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STRONG ARENS IRREGULARITY OF BILINEAR MAPPINGS AND REFLEXIVITY

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Abstract. We provide a sufficient condition for strong (Arens) irregularity of certain bounded bilinear maps, which applies in particular to the adjoint of Banach module actions. We then apply our result to improve several known results concerning to the relation between Arens regularity of certain Banach module actions and reflexivity.

1. Introduction

In the substantial work [1] Arens showed that every bounded bilinear map $f : \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ has two natural but, in general, different extensions f^{***} and f^{r***r} from $\mathscr{X}^{**} \times \mathscr{Y}^{**}$ to $\mathscr{Z}^{**}$. When these extensions coincide on the whole of $\mathscr{X}^{**} \times \mathscr{Y}^{**}$, f is said to be (Arens) regular. f is said to be strongly (Arens) irregular whenever f^{***} and f^{r***r} are equal *only* on $\mathscr{X} \times \mathscr{Y}^{**}$ and $\mathscr{X}^{**} \times \mathscr{Y}$. Regularity and strong irregularity of bounded bilinear maps are investigated by many authors, for example see [6, 8, 10]. The interested reader may also refer to [4, 5] for more information on the subject of Arens regularity.

In this paper, we give a sufficient condition for strong irregularity of certain bounded bilinear maps (Theorem 4.1 infra), then we apply it to determine the topological centers of certain normed module actions. In particular, in Theorem 5.1 for the approximately unital normed $\mathscr{A}-$modules $(\pi_1, \mathscr{X})$ and $(\mathscr{X}, \pi_2)$ we show that π_1^{r*r} and π_2^* are strongly irregular. Our results not only improve

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some older results, but also provide a unified approach to give a simple direct proof for several known results from [2, 3, 6, 7, 8, 10, 11] concerning to the relation between Arens regularity and reflexivity.

2. Notions and notations

First we remark that as usual we regard a normed space $\mathscr{X}$ as a subspace of its second dual $\mathscr{X}^{**}$ in the natural way. We also identify an element of $\mathscr{X}$ with its canonical image in $\mathscr{X}^{**}$. The basic definition of the extensions f^{***} and f^{r***r} of a bounded bilinear mapping f can be found in [1]; see also [4, 5]. To establish our notation, we describe the construction briefly.

Let $\mathscr{X}, \mathscr{Y}$ and $\mathscr{Z}$ be normed spaces and let $f : \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ be a bounded bilinear map. We define the adjoint $f^* : \mathscr{Z}^* \times \mathscr{X} \to \mathscr{Y}^*$of f by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle \qquad (x \in \mathscr{X}, y \in \mathscr{Y} \;\text{ and }\; z^* \in \mathscr{Z}^*).$$

We can define the second and third adjoints f^{**} and f^{***} of f by $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*$, respectively, and so on for the higher adjoints. We also denote by f^r the flip map of f, that is the bounded bilinear map $f^r : \mathscr{Y} \times \mathscr{X} \longrightarrow \mathscr{Z}$ defined by $f^r(y, x) = f(x, y)$ $(x \in \mathscr{X}, y \in \mathscr{Y})$. If we continue the latter process with f^r instead of f, we get to the definition of $f^{r***r} : \mathscr{X}^{**} \times \mathscr{Y}^{**} \to \mathscr{Z}^{**}$.

If $f^{***} = f^{r***r}$ then f is said to be (Arens) regular.

It is easy to verify that f^{***} and f^{r***r} are extensions of f which are w^*-separately continuous on $\mathscr{X} \times \mathscr{Y}^{**}$ and $\mathscr{X}^{**} \times \mathscr{Y}$, respectively. Therefore we define the left (resp. right) topological center $Z_\ell(f)$ (resp. $Z_r(f)$) by

$$Z_\ell(f) = \{x^{**} \in \mathscr{X}^{**};\; y^{**} \mapsto f^{***}(x^{**}, y^{**}) : \mathscr{Y}^{**} \longrightarrow \mathscr{Z}^{**} \text{ is } w^* - \text{continuous}\}$$

(resp.

$$Z_r(f) = \{y^{**} \in \mathscr{Y}^{**};\; x^{**} \mapsto f^{r***r}(x^{**}, y^{**}) : \mathscr{X}^{**} \longrightarrow \mathscr{Z}^{**} \text{ is } w^* - \text{continuous}\}).$$

An standard argument based on the w^*-density of a normed space in its second dual implies that, an element $x^{**} \in \mathscr{X}^{**}$ lies in $Z_\ell(f)$ if and only if $f^{***}(x^{**}, y^{**}) = f^{r***r}(x^{**}, y^{**})$ for every $y^{**} \in \mathscr{Y}^{**}$. Similarly, $y^{**} \in \mathscr{Y}^{**}$ lies in $Z_r(f)$ if and only if $f^{***}(x^{**}, y^{**}) = f^{r***r}(x^{**}, y^{**})$ for every $x^{**} \in \mathscr{X}^{**}$. Therefore the (Arens) regularity of f is equivalent to $Z_\ell(f) = \mathscr{X}^{**}$ as well as $Z_r(f) = \mathscr{Y}^{**}$. The map f is said to be left (resp. right) strongly (Arens) irregular if $Z_\ell(f) = \mathscr{X}$ (resp. $Z_r(f) = \mathscr{Y}$).

The same argument can also be applied to interpret f^{***} and f^{r***r} in terms of the following iterated limit process:

$$f^{***}(x^{**}, y^{**}) = w^* - \lim_\alpha \lim_\beta f(x_\alpha, y_\beta)$$

and

$$f^{r***r}(x^{**}, y^{**}) = w^* - \lim_\beta \lim_\alpha f(x_\alpha, y_\beta),$$

where $\{x_\alpha\}$ and $\{y_\beta\}$ are nets in $\mathscr{X}$ and $\mathscr{Y}$ which converge to x^{**} and y^{**} in the w^*-topologies, respectively.

A familiar example of a bounded bilinear map, whose extensions are of special interest, is the product π of a normed algebra $\mathscr{A}$. In this case the extensions π^{***}

and π^{r***r} are nothing but the so-called Arens products on $\mathscr{A}^{**}$. More information about these products can be found in [4, 5].

3. Approximately unital bilinear maps

Let $\mathscr{X}$ and $\mathscr{A}$ be normed spaces and let $g : \mathscr{X} \times \mathscr{A} \longrightarrow \mathscr{X}$ be a bounded bilinear mapping. Then g is said to be unital (resp. approximately unital) if there exists an $e \in \mathscr{A}$ (resp. a bounded net $\{e_\alpha\}$ in $\mathscr{A}$) such that $g(x, e) = x$ (resp. $\lim_\alpha g(x, e_\alpha) = x$), for all $x \in \mathscr{X}$. The next result reveals the close relation between the situations that g is approximately unital and g^{***} is unital.

Proposition 3.1. *Let $\mathscr{X}$ and $\mathscr{A}$ be normed spaces. A bounded bilinear map $g : \mathscr{X} \times \mathscr{A} \longrightarrow \mathscr{X}$ is approximately unital if and only if $g^{***} : \mathscr{X}^{**} \times \mathscr{A}^{**} \longrightarrow \mathscr{X}^{**}$ is unital.*

Proof. An standard argument shows that $e^{**} \in A^{**}$ is a unit for g^{***} (that is, $g^{***}(x^{**}, e^{**}) = x^{**}$, for all $x^{**} \in \mathscr{X}^{**}$) if and only if e^{**} is a w^*-cluster point of a bounded net $\{e_\alpha\}$ in $\mathscr{A}$ with $\lim_\alpha g(x, e_\alpha) = x$, for all $x \in \mathscr{X}$. $\square$

Remark 3.2. It should be noted that in contrast to the situation occurring for g^{***} in Proposition 3.1, g^{r***r} is not unital, in general. For example, let π be the multiplication of $K(c_0)$, the operator algebra of all compact operators on the sequence space c_0. As it is mentioned in [9, Example 2.5], $K(c_0)$ has a bounded approximate identity and so $K(c_0)^{**}$ enjoys a mixed unit, say e^{**}; i.e.

$$\pi^{***}(x^{**}, e^{**}) = x^{**} = \pi^{r***r}(e^{**}, x^{**}) \quad (x^{**} \in K(c_0)^{**}).$$

However, π^{r***r} is not unital, i.e. the identity $\pi^{r***r}(x^{**}, e^{**}) = x^{**}$ does not hold, in general (indeed, $K(c_0)^*$ factors on the right but not on the left, [9, Example 2.5]).

4. Strong irregularity of certain bilinear maps

We commence with the next result which characterizes the topological center of certain bilinear maps.

Theorem 4.1. *Let $\mathscr{X}$ and $\mathscr{A}$ be normed spaces. Then the adjoint g^* of an approximately unital bounded bilinear map $g : \mathscr{X} \times \mathscr{A} \longrightarrow \mathscr{X}$ is strongly irregular; i.e.*

$$Z_\ell(g^*) = \mathscr{X}^* \quad \text{and} \quad Z_r(g^*) = \mathscr{X}.$$

In particular, g^* is regular if and only if $\mathscr{X}$ is reflexive.

Proof. By Proposition 3.1, g^{***} is unital; i.e. there exists $e^{**} \in \mathscr{A}^{**}$ such that $g^{***}(x^{**}, e^{**}) = x^{**}$, for all $x^{**} \in \mathscr{X}^{**}$. Let $x^{***} \in Z_\ell(g^*)$ then

$$\begin{aligned}
\langle x^{***}, x^{**} \rangle &= \langle x^{***}, g^{***}(x^{**}, e^{**}) \rangle \\
&= \langle g^{****}(x^{***}, x^{**}), e^{**} \rangle \\
&= \langle g^{*r****r}(x^{***}, x^{**}), e^{**} \rangle \qquad (x^{***} \in Z_\ell(g^*)) \\
&= \langle g^{*r**}(x^{***}, e^{**}), x^{**}) \rangle
\end{aligned}$$

Hence $x^{***} = g^{*r**}(x^{***}, e^{**}) \in \mathscr{X}^*$, and so $Z_\ell(g^*) = \mathscr{X}^*$.
To prove right strong irregularity of g^*, let $x^{**} \in Z_r(g^*)$. Suppose that $x^{***} \in \mathscr{X}^{***}$ and $\{x_\alpha\} \subseteq \mathscr{X}$ and $\{x^*_\beta\} \subseteq \mathscr{X}^*$ be bounded nets w^*-converging to x^{**} and x^{***}, respectively. Recall that e^{**} is a w^*-cluster point of a bounded net $\{e_\gamma\}$ in $\mathscr{A}$ such that $\lim_\gamma g(x, e_\gamma) = x,\ (x \in \mathscr{X})$. We therefore have

$$\begin{aligned}
\langle x^{***}, x^{**}\rangle &= \langle x^{***}, g^{***}(x^{**}, e^{**})\rangle \\
&= \langle g^{****}(x^{***}, x^{**}), e^{**}\rangle \\
&= \langle g^{*r****}(x^{***}, x^{**}), e^{**}\rangle \qquad (x^{**} \in Z_r(g^*)) \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle g^*(x^*_\beta, x_\alpha), e_\gamma\rangle \\
&= \lim_\alpha \lim_\beta \lim_\gamma \langle x^*_\beta, g(x_\alpha, e_\gamma)\rangle \\
&= \lim_\alpha \lim_\beta \langle x^*_\beta, x_\alpha\rangle \\
&= \lim_\alpha \lim_\beta \langle x_\alpha, x^*_\beta\rangle \\
&= \lim_\alpha \langle x^{***}, x_\alpha\rangle.
\end{aligned}$$

This means that $\{x_\alpha\} \subseteq \mathscr{X}$ converges to x^{**} in the weak topology. As $\mathscr{X}$ is a (weakly) closed subspace of $\mathscr{X}^{**}$, we get $x^{**} \in \mathscr{X}$. Therefore $Z_r(g^*) = \mathscr{X}$; as required.

If $\mathscr{X}$ is reflexive, then trivially g^* is regular. For the converse, the strong irregularity of g^* together with the regularity imply that $\mathscr{X}^* = Z_\ell(g^*) = \mathscr{X}^{***}$ which forces $\mathscr{X}$ to be reflexive. □

As a consequence of the latter theorem we get the next result of [11].

Corollary 4.2 ([11, Corollary 3.2]). *For every complex normed space X, the bilinear map $f : \mathscr{X}^* \times \mathscr{X} \to \mathbb{C}$ defined by $f(x^*, x) = \langle x^*, x\rangle$, $(x^* \in \mathscr{X}^*,\ x \in \mathscr{X})$, is strongly irregular. In particular, f is regular if and only if $\mathscr{X}$ is reflexive.*

Proof. Define $g : \mathscr{X} \times \mathbb{C} \to \mathscr{X}$ by $g(x, \alpha) = \alpha x$, $(x \in \mathscr{X}, \alpha \in \mathbb{C})$. Then g is a bilinear map with $g(x, 1) = x$, $(x \in \mathscr{X})$. A direct computation shows that $g^* = f$. Now the conclusion follows from Theorem 4.1. Here $\mathbb{C}^*$ is naturally identified to $\mathbb{C}$. □

Part (i) of the following example provides a short proof for [7, Theorem 5] and part (ii) emphasizes that "being approximately unital" in Theorem 4.1 is essential.

Example 4.3 (See [6, Example 4.7]). Let $\mathscr{X}$ be a non-zero normed space and take $\mathscr{A} = \mathscr{X}$. Fix $e \in \mathscr{A}$ and $e^* \in \mathscr{A}^*$ with $\langle e^*, e\rangle = 1$.

(i) Define $g : \mathscr{X} \times \mathscr{A} \to \mathscr{X}$ by $g(x, a) = \langle e^*, a\rangle x$, then $g(x, e) = x$ for each $x \in X$ and so Theorem 4.1 (or a straightforward verification) confirms that g^* is strongly irregular. In particular, g^* is regular if and only if $\mathscr{X}$ is reflexive.

(ii) If we define $h : \mathscr{X} \times \mathscr{A} \to \mathscr{X}$ by $h(x, a) = \langle e^*, x\rangle a$. Then $h^*(x^*, x) = \langle e^*, x\rangle x^*$, $(x^* \in X^*, x \in \mathscr{X})$ and this implies that h^* is regular, that is, $Z_\ell(h^*) = \mathscr{X}^{***}$ and $Z_r(h^*) = \mathscr{X}^{**}$. Therefore h^* is not strongly irregular in the case where $\mathscr{X}$ is not reflexive. Note that h is not approximately unital.

5. Some applications to normed module actions

Let $\mathscr{A}$ be a normed algebra, $\mathscr{X}$ be a normed space and let $\pi_1 : \mathscr{A} \times \mathscr{X} \longrightarrow \mathscr{X}$ be a bounded bilinear map. Then the pair $(\pi_1, \mathscr{X})$ is said to be a (left) normed $\mathscr{A}-$module if π_1 is associative, i.e. $\pi_1(ab, x) = \pi_1(a, \pi_1(b, x))$, for every $a, b \in \mathscr{A}, x \in \mathscr{X}$. A (right) normed $\mathscr{A}-$module $(\mathscr{X}, \pi_2)$ can be defined similarly.

A normed $\mathscr{A}-$module $(\pi_1, \mathscr{X})$ (resp. $(\mathscr{X}, \pi_2)$) is said to be approximately unital if the bilinear map $\pi_1^r : \mathscr{X} \times \mathscr{A} \longrightarrow \mathscr{X}$ (resp. $\pi_2 : \mathscr{X} \times \mathscr{A} \longrightarrow \mathscr{X}$) is approximately unital. Similarly one can define a unital normed $\mathscr{A}-$module.

Trivially $(\pi_2^{r*r}, \mathscr{X}^*)$ and $(\mathscr{X}^*, \pi_1^*)$ are normed $\mathscr{A}-$modules which are called the canonical duals of $(\mathscr{X}, \pi_2)$ and $(\pi_1, \mathscr{X})$, respectively.

As an immediate consequence of Proposition 3.1 one can deduce that: if $(\pi_1, \mathscr{X})$ and $(\mathscr{X}, \pi_2)$ are approximately unital then $(\pi_1^{r***r}, \mathscr{X}^{**})$ and $(\mathscr{X}^{**}, \pi_2^{***})$ are unital. However, as it is emphasized in Remark 3.2, $(\mathscr{X}^{**}, \pi_1^{***})$ and $(\pi_2^{r***r}, \mathscr{X}^{**})$ are not unital, in general.

The following result, which gives a complete characterization of topological centers of π_1^{r*r} and π_2^*, is a generalization of [3, Theorem 2.2 and Theorem 3.4] and [10, Proposition 3.6] with a simple direct proof. It is worth mentioning that, in the proof of the following result we have not used the assumptions that A is a normed algebra and that π_1 and π_2 are module actions. Indeed we merely use the fact that π_1 and π_2 are bilinear.

Theorem 5.1. *Let $(\pi_1, \mathscr{X})$ and $(\mathscr{X}, \pi_2)$ be approximately unital normed $\mathscr{A}-$modules. Then π_1^{r*r} and π_2^* are strongly irregular, i.e.*

$$Z_\ell({\pi_1}^{r*r}) = \mathscr{X} = Z_r({\pi_2}^*) \quad \text{and} \quad Z_r(\pi_1^{r*r}) = \mathscr{X}^* = Z_\ell(\pi_2^*).$$

In particular, π_1^{r*r} is regular if and only if $\mathscr{X}$ is reflexive if and only if π_2^* is regular.

Proof. It is enough to apply Theorem 4.1 for $g = \pi_1^r$ and $g = \pi_2$. For the last part, the regularity and the strong irregularity of π_1^{r*r} imply that $\mathscr{X} = Z_\ell({\pi_1}^{r*r}) = \mathscr{X}^{**}$, i.e. $\mathscr{X}$ is reflexive. Similarly one can show that ${\pi_2}^*$ is regular if and only if $\mathscr{X}$ is reflexive. $\square$

Note that in contrast to the situation in Theorem 5.1, the canonical duals π_2^{r*r} and π_1^* are not necessarily strongly irregular; (see Example 4.3 (ii)). However, the situation is a bit different when we regard π^{r*r} and π^* where π is the multiplication of a normed algebra. As a straightforward application of Theorem 5.1 we bring the next result which studies the strong irregularity of π^{r*r} and π^*. In particular, it presents a generalization of [8, Theorem 2.1, Corollary 2.1 and Corollary 2.4] (see also [6, Proposition 4.5] and [11, Theorem 3.1]).

Corollary 5.2. *Let π denote the multiplication of a normed algebra $\mathscr{A}$. Then*

(i) If $\mathscr{A}$ has a bounded right approximate identity then π^* is strongly irregular. In particular, π^* is regular if and only if $\mathscr{A}$ is reflexive.

(ii) If $\mathscr{A}$ has a bounded left approximate identity then π^{r*r} is strongly irregular. In particular, π^{r*r} is regular if and only if $\mathscr{A}$ is reflexive.

(iii) If $\mathscr{A}$ has a bounded approximate identity then both π^* and π^{r*r} are strongly irregular. In particular, π^* is regular if and only if $\mathscr{A}$ is reflexive if and only if π^{r*r} is regular.

We also present an extension of the main result of [2] with a simple proof.

Corollary 5.3 (See [2, Theorem 4])**.** *Let $\mathscr{A}$ be a normed algebra with a bounded left (or right) approximate identity. Then $\mathscr{A}$ is reflexive if and only if every bounded bilinear map from $A \times \mathscr{X}$ to $\mathscr{X}$ is regular, where $\mathscr{X}$ is a normed space.*

Proof. The necessity is trivial. For sufficiency, let $\mathscr{A}$ has a bounded right approximate identity and let π denote the multiplication of $\mathscr{A}$. By assumption, $\pi^{r*r} : \mathscr{A} \times \mathscr{A}^* \to \mathscr{A}^*$ is regular. Now part (ii) of Corollary 5.2 implies the reflexivity of $\mathscr{A}$. A similar argument can be applied for the case where $\mathscr{A}$ has a bounded left approximate identity. □

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