)$ if and only if $(x_n) \in B_sR(X)$ (Theorem 2.1). Moreover, we will show that their norms are the same, and then conclude that $Y_s^w(X)$ is isometrically isomorphic to $\mathcal{L}(B_s, X)$. If $(x_n) \in B_sR(X)$, then

$$\begin{aligned}
\|(x_n)\|_{B_sR(X)} &= \sup\Big\{\Big\|\sum_{k=1}^{\infty}\alpha_k x_k\Big\| : (\alpha_k) \in B_{B_s}\Big\}\\
&= \sup\Big\{\Big|\sum_{k=1}^{\infty}\alpha_k x^*(x_k)\Big| : (\alpha_k) \in B_{B_s}, x^* \in B_{X^*}\}\\
&= \sup\Big\{\Big|\sum_{k=1}^{\infty}\alpha_k j_s[(x^*(x_n))]e_k\Big| : (\alpha_k) \in B_{B_s}, x^* \in B_{X^*}\}\\
&= \sup\Big\{\Big|j_s[(x^*(x_n))]\Big(\sum_{k=1}^{\infty}\alpha_k e_k\Big)\Big| : (\alpha_k) \in B_{B_s}, x^* \in B_{X^*}\}\\
&= \sup\Big\{\Big\|j_s[(x^*(x_n))]\Big\|_{B_s{}^*} : x^* \in B_{X^*}\Big\}\\
&= \sup\{\|(x^*(x_n))\|_{Y_s} : x^* \in B_{X^*}\} = \|(x_n)\|_{Y_s^w(X)}.
\end{aligned}$$

□

For example, $l_pR(X)$ and $l_{p^*}^w(X)$ $(1 \le p < \infty)$ are isometrically isomorphic to $\mathcal{L}(l_p, X)$, and $c_0R(X)$ and $l_1^w(X)$ are isometrically isomorphic to $\mathcal{L}(c_0, X)$.

Remark 5.3. In Theorem 5.2(a), if (f_n) is a Schauder basis for Y_s, then $B_s^w(X)$ is isometrically isomorphic to the space $\mathcal{L}_{w^*}(X^*, B_s)$ of weak* to weak continuous operators because an operator $T : X^* \to Y$ is weak* to weak continuous if and only if $T^*(Y^*) \subset X$.

Corollary 5.4. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s. Then $B_s^w(X^*)$ is isometrically isomorphic to the space $\mathcal{W}(X, B_s)$ of weakly compact operators and so $B_s^w(X^*)$ is a closed subspace of $B_s^{w^*}(X^*)$ with the same norm.*

Proof. Note that for every Banach space X and Y, the space $\mathcal{W}(X, Y)$ is isometrically isomorphic to the space $\mathcal{L}_{w^*}(X^{**}, Y)$ via $T \mapsto j_Y^{-1}T^{**}$, where $j_Y : Y \to Y^{**}$ is the natural isometry. It follows from Remark 5.3 that $B_s^w(X^*)$ is isometrically isomorphic to the space $\mathcal{W}(X, B_s)$. In view of Theorem 5.2(a), the other part follows. □

Recall that for a sequence (x_n^*) in X^* $(x_n^*) \in Y_s^{w^*}(X^*)$ if and only if $(x_n^*) \in Y_s^w(X^*)$ (Corollary 2.3). Moreover, their norms are also the same.

Corollary 5.5. *Suppose that (e_n) is a Schauder basis for B_s. Then $Y_s^w(X^*)$ and $Y_s^{w^*}(X^*)$ are isometrically isomorphic to $\mathcal{L}(B_s, X^*)$.*

Proof. By Theorem 5.2(b) we only need to show the case $Y_s^{w^*}(X^*)$. But, for every Banach space X and Y, $\mathcal{L}(X, Y^*)$ is isometrically isomorphic to $\mathcal{L}(Y, X^*)$, hence the conclusion follows from Theorem 5.2(a). □

For example, $\|(x_n^*)\|_{l_p^w(X^*)} = \|(x_n^*)\|_{l_p^{w^*}(X^*)}$ $(1 \le p \le \infty)$ for every $(x_n^*) \in l_p^w(X^*)$.

6. Special Bessel and Riesz sequences

In this section we consider the following subspaces of $B_s^w(X)$, $B_s^{w^*}(X^*)$, and $B_sR(X)$, respectively;

$$\check{B}_s^w(X) = \{(x_n) \in B_s^w(X) : \lim_n \|(0,\cdots,0,x_n,x_{n+1},\cdots)\|_{B_s^w(X)} = 0\},$$

$$\check{B}_s^{w^*}(X^*) = \{(x_n^*) \in B_s^{w^*}(X^*) : \lim_n \|(0,\cdots,0,x_n^*,x_{n+1}^*,\cdots)\|_{B_s^{w^*}(X^*)} = 0\},$$

$$\check{B}_sR(X) =$$
$$\Big\{(x_n) \in B_sR(X) : \Big\{\sum_n \alpha_n x_n : (\alpha_n) \in B_{B_s}\Big\} \text{ is relatively compact in } X\Big\}.$$

We will show that they are actually some Banach spaces of compact operators. In order to do this, we need the following lemma.

Lemma 6.1. *Suppose that (e_n) is a Schauder basis for B_s and let K be a bounded subset of B_s. Then K is relatively compact if and only if* $\lim_n \sup_{(k_n)\in K} \|(0,\cdots,0,k_n,k_{n+1},\cdots)\|_{B_s} = 0$.

Proof. Let $P_m : B_s \to B_s$ be the m-th basis projection for each $m \in \mathbb{N}$. Suppose that K is relatively compact. Since for every $(\alpha_n) \in B_s$ $\|P_m(\alpha_n) - (\alpha_n)\|_{B_s} \longrightarrow 0$ as $n \to \infty$ and (P_m) is uniformly bounded, we see that for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that for all $m \geq N$

$$\sup_{(k_n)\in K} \|P_m(k_n) - (k_n)\|_{B_s} \leq \varepsilon,$$

hence the assertion follows.

Suppose the converse. Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that $\|P_N(k_n) - (k_n)\| \leq \varepsilon/2$ for all $(k_n) \in K$. Since K is bounded, $P_N(K)$ is relatively compact in B_s. Let $\{P_N(x_1), \ldots, P_N(x_n)\}$ be an $\varepsilon/2$-net of $P_N(K)$, where $x_j \in K$ for all $1 \leq j \leq n$.

Then for each $x \in K$, there is a j $(1 \leq j \leq n)$ such that $\|P_N x - P_N x_j\| \leq \varepsilon/2$, and so $\|x - P_N x_j\| \leq \varepsilon$. This means that the set $\{P_N(x_1), \ldots, P_N(x_n)\}$ is an ε-net of K, hence K is relatively compact. ☐

Theorem 6.2. *Suppose that (e_n) is a Schauder basis for B_s. Then for every Banach space X, the following statements hold.*

(a) *$\check{B}_s^{w^*}(X^*)$ is isometrically isomorphic to the space $\mathcal{K}(X, B_s)$ of compact operators.*
(b) *$\check{B}_sR(X)$ is isometrically isomorphic to $\mathcal{K}(B_s, X)$.*
(c) *If (f_n) is a Schauder basis for Y_s, then $\check{Y}_s^w(X)$ is isometrically isomorphic to $\mathcal{K}(B_s, X)$.*

Proof. (a) Consider the map from $\check{B}_s^{w^*}(X^*)$ to $\mathcal{K}(X, B_s)$ via $(x_n^*) \mapsto F_{(x_n^*)}$. Then by Lemma 6.1 this map is well defined and clearly a linear isometry. If $T \in \mathcal{K}(X, B_s)$, then by using Lemma 6.1 it is easy to check that $(T^* e_n^*) \in \check{B}_s^{w^*}(X^*)$ and $F_{(T^* e_n^*)} = T$. Hence the map is surjective.

(b) Consider the map from $\check{B}_sR(X)$ to $\mathcal{K}(B_s, X)$ via $(x_n) \mapsto R_{(x_n)}$. Then it is easy to check that this map is a surjective linear isometry.

(c) By Theorem 6.6(a) $\check{B}_s R(X) = \check{Y}_s^w(X)$ and by the proof of Theorem 5.2(b) their norms are also the same. Hence the conclusion follows from (b). $\square$

For example, $\check{l}_p R(X)$ and $\check{l}_{p^*}^w(X)$ $(1 < p < \infty)$ are isometrically isomorphic to $\mathcal{K}(l_p, X)$, and $\check{c}_0 R(X)$ and $\check{l}_1^w(X)$ are isometrically isomorphic to $\mathcal{K}(c_0, X)$.

Proposition 6.3. *Suppose that (e_n) is a Schauder basis for B_s. Then $\check{B}_s^w(X)$ is a closed subspace of $B_s^w(X)$.*

Proof. Let $((x_n^{(k)}))_k$ be a sequence in $\check{B}_s^w(X)$ and let $(x_n) \in B_s^w(X)$ with $\|(x_n^{(k)}) - (x_n)\|_{B_s^w(X)} \longrightarrow 0$ as $k \to \infty$. Consider the analysis operators $F_{(x_n^{(k)})}, F_{(x_n)} : X^* \to B_s$. Then by Lemma 6.1 each $F_{(x_n^{(k)})}$ is a compact operator. Since $\|F_{(x_n^{(k)})} - F_{(x_n)}\| = \|(x_n^{(k)}) - (x_n)\|_{B_s^w(X)} \longrightarrow 0$ as $k \to \infty$, $F_{(x_n)}$ is a compact operator. Hence by Lemma 6.1 $(x_n) \in \check{B}_s^w(X)$. $\square$

Now recall the *injective tensor product* $X \check{\otimes} Y$ of Banach spaces X and Y (see [18, Chapter 3] or [13, Section 1.1]). Then $X \check{\otimes} Y$ is isometrically isomorphic to the operator norm closure $\overline{\mathcal{F}_{w^*}(X^*, Y)}$ of the space of weak* to weak continuous finite rank operators from X^* to Y and, if X has the AP, then $\overline{\mathcal{F}_{w^*}(X^*, Y)} = \mathcal{K}_{w^*}(X^*, Y)$, the space of weak* to weak continuous compact operators. Then we have

Theorem 6.4. *Suppose that (e_n) is a Schauder basis for B_s. Then for every Banach space X, $\check{B}_s^w(X)$ is isometrically isomorphic to $\mathcal{K}_{w^*}(B_s^*, X)$.*

Proof. By the note above, it is enough to show that $B_s \check{\otimes} X$ is isometrically isomorphic to $\check{B}_s^w(X)$.

We define the map $J : (B_s \otimes X, \|\cdot\|_\vee) \to \check{B}_s^w(X)$ by

$$J\Big(\sum_{j \le n} (\lambda_i^j)_i \otimes x_j\Big) = \Big(\sum_{j \le n} \lambda_i^j x_j\Big)_i.$$

Then we see that $J(B_s \otimes X) \subset B_s^w(X)$, and

$$\begin{aligned}
&\lim_m \Big\| \Big(0, \cdots, 0, \sum_{j \le n} \lambda_m^j x_j, \sum_{j \le n} \lambda_{m+1}^j x_j, \cdots \Big) \Big\|_{B_s^w(X)} \\
&= \lim_m \sup \Big\{ \Big\| \Big(0, \cdots, 0, \sum_{j \le n} \lambda_m^j x^*(x_j), \sum_{j \le n} \lambda_{m+1}^j x^*(x_j), \cdots \Big) \Big\|_{B_s} : x^* \in B_{X^*} \Big\} \\
&= \lim_m \sup \Big\{ \Big| \gamma \Big[\Big(0, \cdots, 0, \sum_{j \le n} \lambda_m^j x^*(x_j), \sum_{j \le n} \lambda_{m+1}^j x^*(x_j), \cdots \Big) \Big] \Big| : x^* \in B_{X^*}, \gamma \in B_{B_s^*} \Big\} \\
&= \lim_m \sup \Big\{ \Big| \sum_{j \le n} \gamma[(0, \cdots, 0, \lambda_m^j x^*(x_j), \lambda_{m+1}^j x^*(x_j), \cdots)] \Big| : x^* \in B_{X^*}, \gamma \in B_{B_s^*} \Big\} \\
&\le \lim_m \sum_{j \le n} \sup \Big\{ \Big| \gamma \Big(\sum_{i \ge m} \lambda_i^j x^*(x_j) e_i \Big) \Big| : x^* \in B_{X^*}, \gamma \in B_{B_s^*} \Big\} \\
&\le \lim_m \sum_{j \le n} \|x_j\| \sup \Big\{ \Big| \gamma \Big(\sum_{i \ge m} \lambda_i^j e_i \Big) \Big| : \gamma \in B_{B_s^*} \Big\} \\
&= \lim_m \sum_{j \le n} \|x_j\| \Big\| \sum_{i \ge m} \lambda_i^j e_i \Big\|_{B_s} = 0.
\end{aligned}$$

Thus J is well defined and linear. Since for every $(x_n) \in \check{B}_s^w(X)$ and every m

$$(x_1, \cdots, x_m, 0, \cdots) = J\Big(\sum_{j \le m} e_j \otimes x_j \Big)$$

and $\lim_m \|(0, \cdots, 0, x_m, x_{m+1}, \cdots)\|_{B_s^w(X)} = 0$, $J(B_s \otimes X)$ is dense in $\check{B}_s^w(X)$. Now

$$\begin{aligned}
&\Big\| \sum_{j \le n} (\lambda_i^j)_i \otimes x_j \Big\|_{\vee} \\
&= \Big\{ \Big| \sum_{j \le n} \gamma[(\lambda_i^j)_i] x^*(x_j) \Big| : x^* \in B_{X^*}, \gamma \in B_{B_s^*} \Big\} \\
&= \Big\{ \Big| \gamma \Big(\sum_{j \le n} x^*(x_j)(\lambda_i^j)_i \Big) \Big| : x^* \in B_{X^*}, \gamma \in B_{B_s^*} \Big\} \\
&= \Big\{ \Big\| \sum_{j \le n} x^*(x_j)(\lambda_i^j)_i \Big\|_{B_s} : x^* \in B_{X^*} \Big\} \\
&= \Big\{ \Big\| \Big(x^* \Big(\sum_{j \le n} x_j \lambda_i^j \Big) \Big)_i \Big\|_{B_s} : x^* \in B_{X^*} \Big\} \\
&= \Big\| \Big(\sum_{j \le n} x_j \lambda_i^j \Big)_i \Big\|_{B_s^w(X)}.
\end{aligned}$$

Thus J is an isometry and so there exists an extension $\check{J} : B_s \check{\otimes} X \to \check{B}_s^w(X)$ of J such that $\check{J}$ is surjective and an isometry. □

For example, $\check{l}_p^w(X)$ $(1 \le p < \infty)$ (resp. $c_0(X)$) is isometrically isomorphic to $\mathcal{K}_{w^*}(l_{p^*}, X)$ (resp. $\mathcal{K}_{w^*}(l_1, X)$).

Corollary 6.5. *Suppose that (e_n) is a Schauder basis for B_s. Then for every Banach space X, the following statements hold.*
(a) $\check{B}_s^{w^*}(X^*) = \check{B}_s^w(X^*)$ *with the same norm.*
(b) $\check{Y}_s^{w^*}(X^*) = \check{Y}_s^w(X^*)$ *with the same norm.*

Proof. (b) follows from Corollary 5.5, and (a) is a result of Theorems 6.2(a) and 6.4 because $\mathcal{K}(X, B_s)$ is isometrically isomorphic to $\mathcal{K}_{w^*}(B_s^*, X^*)$. $\square$

For example, $\check{l}_p^w(X^*) = \check{l}_p^{w^*}(X^*)$ $(1 \le p \le \infty)$.

Finally we establish relationships between the special Bessel and Riesz sequences.

Theorem 6.6. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s. Let (x_n) and (x_n^*) be sequences in X and X^*, respectively. Then the following statements hold.*
(a) $(x_n) \in \check{Y}_s^w(X)$ *if and only if* $(x_n) \in \check{B}_s R(X)$.
(b) $(x_n^*) \in \check{Y}_s^{w^*}(X^*)$ *if and only if* $(x_n^*) \in \check{B}_s R(X^*)$.

Proof. (b) follows from (a) and Corollary 6.5(b). To show (a), consider the analysis operator $F_{(x_n)} : X^* \to Y_s$ and synthesis operator $R_{(x_n)} : B_s \to X$. Then in the proof of Theorem 2.1 $j_s F_{(x_n)} = R^*_{(x_n)}$. If $(x_n) \in \check{Y}_s^w(X)$, then by Lemma 6.1 $F_{(x_n)}$ is a compact operator and so is $R_{(x_n)}$. Thus $(x_n) \in \check{B}_s R(X)$. Conversely, if $(x_n) \in \check{B}_s R(X)$, then $R_{(x_n)}$ is a compact operator and so is $F_{(x_n)}$. Hence $(x_n) \in \check{Y}_s^w(X)$ by Lemma 6.1. $\square$

From Theorem 6.6, for a sequence (x_n) in X, $(x_n) \in \check{l}_{p^*}^w(X)$ if and only if $(x_n) \in \check{l}_p R(X)$ $(1 < p < \infty)$, and $(x_n) \in \check{l}_1^w(X)$ if and only if $(x_n) \in \check{c}_0 R(X)$.

Interchanging B_s with Y_s we have the following result.

Theorem 6.7. *Suppose that (e_n) is a Schauder basis for B_s and (f_n) is a Schauder basis for Y_s. Let (x_n) and (x_n^*) be sequences in X and X^*, respectively. Then the following statements hold.*
(a) *If* $(x_n) \in B_s^w(X)$, *then* $(x_n) \in \check{B}_s^w(X)$ *if and only if* $(x_n) \in \check{Y}_s R(X)$.
(b) *If* $(x_n^*) \in B_s^{w^*}(X^*)$, *then* $(x_n^*) \in \check{B}_s^{w^*}(X^*)$ *if and only if* $(x_n^*) \in \check{Y}_s R(X^*)$.

Proof. (a) Let $(x_n) \in B_s^w(X)$. Then by the proof of Theorem 2.4 the analysis operator $F_{(x_n)} : X^* \to B_s$ and synthesis operator $R_{(x_n)} : Y_s \to X^{**}$ is well defined and $F^*_{(x_n)} = R_{(x_n)} j_s^{-1}$. Then the conclusion follows from the same argument of the proof of Theorem 6.6.

(b) Let $(x_n^*) \in B_s^{w^*}(X^*)$. Then by Corollary 2.6 the analysis operator $F_{(x_n^*)}$: $X \to B_s$ and synthesis operator $R_{(x_n^*)} : Y_s \to X^*$ is well defined, and we see that $F^*_{(x_n^*)} = R_{(x_n^*)} j_s^{-1}$. Hence the conclusion follows. $\square$

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References

1. F. Albiac and N.J. Kalton, *Topics in Banach Space Theory*, Springer, New York, 2006.
2. A. Aldroubi, Q. Sun and W. Tang, *p-Frames and shift invariant subspaces of L^p*, J. Fourier Anal. Appl. **7** (2001), 1–22.
3. D. Carando and S. Lassalle, *Duality, reflexive and atomic decomposition in Banach spaces*, Studia Math. **191** (2009), 67–80.
4. D. Carando, S. Lassalle and P. Schmidberg, *The reconstruction formula for Banach frames and duality*, J. Approx. Theory **163** (2011), 640–651.
5. P.G. Casazza, *Approximation Properties*, Handbook of the geometry of Banach spaces, Vol. 1, W. B. Johnson and J. Lindenstrauss, eds, Elsevier, Amsterdam (2001), 271–316.
6. P.G. Casazza, *The art of frame theory*, Taiwanese J. Math. **4** (2000), 129–201.
7. P.G. Casazza, O. Christensen, D. Stoeva, *Frame expansions in separable Banach spaces*, J. Math. Anal. Appl. **307** (2005), 710–723.
8. P.G. Casazza, D. Han and D.R. Larson, *Frames for Banach spaces*, Contemp. Math. **247** (1999), 149–182.
9. O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhäuser, 2003.
10. O. Christensen and C. Heil, *Perturbations of Banach frames and atomic decompositions*, Math. Nachr. **185** (1997), 33–47.
11. O. Christensen and D. Stoeva, *p-Frames in separable Banach spaces*, Adv. Comput. Math. **18** (2003), 117–126.
12. A.M. Davie, *The approximation problem for Banach spaces*, Bull. London Math. Soc. **5** (1973), 261–266.
13. J. Diestel, J.H. Fourie and J. Swart, *The Metric Theory of Tensor Products*, AMS Providence, 2008.
14. J. Diestel and J.J. Uhl, Jr., *Vector Measures*, AMS Providence, 1977.
15. K. Gröchenig, *Describing functions: Atomic decompositions versus frames*, Monatsh. Math. **112** (1991), 1–41.
16. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Springer, Berlin, 1977.
17. R.E. Megginson, *An Introduction to Banach Space Theory*, Springer, New Nork, 1998.
18. R.A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer, Berlin, 2002.
19. D.T. Stoeva, *On p-frames and reconstruction series in separable Banach spaces*, Integral Transforms Spec. Funct. **17** (2006), 127–133.

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