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D-SYMMETRIC OPERATORS: COMMENTS AND SOME OPEN PROBLEMS

SALAH MECHERI

Submitted by M. S. Moslehian

ABSTRACT. In this paper, we show that the class of D-symmetric operators is norm dense in $\mathcal{L}(H)$. It is known that the direct sum of two D- symmetric operators are not D-symmetric in general. Here we will show that the direct sum of two D-symmetric operators is D-symmetric if their spectrums do not meet each other. As a consequence, we show that the set $\{T+K : T\,\text{is}\,D-\text{symmetric}\,\text{and}\,K\,\text{is compact}\}$ is norm dense. Some open problems are also presented.

1. INTRODUCTION

Let H be a separable infinite dimensional complex Hilbert space, and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H. Let $A, B \in \mathcal{L}(H)$. We define the generalized derivation $\delta_{A,B} : \mathcal{L}(H) \mapsto \mathcal{L}(H)$ by

$$\delta_{A,B}(X) = AX - XB.$$

If $A = B$, then $\delta_{A,A}(X) = \delta_A(X) = AX - XA$ is called the inner derivation implemented by $A \in \mathcal{L}(H)$. These concrete operators on $\mathcal{L}(H)$ occur in many settings in mathematical analysis and applications, and their properties have been studied already during many decades. In [2] J.H. Anderson *et.al* showed that if A is D-symmetric, i.e., $\overline{R(\delta_A)} = \overline{R(\delta_{A^*})}$, where $\overline{R(\delta_A)}$ denotes the norm closure of the range of δ_A, then for $T \in C_1(H)$, $AT = TA$ implies $A^*T = TA^*$. In this paper, we show that the class of D-symmetric operators is norm dense in $\mathcal{L}(H)$. It is known that the direct sum of two D-symmetric operators are not D-symmetric in

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general [8]. Here we will show that the direct sum of two D-symmetric operators is D-symmetric if their spectrum do not meet each other. As a consequence, we show that the set $\{T+K : T\,\text{is}\,D-\text{symmetric and}\,K\,\text{is compact}\}$ is norm dense. Some open problems are also presented.

2. D-symmetric operators

We begin this section by the following definitions and notations.

Definition 2.1. An operator $A \in \mathcal{L}(H)$ is called D-symmetric, if

$$\overline{R(\delta_A)} = \overline{R(\delta_{A^*})}.$$

The Trace class operators, denoted by $C_1(H)$, is the set of all compact operators $A \in \mathcal{L}(H)$, for which the eigenvalues of $(TT^*)^{\frac{1}{2}}$, counted according to their multiplicity, are summable. The ideal $C_1(H)$ of $\mathcal{L}(H)$ admits a trace function $tr(T)$, given by $tr(T) = \sum_n (Te_n, e_n)$ for any complete orthonormal system (e_n) in H. As a Banach space, $C_1(H)$ can be identified with the dual of the ideal K of compact operators by means of the linear isometry $T \mapsto f_T$, where $f_T = tr(XT)$. Moreover, $\mathcal{L}(H)$ is the dual of $C_1(H)$, the ultra weakly continuous linear functionals on $\mathcal{L}(H)$ which are of the form f_T for $T \in C_1(H)$ and the weakly continuous linear functionals which are of the form f_T with T is of finite rank.

Definition 2.2. An operator $A \in \mathcal{L}(H)$ is called p-symmetric, if $AT = TA,\ T \in C_1(H)$ implies $A^*T = TA^*$.

Theorem 2.3. [2] *If $A \in \mathcal{L}(H)$, then the following two statements are equivalent*
i) A is D-symmetric
ii) (a) $[A]$, its corresponding element of the Calkin algebra, is D-symmetric and (b) $T \in C_1(H)$, $AT = TA$ implies $A^*T = TA^*$.

In [1], Anderson has shown that the direct sum of two D-symmetric operators is not D-symmetric in general. It is interesting to ask the following question:

Question. Under what conditions the direct sum of two D-symmetric operators is a D-symmetric operator?

In the following theorem we will provide a suitable condition.

Theorem 2.4. *Let A and B be two D-symmetric operators such that $\sigma(A) \cap \sigma(B) = \phi$, where $\sigma(A)$ and $\sigma(B)$ are the spectrums of A and B, respectively. Then $A \oplus B$ is also D-symmetric.*

Proof. Since A and B are two D-symmetric operators, by using Theorem 2.3 we obtain that A and B are p-symmetric operators. Let

$$S = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Since A and B are both p-symmetric,

$$AT = TA\,\text{implies}\,AT^* = T^*A$$

and

$$BT = TB\,\text{implies}\,BT^* = T^*B,$$

$\forall T \in C_1(H)$. Let
$$C = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$$
be a trace class operator. If $SC = CS$, then
$$\begin{pmatrix} AT_1 - TA_1 & AT_2 - T_2B \\ BT_3 - T_3A & BT_4 - T_4B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$
Since $\sigma(A) \cap \sigma(B) = \phi$, by using [9] the two equations $AT_2 - T_2B = 0$ and $BT_3 - T_3A = 0$ imply that $T_2 = T_3 = 0$. Hence $T_1^*A = AT_1^*$ and $T_4^*B = BT_4^*$. Note that when
$$C = \begin{pmatrix} T_1 & 0 \\ 0 & T_4 \end{pmatrix},$$
then
$$C^* = \begin{pmatrix} T_1^* & 0 \\ 0 & T_4^* \end{pmatrix}.$$
So, we have obtained that $C^*S = SC^*$. Hence $A \oplus B$ is p-symmetric. Note that (a) of Theorem 2.3 is also satisfied for $A \oplus B$. □

In the following example we will give two D-symmetric operators A and B with $\sigma(A) \cap \sigma(B) = \phi$.

Example 2.5. Let $H = E \oplus F$, where E is a finite dimensional Hilbert space and F is a complex separable Hilbert space. Define $T \in \mathcal{L}(H)$ by $T = A \oplus B$, where $||A|| < 1$, $\ker A \neq \{0\}$ and B is the unilateral shift on F, i.e., $Be_n = e_{n+1}$, $n \geq 1$. It is known that A and B are D-symmetric [5, 8], and $\sigma(A) \cap \sigma(B) = \phi$ [4, p.53]. Thus $A \oplus B$ is D-symmetric.

Now we will recall the following question posed by Joel Anderson *et.al* [2].

Is the set $\{T + K : T, D\text{-symmetric}, K \text{ compact } \}$ norm-closed in $\mathcal{L}(H)$?

In the following theorem we will prove that this set is norm dense in $\mathcal{L}(H)$. For this, we need the following definition.

Definition 2.6. We shall say that a certain property (P) of operators acting on a Hilbert space H is a bad property, if

(i) Whenever A satisfies (P), then for $\alpha \in \mathbb{C}$, with $\alpha \neq 0$ and $\beta \in \mathbb{C}$, the operator $\alpha A + \beta$ also satisfies (P);

(ii) If B is similar to A, and A satisfies (P), then B also satisfies (P);

(iii) If A satisfies (P), and if $B \in \mathcal{L}(\mathcal{H})$, such that $\sigma(A) \cap \sigma(B) = \phi$, then $A \oplus B$ satisfies (P).

It is known [4, Theorem 3.5.1] that a set satisfying a bad property is norm-dense in $\mathcal{L}(\mathcal{H})$.

Theorem 2.7. *The set*
$$S = \{T + K : T \; is \, D - symmetric \; and \, K \; is \; compact\},$$
is norm-dense .

Proof. It suffices to show that S is a bad property.

1) Assume that $A = T + K \in S$. We have to prove that $\alpha A + \beta I = (\alpha T + \beta I) + \alpha K \in S$. Since $R(\delta_{\alpha T+\beta I}) = R(\delta_T)$ for $\alpha \in \mathbb{C}$, $\alpha \neq 0$ and $\beta \in \mathbb{C}$, the operator $\alpha T + \beta I$ is D-symmetric. Thus $\alpha A + \beta I \in S$. This proves the first condition .

2) Suppose that $A = T + K \in S$ and B is similar to A, i.e, there exists an invertible operator U such that $B = U^{-1}AU$. We claim that $B \in S$. Indeed, note that $U^{-1}AU = U^{-1}TU + U^{-1}KU$. Moreover note that if $W \in \overline{R(\delta_{U^{-1}TU})}$, then there is a sequence $(y_n)_{n\in\mathbb{N}}$ such that

$$(U^{-1}TU)y_n - y_n U^{-1}TU \to W$$

Observe that $\overline{U^{-1}R(\delta_T)U} = \overline{R(\delta_{U^{-1}TU})}$. Since T is D-symmetric, we have $\overline{R(\delta_T)} = \overline{R(\delta_{T^*})}$, thus

$$\overline{R(\delta_{U^{-1}T^*U})} = \overline{U^{-1}R(\delta_{T^*})U} = \overline{U^{-1}R(\delta_T)U} = \overline{R(\delta_{U^{-1}TU})}$$

Thus $U^{-1}TU$ is D-symmetric, and the second condition holds.

3) Let A and B be any two D-symmetric operators such that $\sigma(A) \cap \sigma(B) = \emptyset$. Let $\mathcal{A}$ be Calkin algebra. By using Theorem 2.3, the corresponding elements of the Calkin algebra $\mathcal{A}$, $[A]$ and $[B]$ are also D-symmetric. Thus by using Theorem 2.4, we conclude that $[A] \oplus [B]$, is also D-symmetric. Hence the third condition holds, and S is norm-dense. □

3. Derivation ranges

In one of his paper J.H. Anderson [1] has proved the remarkable result that there exists $A \in \mathcal{L}(H)$ such that the identity operator $I_H \in \overline{R(\delta_A)}$. The classical Brown-Pearcy characterization of the commutators $AX - XA$ on $\mathcal{L}(H)$ as the operators which are not of the form $\lambda I + K$, for $\lambda \neq 0$ and K a compact operator, is a natural motivation for Anderson's result.

Here is a problem that might of interest. Recall from [6] that if $T : X \to Y$, where X and Y are Banach spaces, then

$$\overline{T(X)} = \{\lim_n Tx_n : \sup_n \|x_n\| < \infty\}.$$

It is not the usual closure since its points have to be the limits of images of bounded sequences of vectors. So

Question. For which operators T on Hilbert space H do we have

$$I \in \overline{R(\delta_T)}?$$

Let $\mathcal{N}$ be the set

$$\left\{A \in \mathcal{L}(H) : \ I \notin \overline{R(\delta_A)}\right\}.$$

Recall that H.Yang in [10], shows that the set $\mathcal{N}$ is norm-dense in $\mathcal{L}(H)$. Let

$$\mathcal{M}_w = \{A \in \mathcal{L}(H) : \ I \notin \overline{R(\delta_A)}^w, \forall K \in K(H)\}.$$

Since $\mathcal{M}_w \subset \mathcal{N}$ [3, Remark 3.4], S.N. Elalami [3, Theorem 3.4] generalized Yang's results by proving that the set $\mathcal{M}_w$ is norm dense in $\mathcal{L}(H)$. Here we will show that the set

$$\mathcal{J}_w = \left\{ A \in \mathcal{L}(H) : I + K \notin \overline{R(\delta_A)}^w, \forall K \in K(H) \right\}$$

is also norm-dense in $\mathcal{L}(H)$.

Theorem 3.1. *The set*

$$\mathcal{J}_w = \left\{ A \in \mathcal{L}(H) \ : I + K \notin \overline{R(\delta_A)}^w, \forall\, K \in K(H) \right\}$$

is norm-dense in $\mathcal{L}(H)$.

Proof. By using [4, Theorem 3.5.1], it suffices to prove that the property $A \in \mathcal{J}_w$ is a bad property. It is easy to see that $R(\delta_A) = R(\delta_{\alpha A+\beta})$, for $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, $\alpha \neq 0$, and $X \in \mathcal{L}(H)$. Hence if $A \in \mathcal{J}_w$, then

$$\alpha A + \beta \in \mathcal{J}_w.$$

Now if $S \in \mathcal{L}(H)$ and S is invertible, then for all $X \in \mathcal{L}(H)$,

$$S(AX - XA)S^{-1} = (SAS^{-1})(SXS^{-1}) - (SXS^{-1})(SAS^{-1}).$$

Thus

$$S\overline{R(\delta_A)}^w S^{-1} = \overline{R(\delta_{SAS^{-1}})}^w.$$

Hence if $I + K \in \overline{R(\delta_A)}^w$, then

$$I + SKS^{-1} \in \overline{R(\delta_{SAS^{-1}})}^w.$$

It follows by the above argument that if $\overline{R(\delta_A)}^w$ contains $I + K$, then it also true for all operators similar to A. Hence $A \in \mathcal{J}_w$ is invariant by similarity.

Let $E = H \oplus H$ and $B = A \oplus C$, suppose that there exists a generalized sequence $\{X_\alpha\} \subset B(E)$ such that

$$(A \oplus C)X_\alpha - X_\alpha(A \oplus C) \xrightarrow{w} I_E \oplus K.$$

Let P_0 be the orthogonal projection on H, K_1 denotes the compression of K to H, i.e, $K_1 = P_0 K P_0 \mid H$ and X_{α_1} denotes the compression of X_α to H. Then

$$AX_{\alpha_1} - X_{\alpha_1}A \xrightarrow{w} I_H \oplus K_1.$$

So if $A \oplus C \notin \mathcal{J}_w$, then $A \notin \mathcal{J}_w$. □

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Department of Mathematics, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia.

E-mail address: mecherisalah@hotmail.com