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GENERALIZATION OF SĂLĂGEAN OPERATOR FOR CERTAIN ANALYTIC FUNCTIONS

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ABSTRACT. For analytic functions f in the open unit disc $\mathbb{U}$, a generalization operator $D^\lambda f(z)$ of Sălăgean operator is introduced. Some properties for $D^\lambda f(z)$ are discussed in the present paper.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For $f \in \mathcal{A}$, Sălăgean[5] has defined the following operator $D^n f(z)$ by

(i) $D^0 f(z) = f(z)$

(ii) $D^1 f(z) = Df(z) = zf'(z) = z + \sum_{k=2}^{\infty} k a_k z^k$,

(iii) $D^n f(z) = D(D^{n-1} f(z)) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad (n = 1, 2, 3...)$.

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In view of the Sălăgean operator, we introduce

$$D^{\lambda} f(z) = z + \sum_{k=2}^{\infty} k^{\lambda} a_k z^k, \quad (\lambda \in \Re)$$

for $f \in \mathcal{A}$. Then for any real $\lambda \in \Re$ we see that

$$D^{\lambda+1} f(z) = z + \sum_{k=2}^{\infty} k^{\lambda+1} a_k z^k = z(D^{\lambda} f(z))'$$

and

$$D^{\lambda-1} f(z) = z + \sum_{k=2}^{\infty} k^{\lambda-1} a_k z^k = \int_0^z \frac{D^{\lambda} f(t)}{t} dt.$$

It is easy to see that

$$D^{\lambda_1+\lambda_2} f(z) = D^{\lambda_2}(D^{\lambda_1} f(z)) = D^{\lambda_1}(D^{\lambda_2} f(z))$$

for any real λ_1 and λ_2.

To discuss our new problem, we have to recall here the following lemma by Jack [1] (also by Miller and Mocanu [3]).

Lemma 1.1. *Let $w(z)$ be non-constant and analytic in $\mathbb{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r$ at the point $z_0 \in \mathbb{U}$, then we have $z_0 w(z_0)' = k w(z_0)$ where $k \geq 1$ is real.*

2. Properties of the operator $D^{\lambda} f(z)$

Our first result for the operator $D^{\lambda} f(z)$ is contained in the following theorem.

Theorem 2.1. *If $f \in \mathcal{A}$ satisfies*

$$\left| \frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} - 1 \right|^{\alpha} \left| z \left(\frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} \right)' \right|^{\beta} < \left(\frac{1}{2} \right)^{\beta} \quad (z \in \mathbb{U}) \tag{2.1}$$

for some real α, β with $\alpha + 2\beta \geq 0$ and for any real λ, then

$$\Re \left(\frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} \right) > 0 \quad (z \in \mathbb{U}).$$

Proof. Let us define $w(z)$ by

$$\frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} = \frac{1 + w(z)}{1 - w(z)} \quad (w(z) \neq 1).$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(z) = 0$.

Since

$$z \left(\frac{D^{\lambda+1} f(z)}{D^{\lambda} f(z)} \right)' = \frac{2 z w'(z)}{(1 - w(z))^2},$$

we obtain that

$$\left|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\right|^{\alpha}\left|z\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)'\right|^{\beta}=\left|\frac{2w(z)}{1-w(z)}\right|^{\alpha}\left|\frac{2zw(z)'}{(1-w(z))^2}\right|^{\beta}<\left(\frac{1}{2}\right)^{\beta}$$

for all $z\in\mathbb{U}$. If there exists a point $z_0\in\mathbb{U}$ such that $\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1$, then Lemma 1.1 gives us that $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=ke^{i\theta}$ $(k\ge1)$.
This implies that

$$\begin{aligned}\left|\frac{D^{\lambda+1}f(z_0)}{D^{\lambda}f(z_0)}-1\right|^{\alpha}\left|z_0\left(\frac{D^{\lambda+1}f(z_0)}{D^{\lambda}f(z_0)}\right)'\right|^{\beta} &= \left|\frac{2e^{i\theta}}{1-e^{i\theta}}\right|^{\alpha}\left|\frac{2ke^{i\theta}}{(1-e^{i\theta})^2}\right|^{\beta}\\ &= \frac{2^{\alpha+\beta}k^{\beta}}{|1-e^{i\theta}|^{\alpha+2\beta}}\ge\left(\frac{k}{2}\right)^{\beta}\ge\left(\frac{1}{2}\right)^{\beta}\end{aligned}$$

for all $z\in\mathbb{U}$, which contradicts the condition of the theorem. This show that there is no $z_0\in\mathbb{U}$ such that $|w(z_0)|=1$. Therefore $|w(z)|<1$ for all $z\in\mathbb{U}$ which implies that

$$\Re\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)>0\quad(z\in\mathbb{U}).$$

This completes the proof of the theorem.

Noting that if $f\in\mathcal{A}$ is starlike in $\mathbb{U}$ which is equivalent to

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>0\quad(z\in\mathbb{U}),$$

then

$$|a_k|\le k\quad(k=2,3,4,...)$$

and equality holds true for Koebe function f given by $f(z)=\dfrac{z}{(1-z)^2}$ which is the extremal function for the class of starlike functions in $\mathbb{U}$.
Thus we have

Corollary 2.2. *If $f\in\mathcal{A}$ satisfies the inequality* (2.1) *for some real α, β with $\alpha+2\beta\ge0$ and for any real λ, then*

$$|a_k|\le k^{1-\lambda}\quad(k=2,3,4,...).$$

Equality holds true for Koebe function.

By the Marx-Strohhäcker theorem ([2], [6]), we know that if $f\in\mathcal{A}$ satisfies

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0\quad(z\in\mathbb{U}),$$

then

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\frac{1}{2}\quad(z\in\mathbb{U}).$$

If we define the function $F(z)$ by $F(z)=D^{\lambda-1}f(z)$, then $zF'(z)=D^{\lambda}f(z)$ and $zF'(z)+z^2F''(z)=D^{\lambda+1}f(z)$. Therefore, we have

Corollary 2.3. *If $f \in \mathcal{A}$ satisfies the inequality (2.1) for some real α, β with $\alpha + 2\beta \geq 0$ and for any real λ, then*

$$\Re\left(\frac{D^{\lambda}f(z)}{D^{\lambda-1}f(z)}\right) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

The result is sharp for the function f given by

$$f(z) = z + \sum_{k=2}^{\infty} k^{1-\lambda} z^k$$

which is equivalent to

$$D^{\lambda}f(z) = \frac{z}{(1-z)^2}.$$

Next we prove the following theorem.

Theorem 2.4. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} - 1\right|^{\alpha} \left|z\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)'\right|^{\beta} < \left(\frac{1}{2}\right)^{\beta}(1-\gamma)^{\alpha+\beta} \quad (z \in \mathbb{U}) \tag{2.2}$$

for some real α, β, γ with $\alpha + 2\beta \geq 0$ and $0 \leq \gamma < 1$, then

$$\Re\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right) > \gamma \quad (z \in \mathbb{U}).$$

Proof. Defining the function $w(z)$ by

$$\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} = \frac{1+(1-2\gamma)w(z)}{1-w(z)} \quad (w(z) \neq 1),$$

we see that $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Note that

$$z\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)' = \frac{2(1-\gamma)zw'(z)}{(1-w(z))^2}.$$

Thus we have that

$$\begin{aligned} \left|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)} - 1\right|^{\alpha} \left|z\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)'\right|^{\beta} &= \left|\frac{2(1-\gamma)w(z)}{1-w(z)}\right|^{\alpha} \left|\frac{2(1-\gamma)zw(z)'}{(1-w(z))^2}\right|^{\beta} \\ &< \left(\frac{1}{2}\right)^{\beta}(1-\gamma)^{\alpha+\beta} \quad (z \in \mathbb{U}). \end{aligned}$$

If there exists a point $z_0 \in \mathbb{U}$ such that $\max_{|z| \leq |z_0|}|w(z)| = |w(z_0)| = 1$, then $w(z)$ satisfies $w(z_0) = e^{i\theta}$ and $z_0 w'(z_0) = ke^{i\theta}$ $(k \geq 1)$ by Lemma 1.1.

This gives us that

$$\begin{aligned}\left|\frac{D^{\lambda+1}f(z_0)}{D^{\lambda}f(z_0)}-1\right|^{\alpha}\left|z_0\left(\frac{D^{\lambda+1}f(z_0)}{D^{\lambda}f(z_0)}\right)'\right|^{\beta} &= \left|\frac{2(1-\gamma)e^{i\theta}}{1-e^{i\theta}}\right|^{\alpha}\left|\frac{2(1-\gamma)ke^{i\theta}}{(1-e^{i\theta})^2}\right|^{\beta} \\ &= \frac{2^{\alpha+\beta}k^{\beta}(1-\gamma)^{\alpha+\beta}}{|1-e^{i\theta}|^{\alpha+2\beta}} \\ &\geq \left(\frac{k}{2}\right)^{\beta}(1-\gamma)^{\alpha+\beta} \\ &\geq \left(\frac{1}{2}\right)^{\beta}(1-\gamma)^{\alpha+\beta} \quad (z\in\mathbb{U})\end{aligned}$$

which contradicts the condition of the theorem. This show that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. Therefore $|w(z)| < 1$ for all $z \in \mathbb{U}$.

Thus we conclude that

$$\Re\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right) > \gamma \quad (z \in \mathbb{U}).$$

Noting that if $f \in \mathcal{A}$ satisfies

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \gamma \quad (z \in \mathbb{U}),$$

then

$$|a_k| \leq \frac{\prod_{j=2}^{k}(j-2\gamma)}{(k-1)!} \quad (k = 2, 3, 4, ...)$$

and equality holds true for the functions f given by

$$f(z) = \frac{z}{(1-z)^{2(1-\gamma)}}$$

which is the extremal function for the class of starlike of order γ in $\mathbb{U}$ (cf. Robertson[4]).

In view of the above, we give direct corollary as follows:

Corollary 2.5. *If $f \in \mathcal{A}$ satisfies the inequality (2.2) for some real α, β, γ with $\alpha + 2\beta \geq 0$ and $0 \leq \gamma < 1$, then*

$$|a_k| \leq \frac{\prod_{j=2}^{k}(j-2\gamma)}{k^{\lambda}(k-1)!} \quad (k = 2, 3, 4, ...).$$

Equality holds true for the function f given by

$$f(z) = z + \sum_{k=2}^{\infty}\frac{\prod_{j=2}^{k}(j-2\gamma)}{k^{\lambda}(k-1)!}z^k$$

which is equivalent to

$$D^{\lambda}f(z) = \frac{z}{(1-z)^{2(1-\gamma)}}.$$

Finally, we derive the following:

Theorem 2.6. *If $f \in \mathcal{A}$ satisfies*

$$\left|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\right|^{\alpha}\left|z\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)'\right|^{\beta}<\left(\frac{\gamma}{2}\right)^{\beta}\quad(z\in\mathbb{U})$$

for some real α, β, and $\gamma=\dfrac{\beta}{\alpha+\beta}$, then

$$\Re\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)^{\frac{1}{\gamma}}>0\quad(z\in\mathbb{U}).$$

Proof. Defining the function $w(z)$ by

$$\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}=\left(\frac{1+w(z)}{1-w(z)}\right)^{\gamma}\quad(w(z)\neq 1)$$

with $\gamma=\dfrac{\beta}{\alpha+\beta}$, we see that $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. Noting that

$$z\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)'=\frac{2\gamma zw'(z)}{(1-w(z))^{2}}\left(\frac{1+w(z)}{1-w(z)}\right)^{\gamma-1},$$

we have that

$$\begin{aligned}\left|\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}-1\right|^{\alpha}\left|z\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)'\right|^{\beta} &= \left|\frac{1+w(z)}{1-w(z)}\right|^{\alpha\beta+\beta(\gamma-1)}\left|\frac{2\gamma zw(z)'}{(1-w(z))^{2}}\right|^{\beta}\\ &= \left|\frac{2\gamma zw(z)'}{(1-w(z))^{2}}\right|^{\beta}\\ &< \left(\frac{\gamma}{2}\right)^{\beta}\quad(z\in\mathbb{U})\end{aligned}$$

since $\gamma=\dfrac{\beta}{\alpha+\beta}$. Now, suppose that there exists a point $z_0\in\mathbb{U}$ such that $\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1$. Then, by Lemma 1.1, we have that $w(z_0)=e^{i\theta}$ and $z_0w'(z_0)=ke^{i\theta}$ $(k\geq 1)$.

This gives us that

$$\begin{aligned}\left|\frac{D^{\lambda+1}f(z_0)}{D^{\lambda}f(z_0)}-1\right|^{\alpha}\left|z_0\left(\frac{D^{\lambda+1}f(z_0)}{D^{\lambda}f(z_0)}\right)'\right|^{\beta} &= \left|\frac{2\gamma k e^{i\theta}}{(1-e^{i\theta})^2}\right|^{\beta}\\ &= \frac{2^{\beta}k^{\beta}\gamma^{\beta}}{|(1-e^{i\theta})^2|^{\beta}}\\ &\geq \left(\frac{k\gamma}{2}\right)^{\beta}\\ &\geq \left(\frac{\gamma}{2}\right)^{\beta} \quad (z\in\mathbb{U})\end{aligned}$$

which contradicts the condition of the theorem. This show that there is no $z_0 \in \mathbb{U}$ such that $|w(z_0)| = 1$. Therefore, we conclude that $|w(z)| < 1$ for all $z \in \mathbb{U}$, that is, that

$$\Re\left(\frac{D^{\lambda+1}f(z)}{D^{\lambda}f(z)}\right)^{\frac{1}{\gamma}} > 0 \quad (z\in\mathbb{U}).$$

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