Banach J. Math. Anal. 3 (2009), no. 1, 36–43

Banach Journal of Mathematical Analysis
ISSN: 1735-8787 (electronic)
http://www.math-analysis.org

HYERS–ULAM–RASSIAS STABILITY FOR A CLASS OF NONLINEAR VOLTERRA INTEGRAL EQUATIONS

L.P. CASTRO[1]* AND A. RAMOS[2]

Communicated by S.-M. Jung

Abstract. The paper is devoted to the study of Hyers, Ulam and Rassias types of stability for a class of nonlinear Volterra integral equations. Both Hyers–Ulam–Rassias stability and Hyers–Ulam stability are obtained for such a class of Volterra integral equations when considered on a finite interval. In addition, for corresponding Volterra integral equations on infinite intervals the Hyers–Ulam–Rassias stability is also obtained.

1. Introduction

Volterra integral equations have been studied in a quite extensive way since the four fundamental papers of Vito Volterra in 1896, and specially since 1913 when Volterra's book *Leçons sur les Équations Intégrales et les Équations Intégro-différentielles* appeared. Part of this interest arises directly from the applications where this kind of equations appears. This is the case in elasticity, semiconductors, scattering theory, seismology, heat conduction, fluid flow, chemical reactions, population dynamics, etc. (see [1, 2, 3, 6]).

Equation stability is an important subject in the applications. In general terms, we may say that the main issue in the stability of functional equations is to answer the question of when the solutions of an equation, differing slightly from a given one, must be close to a solution of the given equation.

Date: Received: 23 June 2008; Revised: 4 July 2008; Accepted: 16 July 2008.

* Corresponding author.

2000 *Mathematics Subject Classification.* Primary 45D05; Secondary 26D10, 34K20, 47H10.

Key words and phrases. Hyers–Ulam–Rassias stability, Volterra integral equation, fixed point.

Despite the large amount of works on Volterra integral equations, up to our knowledge only the work [5] studies conditions which ensure *Hyers–Ulam–Rassias stability* and *Hyers–Ulam stability* of a certain type of Volterra integral equations (see §2 and [4, 5, 7, 8]).

In the present work we propose a Hyers–Ulam–Rassias stability study for the nonlinear Volterra integral equations of the form

$$y(x) = \int_a^x f(x, \tau, y(\tau))\, d\tau, \qquad -\infty < a \le x \le b < +\infty, \tag{1.1}$$

where a and b are fixed real numbers and f is a continuous function. We note that this class of functional equations is more global than the one considered in [5]. Anyway, we follow the fixed point arguments used in [5] and prove the Hyers–Ulam–Rassias stability and the Hyers–Ulam stability of the Volterra integral equation (1.1) for the case of compact domains. In addition, conditions for the Hyers–Ulam–Rassias stability of (1.1) in the case when x is not belonging to a finite interval are also here obtained. We note however that for such an infinite interval setting, the problem of finding conditions which ensure the corresponding equation (1.1) to have the Hyers–Ulam stability remains open.

2. Basic concepts

In this section we introduce the basic definitions and a Banach fixed point result which will be used throughout all the work.

Definition 2.1. If for each function y satisfying

$$\left| y(x) - \int_a^x f(x, \tau, y(\tau))\, d\tau \right| \le \psi(x),$$

where ψ is a non-negative function, there exists a solution y_0 of the Volterra integral equation (1.1) and a constant $C_1 > 0$ independent of y and y_0 such that

$$|y(x) - y_0(x)| \le C_1 \psi(x),$$

for all x, then we say that the integral equation (1.1) has the *Hyers–Ulam–Rassias stability*.

Definition 2.2. In the particular case of Definition 2.1 when ψ is just a constant function in the above inequalities, we say that the integral equation (1.1) has the *Hyers–Ulam stability*.

For a nonempty set X, let us recall the definition of generalized metric on X.

Definition 2.3. A function $d : X \times X \to [0, +\infty]$ is called a *generalized metric on* X if and only if d satisfies the following three propositions:

$$\begin{aligned} &\text{(P1)} \qquad d(x,y) = 0 \quad \text{if and only if} \quad x = y; \\ &\text{(P2)} \qquad d(x,y) = d(y,x) \quad \text{for all} \quad x, y \in X; \\ &\text{(P3)} \qquad d(x,z) \le d(x,y) + d(y,z) \quad \text{for all} \quad x, y, z \in X. \end{aligned}$$

In the just presented setting of generalized metrics, we recall here the *Banach Fixed Point Theorem* which will play an important role in proving our main theorems.

Theorem 2.4. *Let (X, d) be a generalized complete metric space and $T : X \to X$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following propositions hold true:*

(A) *the sequence $(T^n x)_{n\in\mathbb{N}}$ converges to a fixed point x^* of T;*
(B) *x^* is the unique fixed point of T in*
$$X^* = \{y \in X \mid d(T^k x, y) < \infty\};$$
(C) *if $y \in X^*$, then*
$$d(y, x^*) \le \frac{1}{1-L} d(Ty, y).$$

3. The Hyers–Ulam–Rassias stability of the Volterra integral equation

This section is totaly devoted to find out conditions under which the Volterra integral equation (1.1) admits the Hyers–Ulam–Rassias stability. This is assembled in the next theorem.

Theorem 3.1. *Let C and L be positive constants with $0 < CL < 1$ and assume that $f : [a, b] \times [a, b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function which additionally satisfies the Lipschitz condition*
$$|f(x, \tau, y) - f(x, \tau, z)| \le L|y - z| \tag{3.1}$$
for any $x, \tau \in [a, b]$ and all $y, z \in \mathbb{C}$.

If a continuous function $y : [a, b] \to \mathbb{C}$ satisfies
$$\left| y(x) - \int_a^x f(x, \tau, y(\tau))\, d\tau \right| \le \varphi(x) \tag{3.2}$$
for all $x \in [a, b]$, and where $\varphi : [a, b] \to (0, \infty)$ is a continuous function with
$$\left| \int_a^x \varphi(\tau)\, d\tau \right| \le C\varphi(x) \tag{3.3}$$
for each $x \in [a, b]$, then there exists a unique continuous function $y_0 : [a, b] \to \mathbb{C}$ such that
$$y_0(x) = \int_a^x f(x, \tau, y_0(\tau))\, d\tau \tag{3.4}$$
$$|y(x) - y_0(x)| \le \frac{1}{1 - CL}\varphi(x) \tag{3.5}$$
for all $x \in [a, b]$.

Proof. We will consider the space of continuous functions
$$X = \{g : [a, b] \to \mathbb{C} \mid g \text{ is continuous}\} \tag{3.6}$$
endowed with the generalized metric on X defined by
$$d(g, h) = \inf\{C \in [0, \infty] \mid |g(x) - h(x)| \le C\varphi(x)\,, \text{ for all } x \in [a, b]\}.$$
It is known that (X, d) is a complete generalized metric space (cf. [5]).

Let us now introduce the operator $T : X \to X$ which is defined by

$$(Tg)(x) = \int_a^x f(x, \tau, g(\tau))\, d\tau$$

for all $g \in X$ and $x \in [a, b]$. Thus, due to the fact that f is a continuous function, it follows that Tg is also continuous and this ensures that T is a well defined operator. Indeed,

$$\begin{aligned}
|(Tg)(x) - (Tg)(x_0)| &= \left| \int_a^x f(x, \tau, g(\tau))\, d\tau - \int_a^{x_0} f(x_0, \tau, g(\tau))\, d\tau \right| \\
&= \left| \int_a^x f(x, \tau, g(\tau)) - \int_a^x f(x_0, \tau, g(\tau))\, d\tau \right. \\
&\qquad \left. + \int_a^x f(x_0, \tau, g(\tau)) - \int_a^{x_0} f(x_0, \tau, g(\tau))\, d\tau \right| \\
&\leq \left| \int_a^x f(x, \tau, g(\tau)) - \int_a^x f(x_0, \tau, g(\tau))\, d\tau \right| \\
&\qquad + \left| \int_a^x f(x_0, \tau, g(\tau)) - \int_a^{x_0} f(x_0, \tau, g(\tau))\, d\tau \right| \\
&\leq \int_a^x |f(x, \tau, g(\tau)) - f(x_0, \tau, g(\tau))|\, d\tau \\
&\qquad + \left| \int_{x_0}^x f(x_0, \tau, g(\tau))\, d\tau \right| \quad \xrightarrow{x \to x_0} \quad 0.
\end{aligned}$$

We will now verify that T is strictly contractive on X. For any $g, h \in X$, let us consider $C_{gh} \in [0, \infty]$ such that

$$|g(x) - h(x)| \leq C_{gh} \varphi(x) \tag{3.7}$$

for any $x \in [a, b]$. Note that this is always possible due to the definition of (X, d). From the definition of T and (3.1), (3.3) and (3.7), it follows

$$\begin{aligned}
|(Tg)(x) - (Th)(x)| &= \left| \int_a^x [f(x, \tau, g(\tau)) - f(x, \tau, h(\tau))]\, d\tau \right| \\
&\leq \left| \int_a^x |f(x, \tau, g(\tau)) - f(x, \tau, h(\tau))|\, d\tau \right| \\
&\leq L \left| \int_a^x |g(\tau) - h(\tau)|\, d\tau \right| \\
&\leq L C_{gh} \left| \int_a^x \varphi(\tau)\, d\tau \right| \\
&\leq L C_{gh} C \varphi(x)
\end{aligned}$$

for all $x \in [a, b]$. Therefore, $d(Tg, Th) \leq L C_{gh} C$. This allows us to conclude that $d(Tg, Th) \leq LCd(g, h)$ for any $g, h \in X$, and since $CL \in (0, 1)$ the (strictly) contraction property is verified.

Let us take $g_0 \in X$. From the continuous property of g_0 and Tg_0 it follows that there exists a constant $C_1 \in (0, \infty)$ such that

$$\begin{aligned} |(Tg_0)(x) - g_0(x)| &= \left| \int_a^x f(x, \tau, g_0(\tau))\, d\tau - g_0(x) \right| \\ &\leq C_1 \varphi(x) \end{aligned}$$

for all $x \in [a, b]$. Note that this occurs also because f and g_0 are bounded on $[a, b]$ and φ is a positive function. Therefore, from the definition of the generalized metric d, it follows that

$$d(Tg_0, g_0) < \infty. \tag{3.8}$$

Consequently, we are in conditions to use the *Banach Fixed Point Theorem* and conclude that there exists a continuous function $y_0 : [a, b] \to \mathbb{C}$ such that

$$T^n g_0 \overset{n\to\infty}{\longrightarrow} y_0 \quad \text{in} \quad (X, d),$$

and $Ty_0 = y_0$.

For any g_0 with the property (3.8) it follows that X can be rewritten the following new form

$$X = \{g \in X \,|\, d(g_0, g) < \infty\}$$

(cf. [5]). Therefore, once again the *Banach Fixed Point Theorem* ensures that y_0 is the unique continuous function with the property (3.4).

Now, from (3.2) it follows that $d(y, Ty) \leq 1$, and so the *Banach Fixed Point Theorem* leads to

$$d(y, y_0) \leq \frac{1}{1 - CL}\, d(Ty, y) \leq \frac{1}{1 - CL}. \tag{3.9}$$

Thus, the last inequality together with the definition of the generalized metric d lead to inequality (3.5). □

4. Hyers–Ulam–Rassias stability of the Volterra integral equation in the infinite interval case

The present section is devoted to the analysis of the Hyers–Ulam–Rassias stability of the Volterra integral equation (1.1) but when considering infinite intervals. Such stability is here obtained for this case under the conditions of the next result. Here the main argument is based on a recurrence procedure due the already obtained result for the corresponding finite interval case.

Theorem 4.1. *Let C and L be positive constants with $0 < CL < 1$ and assume that $f : \mathbb{R} \times \mathbb{R} \times \mathbb{C} \to \mathbb{C}$ is a continuous function which additionally satisfies the Lipschitz condition (3.1), for any $x, \tau \in \mathbb{R}$ and all $y, z \in \mathbb{C}$.*

If a continuous function $y : \mathbb{R} \to \mathbb{C}$ satisfies (3.2), for all $x \in \mathbb{R}$ and for some $a \in \mathbb{R}$, where $\varphi : \mathbb{R} \to (0, \infty)$ is a continuous function satisfying (3.3), for each $x \in \mathbb{R}$, then there exists a unique continuous function $y_0 : \mathbb{R} \to \mathbb{C}$ which satisfies (3.4) and (3.5) for all $x \in \mathbb{R}$.

Proof. First we will prove that y_0 is a continuous function.

For any $n \in \mathbb{N}$, let us define $I_n = [a-n, a+n]$. According to Theorem 3.1, there exists a unique continuous function $y_{0,n} : I_n \to \mathbb{C}$ such that

$$y_{0,n}(x) = \int_a^x f(x, \tau, y_{0,n}(\tau))\, d\tau \tag{4.1}$$

$$|y(x) - y_{0,n}(x)| \leq \frac{1}{1-CL}\varphi(x) \tag{4.2}$$

for all $x \in I_n$. The uniqueness of $y_{0,n}$ implies that if $x \in I_n$ then

$$y_{0,n}(x) = y_{0,n+1}(x) = y_{0,n+2}(x) = \ldots \tag{4.3}$$

For any $x \in \mathbb{R}$, let us define $n(x) \in \mathbb{N}$ as

$$n(x) = \min\{n \in \mathbb{N} \mid x \in I_n\}.$$

We define also a function $y_0 : \mathbb{R} \to \mathbb{C}$ by

$$y_0(x) = y_{0,n(x)}(x),$$

and we can say that y_0 is continuous. Indeed, for any $x_1 \in \mathbb{R}$, let $n_1 = n(x_1)$. Then x_1 belongs to the interior of I_{n_1+1} and there exists an $\epsilon > 0$ such that $y_0(x) = y_{0,n_1+1}(x)$ for all $x \in (x_1-\epsilon, x_1+\epsilon)$. By Theorem 3.1, y_{0,n_1+1} is continuous at x_1, so it is y_0.

We will now prove that y_0 satisfies (3.4) and (3.5) for all $x \in \mathbb{R}$. We must choose $n(x)$ for an arbitrary $x \in \mathbb{R}$. Then $x \in I_{n(x)}$ and from (4.1) it follows that

$$y_0(x) = y_{0,n(x)}(x) = \int_a^x f(x, \tau, y_{0,n(x)}(\tau))\, d\tau = \int_a^x f(x, \tau, y_0(\tau))\, d\tau$$

where the last equality is true because $n(\tau) \leq n(x)$ for any $\tau \in I_{n(x)}$ and it follows from (4.3) that

$$y_0(\tau) = y_{0,n(\tau)}(\tau) = y_{0,n(x)}(\tau).$$

Moreover, (4.2) implies that, for all $x \in \mathbb{R}$,

$$|y(x) - y_0(x)| = \left|y(x) - y_{0,n(x)}(x)\right| \leq \frac{1}{1-CL}\varphi(x).$$

Finally, we will prove that y_0 is unique. Suppose that y_1 is another continuous function which satisfies (3.4) and (3.5), for all $x \in \mathbb{R}$. Since the restrictions $y_0|_{I_{n(x)}} = y_{0,n(x)}$ and $y_1|_{I_{n(x)}}$ both satisfy (3.4) and (3.5) for all $x \in I_{n(x)}$, the uniqueness of $y_0|_{I_{n(x)}} = y_{0,n(x)}$ implies that

$$y_0(x) = y_0|_{I_{n(x)}}(x) = y_1|_{I_{n(x)}}(x) = y_1(x).$$

□

5. The Hyers–Ulam stability of the Volterra integral equation

In this final section, by imposing some stronger assumptions, the Hyers–Ulam stability is obtained for the Volterra integral equation under study (in the finite interval case).

Theorem 5.1. *Let $K = b - a$ and consider L to be a positive constant such that $0 < KL < 1$. Assume that $f : [a, b] \times [a, b] \times \mathbb{C} \to \mathbb{C}$ is a continuous function which fulfils the Lipschitz condition*

$$|f(x, \tau, y) - f(x, \tau, z)| \leq L|y - z| \tag{5.1}$$

for any $x, \tau \in [a, b]$ and $y, z \in \mathbb{C}$.

If a continuous function $y : [a, b] \to \mathbb{C}$ satisfies

$$\left| y(x) - \int_a^x f(x, \tau, y(\tau))\, d\tau \right| \leq \theta$$

for each $x \in [a, b]$ and some $\theta \geq 0$, then there exists a unique continuous function $y_0 : [a, b] \to \mathbb{C}$ such that

$$y_0(x) = \int_a^x f(x, \tau, y_0(\tau))\, d\tau \tag{5.2}$$

$$|y(x) - y_0(x)| \leq \frac{\theta}{1 - KL}$$

for all $x \in [a, b]$.

Proof. Let us consider once more the space of continuous functions presented in (3.6) and endowed with the generalized metric defined by

$$d(g, h) = \inf\{C \in [0, \infty] \mid |g(x) - h(x)| \leq C\,, \text{ for all } x \in [a, b]\}.$$

Let us now introduce the operator $T : X \to X$ which is defined by

$$(Tg)(x) = \int_a^x f(x, \tau, g(\tau))\, d\tau$$

for all $g \in X$ and $x \in [a, b]$. Note that for any continuous function g, Tg is also continuous, as we have seen before.

Let us now verify that the operator T is strictly contractive on X. For any $g, h \in X$, let us consider $C_{gh} \in [0, \infty]$ such that

$$|g(x) - h(x)| \leq C_{gh} \tag{5.3}$$

for any $x \in [a, b]$. From the definition of T, (5.1) and (5.3), it follows

$$\begin{aligned}
|(Tg)(x) - (Th)(x)| &= \left| \int_a^x [f(x, \tau, g(\tau)) - f(x, \tau, h(\tau))]\, d\tau \right| \\
&\leq \left| \int_a^x |f(x, \tau, g(\tau)) - f(x, \tau, h(\tau))|\, d\tau \right| \\
&\leq L \left| \int_a^x |g(\tau) - h(\tau)|\, d\tau \right| \\
&\leq L C_{gh} K
\end{aligned}$$

for all $x \in [a, b]$. Therefore, $d(Tg, Th) \leq LC_{gh}K$. This allows us to conclude that $d(Tg, Th) \leq LKd(g, h)$ for any $g, h \in X$, and since $KL \in (0, 1)$ the (strict) contraction property is verified.

Similarly as in the proof of Theorem 3.1 we can choose $g_0 \in X$ with

$$d(Tg_0, g_0) < \infty. \tag{5.4}$$

Consequently, we are in conditions to use the *Banach Fixed Point Theorem* and conclude that there exists a continuous function $y_0 : [a, b] \to \mathbb{C}$ such that

$$T^n g_0 \overset{n\to\infty}{\longrightarrow} y_0 \quad \text{in} \quad (X, d),$$

and $Ty_0 = y_0$. For any g_0 with the property (5.4) it follows that X can be rewritten in the following new form

$$X = \{g \in X \mid d(g_0, g) < \infty\}$$

(cf. [5]). Therefore, once again the *Banach Fixed Point Theorem* ensures that y_0 is the unique continuous function with the property (5.2). Furthermore, the third proposition of Theorem 2.4 yields

$$|y(x) - y_0(x)| \leq \frac{\theta}{1 - KL},$$

for all $x \in [a, b]$. □

Acknowledgements. This work was supported in part by *Unidade de Investigação Matemática e Aplicações* of Universidade de Aveiro through the Portuguese Science Foundation (*FCT–Fundação para a Ciência e a Tecnologia*).

References

1. T.A. Burton, *Volterra Integral and Differential Equations* (2nd ed.), Mathematics in Science and Engineering 202, Elsevier, Amsterdam, 2005.
2. C. Corduneanu, *Principles of Differential and Integral Equations* (2nd ed.), Chelsea, New York, 1988.
3. G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra Integral and Functional Equations*, Cambridge University Press, Cambridge, 1990.
4. D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Progress in Nonlinear Differential Equations and their Applications 34, Birkhäuser, Boston, 1998.
5. S.-M. Jung, A fixed point approach to the stability of a Volterra integral equation, *Fixed Point Theory Appl.*, **2007**, Article ID 57064, 9 p.
6. V. Lakshmikantham and M.R.M. Rao, *Theory of Integro-differential Equations*, Stability and Control: Theory, Methods and Applications 1, Gordon and Breach Publ., Philadelphia, 1995.
7. Th.M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300.
8. S.M. Ulam, *A Collection of Mathematical Problems*, Interscience Tracts in Pure and Applied Mathematics 8, Interscience Publishers, New York, 1960.

[1,2] Research Unit "Mathematics and Applications", Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

E-mail address: castro@ua.pt and anabela@ua.pt