Banach J. Math. Anal. 5 (2011), no. 1, 51–55

BANACH JOURNAL OF MATHEMATICAL ANALYSIS
ISSN: 1735-8787 (electronic)
www.emis.de/journals/BJMA/

TOTAL DECOMPOSITION AND BLOCK NUMERICAL RANGE

ABBAS SALEMI[1]

Communicated by F. Kittaneh

ABSTRACT. Let $\mathcal{H}$ be a separable Hilbert space and let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$. In this note the notion of a total decomposition is introduced, and it is shown that sometimes the block numerical ranges corresponding to a total decomposition approximate $\sigma(A)$, sometimes not.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{H}$ be a Hilbert space and let $A \in \mathcal{B}(\mathcal{H})$. The numerical range of A is defined as follows (see [1, 3]):

$$W(A) = \{x^*Ax : x \in \mathcal{H}, \|x\| = 1\}.$$

The notion of quadratic numerical range was introduced in [4] and this concept was generalized to the block numerical range in [6]. Let $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$, where $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_m$ are Hilbert spaces. With respect to this decomposition, the block operator matrix $\mathcal{A}$ on $\mathcal{H}$ has the following representation:

$$\mathcal{A} := \begin{bmatrix} A_{11} & \cdots & A_{1m} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mm} \end{bmatrix}, \tag{1.1}$$

Date: Received: 24 May 2010; Accepted: 10 June 2010.
2010 *Mathematics Subject Classification.* Primary 47A12; Secondary 49M27.
Key words and phrases. Block numerical range, total decomposition , spectrum.

where $A_{ij} \in \mathcal{B}(\mathcal{H}_j, \mathcal{H}_i)$ for all $(i, j = 1, \dots, m)$. For $\mathbf{x} = (x_1, \dots, x_m) \in \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots \oplus \mathcal{H}_m$, define $\mathcal{A}_{\mathbf{x}} \in M_m(\mathbb{C})$ (the space of $m \times m$ matrices over $\mathbb{C}$) as follows:

$$\mathcal{A}_{\mathbf{x}} := \begin{bmatrix} \langle A_{11}x_1, x_1\rangle & \cdots & \langle A_{1m}x_m, x_1\rangle \\ \vdots & \ddots & \vdots \\ \langle A_{m1}x_1, x_m\rangle & \cdots & \langle A_{mm}x_m, x_m\rangle \end{bmatrix}.$$

The *block numerical range of the block operator matrix* $\mathcal{A}$ as in (1.1) is the set

$$W^m(\mathcal{A}) := \{\lambda \in \mathbb{C} : \lambda \in \sigma(\mathcal{A}_{\mathbf{x}}),\ \mathbf{x} \in \mathcal{S}_m\},$$

where $\mathcal{S}_m = \{(x_1, \dots, x_m) \in \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m : \| x_1 \| = \cdots = \| x_m \| = 1\}$.

In the following Lemma we state some properties. (For details see [6].)

Lemma 1.1. *Let $\mathcal{A}$ as in (1.1) be a block operator matrix on $\mathcal{H}$. Then*

(1) $\sigma_p(\mathcal{A}) \subseteq W^m(\mathcal{A})$, *where $\sigma_p(\mathcal{A})$ is the point spectrum of $\mathcal{A}$.*
(2) $\sigma(\mathcal{A}) \subseteq \overline{W^m(\mathcal{A})}$, *where $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$.*
(3) $W^m(\mathcal{A}) \subseteq W(\mathcal{A})$,
(4) $W^m(\mathcal{A}^*) := \{\lambda : \bar{\lambda} \in W^m(\mathcal{A})\}$.

Let $\mathcal{H} = \widehat{\mathcal{H}}_1 \oplus \cdots \oplus \widehat{\mathcal{H}}_{\widehat{m}} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$. Then $\widehat{\mathcal{H}}_1 \oplus \cdots \oplus \widehat{\mathcal{H}}_{\widehat{m}}$ is a refinement of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$, if $m \leq \widehat{m}$ and there exist $0 = i_0 < i_1 < \cdots < i_m = \widehat{m}$ such that $\mathcal{H}_j = \widehat{\mathcal{H}}_{i_{j-1}+1} \oplus \cdots \oplus \widehat{\mathcal{H}}_{i_j},\ 1 \leq j \leq m$.

Proposition 1.2. [6, Theorem 3.5] *Let $\widehat{\mathcal{H}}_1 \oplus \cdots \oplus \widehat{\mathcal{H}}_{\widehat{m}}$ be a refinement of $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$. Then $W^{\widehat{m}}(\mathcal{A}) \subseteq W^m(\mathcal{A})$.*

Notice that, we consider $\mathcal{A}$ with respect to the decompositions $\mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m$ and $\widehat{\mathcal{H}}_1 \oplus \cdots \oplus \widehat{\mathcal{H}}_{\widehat{m}}$ to define $W^m(\mathcal{A})$ and $W^{\widehat{m}}(\mathcal{A})$ respectively.

Throughout the paper we will fix these notations: Let $\mathbb{T}$ and $\overline{D}$ be the unit circle and closed unit disc in the complex plane, respectively.

2. Main results

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$. In this section the notion of a total decomposition of $\mathcal{H}$ is introduced. Also, we define an estimable decomposition of $\mathcal{H}$ for $\sigma(\mathcal{A})$. By using an estimable decomposition, we will approximate the spectrum of $\mathcal{A}$ by block numerical ranges of $\mathcal{A}$.

Definition 2.1. Let $\mathcal{H}$ be a separable Hilbert space. A *total decompostion* of $\mathcal{H}$ is a sequence of decompositions $\{\mathcal{H} = \mathcal{H}_1^k \oplus \mathcal{H}_2^k \oplus \cdots \oplus \mathcal{H}_{n_k}^k\}_{k=1}^{\infty}$ with the $(k+1)^{th}$ being a refinement of k^{th} and there is no subspace V with $\dim(V) > 1$ such that for all $k \in \mathbb{N}$, there exists $1 \leq l_k \leq n_k$ such that $V \subseteq \mathcal{H}_{l_k}^k$.

Lemma 2.2. *Every separable Hilbert space $\mathcal{H}$ has a total decomposition.*

Proof. Let $\mathcal{H}$ be a separable Hilbert space. Then $\mathcal{H}$ has an orthonormal basis $\mathcal{B} = \{\alpha_1, \alpha_2, \dots\}$. Now, we define a sequence of decompositions for the Hilbert space $\mathcal{H}$, $\left\{\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_m \oplus \widehat{\mathcal{H}}_{m+1}\right\}_{m=1}^{\infty}$, where $\mathcal{H}_i$ is the subspace generated by $\{\alpha_i\}$, $i = 1, \dots m$ and $\widehat{\mathcal{H}}_{m+1}$ is the subspace generated by $\{\alpha_{m+1}, \alpha_{m+2}, \dots\}$. It is readily seen that this sequence of decompositions is a total decomposition. $\square$

Let $\{\mathcal{H} = \mathcal{H}_1^k \oplus \mathcal{H}_2^k \oplus \cdots \oplus \mathcal{H}_{n_k}^k\}_{k=1}^{\infty}$ be a total decomposition of $\mathcal{H}$ and let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$. Then, by Proposition 1.2, and Lemma 1.1(2,3), we have the following:

$$\sigma(\mathcal{A}) \subseteq \overline{W^{n_{k+1}}(\mathcal{A})} \subseteq \overline{W^{n_k}(\mathcal{A})} \subseteq \overline{W(\mathcal{A})},\ k \in \mathbb{N}. \tag{2.1}$$

Hence, $\sigma(\mathcal{A}) \subseteq \bigcap_{k=1}^{\infty} \overline{W^{n_k}(\mathcal{A})}$. In the following it is shown that sometimes for m large enough $\overline{W^{n_m}(\mathcal{A})}$ is a good approximation of $\sigma(A)$ and sometimes not.

Definition 2.3. Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{A} \in \mathcal{B}(\mathcal{H})$. A total decomposition $\{\mathcal{H} = \mathcal{H}_1^k \oplus \mathcal{H}_2^k \oplus \cdots \oplus \mathcal{H}_{n_k}^k\}_{k=1}^{\infty}$ is called an *estimable decomposition* of $\mathcal{H}$ for $\sigma(\mathcal{A})$, if $\sigma(\mathcal{A}) = \bigcap_{k=1}^{\infty} \overline{W^{n_k}(\mathcal{A})}$.

It is obvious that all total decompositions in the finite dimensional cases are estimable decompositions.

Let $\mathcal{A} \in \mathcal{B}(\mathcal{H})$. Equivalently, a total decomposition of $\mathcal{H}$ is an estimable decomposition for $\sigma(\mathcal{A})$, if

$$\forall \varepsilon > 0, \exists\ M > 0 \ni\ \mathbf{d}\left(\sigma(\mathcal{A}), \overline{W^{n_m}(\mathcal{A})}\right) < \varepsilon, \forall m \geq M,$$

where, $\mathbf{d}$ is the Hausdorff metric [2, page 117] for compact subsets of the complex plane $\mathbb{C}$.

Theorem 2.4. *Let $\mathcal{S}_+$ be the unilateral shift operator on a separable infinite dimensional Hilbert space $\mathcal{H}$. All total decompositions of $\mathcal{H}$ are estimable for $\sigma(\mathcal{S}_+)$.*

Proof. We know that $\sigma(\mathcal{S}_+) = \overline{D}$ and $\overline{W(\mathcal{S}_+)} = \overline{D}$, (see [2, 5]). By Lemma 1.1(2), $\sigma(\mathcal{S}_+) \subseteq \overline{W^k(\mathcal{S}_+)} \subseteq \overline{W(\mathcal{S}_+)}$. Then, $\sigma(\mathcal{S}_+) = \bigcap_{k=1}^{\infty} \overline{W^k(\mathcal{S}_+)} = \overline{D}$, and hence all total decompositions of $\mathcal{H}$ are estimable for $\sigma(\mathcal{S}_+)$. □

In the following Theorem we show that for the bilateral shift operator $\mathcal{S}$, there exists a total decomposition which is not estimable for $\sigma(\mathcal{S})$.

Theorem 2.5. *Let $\mathcal{S}$ be the bilateral shift operator on a separable infinite dimensional Hilbert space $\mathcal{H}$. Then, there exists a total decomposition of $\mathcal{H}$, which is not estimable for $\sigma(\mathcal{S})$.*

Proof. Let $\{\alpha_0, \alpha_{\pm 1}, \ldots\}$ be an orthonormal basis and let $S\alpha_i = \alpha_{i+1}$ $(i = 0, \pm 1, \ldots)$. We consider $\mathcal{H}_0$, the subspace generated by $\{\alpha_0\}$, and for all $|i| \geq 1$,$\mathcal{H}_{\pm i}$, the subspace generated by $\{\alpha_{\pm i}\}$, and $\widehat{\mathcal{H}}_{\pm i}$, the subspace generated by $\{\alpha_{\pm i}, \alpha_{\pm(i+1)}, \ldots\}$. Then

$$\mathcal{H} = \widehat{\mathcal{H}}_{-m} \oplus \mathcal{H}_{-m+1} \oplus \cdots \oplus \mathcal{H}_{-1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_{m-1} \oplus \widehat{\mathcal{H}}_m,\ m \in \mathbb{N}.$$

It is readily seen that this sequence of decompositions is a total decomposition. Now, we consider the $(2m+1) \times (2m+1)$ block operator matrix $\mathcal{S} = (S_{ij})_{i,j=-m}^{m}$ with respect to the above decomposition. It is easy to show that the block entry S_{mm} is a unilateral shift operator and by [6, Corollary 3.2], $W(S_{mm}) \subseteq W^{2m+1}(\mathcal{S})$ for all $m \geq 1$. We know that $\sigma(\mathcal{S}) = \mathbb{T}$, and $W(S_{mm}) = \overline{D},\ m \geq 1$, (see [1, 2]). Thus, $\sigma(\mathcal{S}) = \mathbb{T} \nsupseteq \overline{D} \subseteq \bigcap_{m=1}^{\infty} \overline{W^{2m+1}(\mathcal{S})}$ and hence this total decomposition is not estimable for $\sigma(\mathcal{S})$. □

In the following Theorem we show that for any separable infinite dimensional Hilbert space $\mathcal{H}$ there exist $T \in \mathcal{B}(\mathcal{H})$ and two total decompositions, which one of them is estimable for $\sigma(T)$ and the other is not.

Theorem 2.6. *Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space. Then there exists an operator $T \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{H}$ have two total decompositions which one of them is estimable for $\sigma(T)$ and the other is not.*

Proof. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \ldots\}$ be an orthonormal basis for $\mathcal{H}$. We define $T \in \mathcal{B}(\mathcal{H})$ such that the representation of T with respect to $\mathcal{B}$ is of the form $[T]_{\mathcal{B}} = \mathrm{diag}(e^{ir_1}, e^{ir_2}, \ldots)$, where $\{r_k\}_{k=1}^{\infty}$ be a sequence of all rational numbers in $[0, 2\pi)$. Now, we consider two total decompositions which one of them is estimable for $\sigma(T)$ and the other is not.

First, we consider a total decomposition,

$$\{\mathcal{H} =< \alpha_1 > \oplus \cdots \oplus < \alpha_{m-1} > \oplus < \alpha_m, \alpha_{m+1}, \ldots >\}_{m=2}^{\infty},$$

where $< S >$ is the subspace generated by S. It is clear that this sequence of decompositions is a total decomposition of $\mathcal{H}$. With respect to these decompositions, T have representations of the forms $T = [e^{ir_1}] \oplus \cdots \oplus [e^{ir_{m-1}}] \oplus \mathrm{diag}(e^{ir_m}, \ldots)$, $1 < m < \infty$. Since for any $m > 1$, the set $\{e^{ir_m}, e^{ir_{m+1}}, \ldots\}$ is a dense subset of the unit circle in the complex plane, we obtain that, $\overline{W(T)} = \overline{W\left(\mathrm{diag}(e^{ir_m}, \ldots)\right)} = \overline{D}$, $m > 1$. Also, by [6, Corollary 3.2], $W\left(\mathrm{diag}(e^{ir_m}, \ldots)\right) \subseteq W^m(T)$. Therefore, $\sigma(T) = \mathbb{T} \nsubseteq \overline{D} \subseteq \bigcap_{m=1}^{\infty} \overline{W^m(T)}$ and hence this total decomposition is not estimable for $\sigma(T)$.

Second, let $\{\mathcal{H} = H_1^m \oplus H_2^m \oplus \cdots \oplus H_{2^m}^m\}_{m=1}^{\infty}$ be a total decomposition, where H_j^m be the subspace generated by $\{\alpha_k : r_k \in \left[\frac{2(j-1)\pi}{2^m}, \frac{2j\pi}{2^m}\right) \cap \mathbb{Q})\}$, where $\{\alpha_k, e^{ir_k}\}$ is an eigenpair of T. For $m \in \mathbb{N}$, define

$$T_j^m := \mathrm{diag}\left(e^{ir_k} : r_k \in \left[\frac{2(j-1)\pi}{2^m}, \frac{2j\pi}{2^m}\right) \cap \mathbb{Q}\right), \ 1 \leq j \leq 2^m.$$

Hence $\overline{W(T_j^m)} = \mathrm{conv}\left(\left\{e^{ir} : r \in \left[\frac{2(j-1)\pi}{2^m}, \frac{2j\pi}{2^m}\right]\right\}\right)$, where $\mathrm{conv}(X)$ is the convex hull of X. Since $T = T_1^m \oplus T_2^m \oplus \cdots \oplus T_{2^m}^m$ for all $m \in \mathbb{N}$, we obtain that $\overline{W^{2^m}(T)} = \overline{W(T_1^m)} \cup \overline{W(T_2^m)} \cup \cdots \cup \overline{W(T_{2^m}^m)}$ is the region between the regular polygon of degree 2^m and it's circumscribed unit circle $\mathbb{T}$. Then, for all $\varepsilon > 0$, there exists $M > 0$ such that $1 - \cos\left(\frac{1}{2^{m+1}}\right) < \varepsilon$, $m > M$. So, $\mathbf{d}\left(\overline{W^{2^m}(T)}, \mathbb{T}\right) < \varepsilon$, for all $m > M$. Therefore, $\sigma(T) = \mathbb{T} = \bigcap_{m=1}^{\infty} \overline{W^{2^m}(T)}$, and hence, this total decomposition of $\mathcal{H}$ for $\sigma(T)$ is estimable. $\square$

It would be nice to solve the following conjecture.

Conjecture. Let $\mathcal{H}$ be a separable Hilbert space. For any $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ there exits an estimable decomposition of $\mathcal{H}$ for $\sigma(\mathcal{A})$.

Acknowledgement. This research has been partially supported by Mahani Mathematical Research Center, Kerman, Iran.

References

1. K. Gustafson and K.M. Rao, *Numerical Range*, Springer-Verlag, New York, 1997.
2. P.R. Halmos, *A Hilbert space problem book*, Springer-Verlag, New York, 1974.
3. R. Horn and C. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, NewYork, 1991.
4. H.Langer and C. Tretter, *Spectral decomposition of some nonselfadjoint block operator matrices*, J. Operator Theory **39** (1998), no. 2, 339–359.
5. W. Rudin, *Functional Analysis*, Mc-Graw Hill, Inc., 1973.
6. C. Tretter and M. Wagenhofer, *The block numerical range of an $n \times n$ block operator matrix*, SIAM J. Matrix Anal. Appl. **22**, 4, (2003), 1003-1017.

[1] Department of Mathematics, Shahid Bahonar University of Kerman, Kerman, Iran. The SBUK Center of Excellence in Linear Algebra and Optimization, Kerman, Iran.

E-mail address: salemi@mail.uk.ac.ir