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ON EXISTENCE OF HYPERINVARIANT SUBSPACES FOR LINEAR MAPS

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Abstract. Let X be an infinite dimensional complex vector space. We show that a non-constant endomorphism of X has a proper hyperinvariant subspace if and only if its spectrum is non-void. As an application we show that each non-constant continuous endomorphism of the locally convex space (s) of all complex sequences has a proper closed hyperinvariant subspace.

1. Introduction

In a short note Schaefer [3] has observed that any linear map from an infinite dimensional complex vector space into itself has a proper invariant linear subspace. A similar result concerning hyperinvariant subspaces we obtain here. The presented here proof of this result is simple and can be offered during the first course of linear algebra. Nevertheless the result is new. We apply this result for obtaining a simpler proof of the result already observed in [6], which states that all non-constant continuous endomorphisms of the locally convex space (s) of all complex sequences have proper closed hyperinvariant subspaces (Corollary 2.3).

2. The results

Denote by $\mathfrak{L}(X)$ the family (algebra) of all linear maps (operators) from a vector space X into itself. The spectrum $\sigma(T)$ of an operator $T \in \mathfrak{L}(X)$ is defined as

$$\sigma(T) = \{\lambda \in \mathbf{C} : T - \lambda I \in \mathfrak{L}(X)^{-1}\},$$

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where I is the identity map on X and $\mathfrak{L}(X)^{-1}$ is the set of all invertible maps in $\mathfrak{L}(X)$. The commutant T' is the set $\{S \in \mathfrak{L}(X) : ST = TS\}$ and a hyperinvariant subspace for T is a linear subspace $X_0 \subset X$ which is an invariant subspace for all operators in T', i.e. $SX_0 \subset X_0$ for all S in T'. Such a subspace is said proper if $(0) \neq X_0 \neq X$. Call T in $\mathfrak{L}(X)$ non-constant if $T \neq \lambda I$ for all scalars λ. Clearly, any constant operator has no proper hyperinvariant subspace. Our result reads as follows.

Theorem 2.1. *Let X be an infinite dimensional complex vector space and let T be a non-constant operator in $\mathfrak{L}(X)$. Then T has a proper hyperinvariant subspace X_0 if and only if the spectrum $\sigma(T)$ is non-void.*

Proof. Assume first that $\lambda_0 \in \sigma(T)$, so that either $T - \lambda_0 I$ is not one-to-one, or it is not a map onto. In the first case the kernel

$$X_0 = \{x \in X : (T - \lambda_0 I)x = 0\}$$

is a non-zero vector subspace of X, and since T is non-constant, it is also different from X. Thus X_0 is a proper subspace of X. For an arbitrary S in T' and x in X_0 we have

$$0 = S(T - \lambda_0 I)x = (T - \lambda_0 I)Sx,$$

so that $Sx \in X_0$. Thus X_0 is invariant for all S in T'.

In the case when the operator $T - \lambda_0 I$ is one-to-one but not onto, the vector subspace

$$X_0 = (T - \lambda_0 I)X$$

is proper, and for any S in T' we have

$$SX_0 = S(T - \lambda_0 I)X = (T - \lambda_0 I)SX \subset (T - \lambda_0 I)X = X_0.$$

Thus X_0 is hyperinvariant for T.

It remains to be shown that if $\sigma(T)$ is void, then T has no proper hyperinvariant subspace. Suppose then, that T has such a subspace X_0 and try to get a contradiction. Since $\sigma(T) = \emptyset$, then for each non-zero complex rational function φ the operator $\varphi(T)$ is a well defined element of $\mathfrak{L}(X)^{-1}$. Denote by Q the set (field) of all complex rational functions and by $Q(T)$ the set of all operators of the form $\varphi(T)$. Clearly $Q(T) \subset T'$. Fix now a non-zero element x_0 in X_0 and put

$$Y_0 = Q(T)x_0 = \{Sx_0 : S \in Q(T)\}.$$

Thus Y_0 is a non-zero vector subspace of X_0. Since X_0 is proper, there is an element $x_1 \in X \setminus X_0$. Put $Y_1 = Q(T)x_1$, it is a non-zero vector subspace of X. We have

$$Y_0 \cap Y_1 = (0),$$

otherwise $\varphi_0(T)x_0 = \varphi_1(T)x_1$ for some non-zero $\varphi_0, \varphi_1 \in Q$, and consequently $x_1 = \varphi_1(T)^{-1}\varphi_0(T)x_0 \in Y_0 \subset X_0$, which is a contradiction. □

Observe now, that we can treat X not only as a complex vector space, but also as a vector space over the field Q, if we define there the scalar multiplication by the formula

$$\varphi x = \varphi(T)x, \quad (\varphi \in Q, x \in X).$$

With this interpretation, the elements x_0 and x_1 are Q-linearly independent, and the subspaces Y_0 and Y_1 are Q-one dimensional. The elements x_0 and x_1 can be imbedded into a Q-Hamel basis (x_α) for X treated as a Q-vector space. Consequently we obtain a Q-direct sum decomposition

$$X = Y_0 + Y_1 + Y_2, \tag{2.1}$$

where Y_2 is the Q-span of $(x_\alpha)\backslash\{x_0, x_1\}$ (it can happen that Y_2 is the zero-subspace of X). In particular we have

$$QY_i = Y_i \qquad (i = 0, 1, 2). \tag{2.2}$$

Denote by $\mathfrak{L}_Q(X)$ the set of all Q-linear operators on X, i.e. the set of all operators S in $\mathfrak{L}(X)$ satisfying

$$S\varphi x = \varphi S x, x \in X, \varphi \in Q.$$

Clearly $\mathfrak{L}_Q(X) \subset T'$ (in fact $\mathfrak{L}_Q(X) = T'$). By the decomposition (2.1) we can write every element x in X in the form

$$x = \varphi_0^{(x)} x_0 + \varphi_1^{(x)} x_1 + y_2, \varphi_0^{(x)}, \varphi_1^{(x)} \in Q, y_2 \in Y_2. \tag{2.3}$$

Define an operator S_0 in $\mathfrak{L}(X)$ setting for an x of the form (2.3)

$$S_0(x) = \varphi_0^{(x)} x_1 + \varphi_1^{(x)} x_0\,.$$

By the formulas (2.1), (2.2) and (2.3) it is a well defined operator in $\mathfrak{L}_Q(X) \subset T'$. Since the subspace X_0 is hyperinvariant for T and $x_0 \in X_0$, we have $S_0 x_0 \in X_0$. But $S_0(x_0) = x_1 \notin X_0$. The obtained contradiction shows that T has no proper hyperinvariant subspace and our conclusion follows.

Call a subset $U \subset \mathfrak{L}(X)$ transitive, if for all $x_0, x_1 \in X$, $x_0 \neq 0$, there is an operator T in U with $Tx_0 = x_1$.

The following corollary is in fact equivalent to the above result.

Corollary 2.2. *Let X be a complex vector space and $T \in \mathfrak{L}(X)$. Then the commutant T' is transitive if and only if either of the following conditions hold true.*

(i) The operator T is a scalar multiple of the identity operator I,

or

(ii) the spectrum $\sigma(T)$ is empty.

Finally, as another corollary, we obtain the following, already observed result ([6, Theorem 2.2]) concerning the locally convex space X of all sequences with the topology of coordinatewise convergence (denoted in Banach [1]) by (s) and in Köthe [2] by ω).

Corollary 2.3. *All non-constant continuous linear operators on the complex space of all sequences have proper closed hyperinvariant subspaces.*

For the reader's convenience we give details of the proof. We shall be using the following facts concerning the space (s). It is a completely metrizable locally convex space, whose topology can be given by means of the sequence of seminorms

$$||x||_n = \max\{|\xi_1(x)|, \dots, |\xi_n(x)|\}, n = 1, 2, \dots,$$

where $x = (\xi_i(n))_1^\infty$ is an element of (s).

The dual space $(s)'$, i.e. the linear space of all continuous linear functionals on (s), provided with the topology of uniform convergence on bounded subsets of (s), consists of all eventually zero sequences $f = (\eta_i(f))_1^\infty$ ($\eta_i(f) = 0$ for $i > i_0(f)$) and the topology of $(s)'$ turns out to be the maximal locally convex topology defined by the means of all seminorms on $(s)'$. Under this topology $(s)'$ is a complete locally convex space with all linear subspaces closed and with all linear functionals and all operators (endomorphisms) continuous. Note that $(s)'$ has a countable Hamel basis, e.g. consisting of all vectors $e_i = (\delta_{i,j})_{j=1}^\infty, i = 1, 2, \dots,$ where $\delta_{i,j}$ is the Kronecker symbol. An essential fact, used in the sequel, is that (s) is a reflexive space, i.e. every its continuous linear functional F is of the form

$$F(f) = f(x_F) \qquad (x_F \in (s))$$

and the topology of s'', identified by the above formula with (s), coincides with the original topology of (s). The duality between (s) and $(s)'$ is given by the bilinear form

$$(x, f) = \sum_i \xi_i(x)\eta_i(f).$$

For the proofs of above facts the reader is referred to [2, pp. 287–292], (see also [2, §18.5, p. 214]) and to [4, Example on p. 56].

We pass now to the proof of the Corollary 2.3. Denote by $L((s))$ and $L((s)')$ the algebras of all continuous operators (endomorphisms) respectively on the spaces (s) and $(s)'$. Let $T \in L(s)$. Its dual, denoted by T^* is an operator in $L((s)')$ is given by the formula

$$(T^*f)x = f(Tx), x \in (s), f \in (s)'.$$

Since (s) is reflexive, we have $T^{**} = T$. Thus all elements in $L((s)')$ are of the form T^* with $T \in L((s))$.

We shall first prove this Corollary with (s) replaced by $(s)'$ and then we pass, by duality, to the space (s). In order to show that every non constant operator T^* in $L((s)')$ has a proper closed hyperinvariant subspace it is sufficient to show that its spectrum is non-void and apply our Theorem, since all linear subspaces of $(s)'$ are closed. So suppose, towards contradiction, that for some T in $L(s)$ its dual T^* has a void spectrum, i.e. for all complex λ there are inverses $(T^* - \lambda I^*)^{-1}$, where I is the identity operator on (s). These inverses are automatically continuous since all linear operators on $(s)'$ are continuous. Fix any non-zero functional f_0 in $(s)'$. Since each Hamel basis in $(s)'$ is countable and the family

$$\{(T^* - \lambda I^*)^{-1} f_0 \in (s)' : \lambda \in \mathbf{C}\}$$

is uncountable, it is linearly dependent. Thus there are $\lambda_i, \alpha_i \in \mathbf{C}$, $1 \le i \le k$, $\alpha_i \neq 0$, such that

$$\Big(\sum_{i=1}^k \alpha_i (T^* - \lambda_i I^*)^{-1}\Big) f_0 = \sum_{i=1}^k \alpha_i (T^* - \lambda_i I^*)^{-1} f_0 = 0. \tag{2.4}$$

The invertibility of $T^* - \lambda I^*$ for all complex λ implies that for each non-zero complex polynomial p the operator $p(T^*)$ is invertible. Since the left-hand operator in (2.4) can be written as a quotient of two polynomials $p(T^*)/q(T^*)$ with $q \neq 0$, it is invertible, which implies $f_0 = 0$. This contradiction shows that every non-constant operator T^* in $L((s)')$ has a proper closed hyperinvariant subspace Y_{T^*}.

Let now T be a non constant operator in $L((s))$, so that T^* is non-constant too. Put

$$Y_T = \{x \in (s) : f(x) = 0 \quad \text{for all } f \text{ in } \quad Y_{T^*}\}, \tag{2.5}$$

where Y_{T^*} is a proper closed hyperinvariant subspace for T^*. We shall show that Y_T is a proper closed hyperinvariant subspace for T. It is closed, because whenever $y = \lim_i x_i, x_i \in Y_T$ we have $f(x_i) = 0$ for all f in Y_{T^*}, and so $f(y) = 0$ by the continuity of f. Consequently $f(y) = 0$ for all $f \in Y_{T^*}$ and y is in Y_T. Since $Y_{T^*} \neq (s)'$, there is a functional $f_0 \notin Y_{T^*}$. By the Hahn–Banach theorem for locally convex spaces, we can find a continuous linear functional on $(s)'$ represented by an element x_0 in (s) such that $f(x_0) = 0$ for all f in Y_{T^*} and $f_0(x_0) = 1$. Thus $x_0 \in Y_T$ and $x_0 \neq 0$. Consequently $Y_T \neq (0)$. Since $Y_{T^*} \neq (0)$, there is a functional x_0 on $(s)'$ with $f_0(x_0) = 1$ for some f_0 in Y_{T^*}. Thus $x_0 \notin Y_T$ and $Y_T \neq (s)$. Thus Y_T is a proper closed subspace of (s). It remains to be shown that it is hyperinvariant for T. Let $S \in L((s))$ with $ST = TS$. Since $(TS)^* = S^*T^*$, we have $S^*T^* = T^*S^*$, and so the subspace Y_{T^*} is invariant for S^*. Let now $f \in Y_{T^*}$ and $x \in Y_T$. Since Y_{T^*} is invariant for S^*, we have $S^*f \in Y_{T^*}$ and so $(S^*f)x = f(Sx) = 0$. The last equality holds for all f in Y_{T^*}, and so, by the formula (2.5), $Sx \in Y_T$. Consequently Y_T is invariant for S and since S was an arbitrary operator in the commutant of T, it is hyperinvariant for T. Since T was an arbitrary non-constant operator on (s) the conclusion follows.

The space (s) is the only known complete metric infinite dimensional locally convex space (a B_0 space in terminology of the Banach school), for which the Corollary 2.3 holds true, and even the only known B_0-space for each every its continuous endomorphism has a proper closed invariant subspace (by a result in [5] every such endomorphism has a proper closed invariant subspace which is either of dimension or codimension 1). In particular, no infinite dimensional Banach space is known for which every continuous endomorphism has a proper closed invariant subspace.

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