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# WIDTH-INTEGRALS AND AFFINE SURFACE AREA OF CONVEX BODIES

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Abstract. The main purposes of this paper are to establish some new Brunn–Minkowski inequalities for width-integrals of mixed projection bodies and affine surface area of mixed bodies, together with their inverse forms.

## 1. Introduction

In recent years some authors including Ball [1], Bourgain [2], Gardner [3], Schneider [4] and Lutwak [5]–[10] *et al* have given considerable attention to Brunn–Minkowski theory, Brunn–Minkowski-Firey theory, and their various generalizations. In particular, Lutwak [7] generalized the Brunn–Minkowski inequality (1.1) to mixed projection bodies and obtain inequality (1.2):

**The Brunn–Minkowski inequality** *If $K, L \in \mathcal{K}^n$, then*

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},$$

*with equality if and only if $K$ and $L$ are homothetic.*

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**The Brunn–Minkowski inequality for mixed projection bodies** *If* $K, L \in \mathcal{K}^n$, *then*

$$V(\Pi(K+L))^{1/n(n-1)} \geq V(\Pi K)^{1/n(n-1)} + V(\Pi L)^{1/n(n-1)},$$

*with equality if and only if* $K$ *and* $L$ *are homothetic.*

On the other hand, width-integrals of convex bodies and affine surface areas play an important role in Brunn–Minkowski theory. Width-integrals were first considered by Blaschke [11] and later by Hadwiger [12]. In addition, Lutwak established the following results for the width-integrals of convex bodies and affine surface areas.

**The Brunn–Minkowski inequality for width-integrals of convex bodies** [10]

*If* $K, L \in \mathcal{K}^n$, $i < n-1$,

$$B_i(K+L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)}, \tag{1.1}$$

*with equality if and only if* $K$ *and* $L$ *have similar width.*

**The Brunn–Minkowski inequality for affine surface area** [9]

*If* $K, L \in \mathcal{K}^n$, *and* $i \in \mathbb{R}$, *then for* $i < -1$,

$$\Omega_i(K \tilde{+} L)^{(n+1)/(n-i)} \leq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)}, \tag{1.2}$$

*with equality if and only if* $K$ *and* $L$ *are homothetic, while for* $i > -1$,

$$\Omega_i(K \tilde{+} L)^{(n+1)/(n-i)} \geq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)}, \tag{1.3}$$

*with equality if and only if* $K$ *and* $L$ *are homothetic.*

The purpose of this paper is two-fold. First, to generalize inequality (1.1) to the context of to mixed projection bodies and also establish its inverse version. Second, to obtain analogs of inequalities (1.2) and (1.3) for affine surface area of mixed bodies.

## 2. Notations and Preliminary works

The setting for this paper is the $n$-dimensional Euclidean space $\mathbb{R}^n (n > 2)$. Let $\mathfrak{C}^n$ denote the set of non-empty convex figures(compact, convex subsets) and $\mathcal{K}^n$ denote the subset of $\mathfrak{C}^n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. For $p \in \mathcal{K}^n$, let $\mathcal{K}^n_p$ denote the subset of $\mathcal{K}^n$ that contains the centered (centrally symmetric with respect to $p$) bodies. We reserve the letter $u$ for unit vectors, and the letter $B$ is reserved for the unit ball centered at the origin. The boundary surface of $B$ is $S^{n-1}$. For $u \in S^{n-1}$, let $E_u$ denote the hyperplane, through the origin, that is orthogonal to $u$. We will use $K^u$ to denote the image of $K$ under an orthogonal projection onto the hyperplane $E_u$.

2.1. **Mixed volumes.** Let $K \in \mathcal{K}^n$. We denote by $V(K)$ the $n$-dimensional volume of $K$. Let $h(K, \cdot) : S^{n-1} \to \mathbb{R}$ denote the support function of $K$, i.e.,

$$h(K, u) := Max\{u \cdot x : x \in K\}, \quad u \in S^{n-1},$$

where $u \cdot x$ denotes the usual inner product of $u$ and $x$ in $\mathbb{R}^n$.

Let $\delta$ denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$\delta(K, L) := |h_K - h_L|_\infty,$$

where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

For any nonnegative scalar $\lambda$, we write $\lambda K := \{\lambda x : x \in K\}$. For $K_i \in \mathcal{K}^n, \lambda_i \geq 0, (i = 1, 2, \ldots, r)$, the Minkowski linear combination $\sum_{i=1}^r \lambda_i K_i \in \mathcal{K}^n$ is defined by

$$\lambda_1 K_1 + \cdots + \lambda_r K_r := \{\lambda_1 x_1 + \cdots + \lambda_r x_r \in K^n : x_i \in K_i\}.$$

It is trivial to verify that

$$h(\lambda_1 K_1 + \cdots + \lambda_r K_r, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_r h(K_r, \cdot). \tag{2.1}$$

If $K_i \in \mathcal{K}^n$ and $\lambda_i (i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in $\lambda_i$ given by [4]

$$V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V_{i_1 \ldots i_n}, \tag{2.2}$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (2.2). It is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is also written as $V(K_{i_1}, \ldots, K_{i_n})$. If $K_{i_1} = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1 \ldots K_n)$ is usually written as $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the $i$th projection measure(Quermassintegral) of $K$ and is written as $W_i(K)$. With this notation, $W_0 = V(K)$, while $nW_1(K)$ is the surface area $S(K)$ of $K$.

2.2. **Width-integrals of convex bodies.** For $u \in S^{n-1}$, $b(K, u)$ is defined to be half the width of $K$ in the direction $u$. Two convex bodies $K$ and $L$ are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$ for all $u \in S^{n-1}$. For $K \in \mathcal{K}^n$ and $p \in intK$, we use $K^p$ to denote the polar reciprocal of $K$ with respect to the unit sphere centered at $p$. The width-integral of index $i$ is defined by Lutwak [10]: For $K \in \mathcal{K}^n$ and $i \in \mathbb{R}$,

$$B_i(K) := \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u), \tag{2.3}$$

where $dS$ is the $(n-1)$-dimensional volume element on $S^{n-1}$.

The width-integral of index $i$ is hence a map

$$B_i : \mathcal{K}^n \to \mathbb{R}$$

which is positive, continuous, homogeneous of degree $n - i$ and invariant under rigid motions. In addition, for $i \leq n$, it is also bounded and monotone under set inclusion.

The following results (cf. [10]) will be used later:

$$b(K + L, u) = b(K, u) + b(L, u), \tag{2.4}$$

$$B_{2n}(K) \leq V(K^p), \tag{2.5}$$

with equality if and only if $K$ is symmetric with respect to $p$.

2.3. **The radial function and the Blaschke linear combination.** The radial function of the convex body $K$ is the function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$ defined for $u \in S^{n-1}$ by

$$\rho(K,\cdot) := Max\{\lambda \geq 0 : \lambda\mu \in K\}.$$

If $\rho(K,\cdot)$ is positive and continuous, $K$ will be called a star body. Let $\varphi^n$ denote the set of star bodies in $\mathbb{R}^n$.

A convex body $K$ is said to have a positive continuous curvature function [5]

$$f(K,\cdot) : S^{n-1} \to [0,\infty),$$

if for each $L \in \varphi^n$, the mixed volume $V_1(K,L)$ has the integral representation

$$V_1(K,L) = \frac{1}{n}\int_{S^{n-1}} f(K,u)h(L,u)dS(u).$$

The subset of $\mathcal{K}^n$ consisting of convex bodies which have a positive continuous curvature function will be denoted by $\kappa^n$. Let $\kappa_c^n$ denote the set of centrally symmetric members of $\kappa^n$.

For $K \in \kappa^n$, it is shown in [6] that

$$\int_{S^{n-1}} uf(K,u)dS(u) = 0.$$

Suppose $K, L \in \kappa^n$ and $\lambda, \mu \geq 0$ be not both zero. From above it follows that the function $\lambda f(K,\cdot) + \mu f(L,\cdot)$ satisfies the hypothesis of Minkowski's existence theorem (see [13]). The solution of the Minkowski problem for this function is denoted by $\lambda \cdot K \tilde{+} \mu \cdot L$, that is,

$$f(\lambda \cdot K \tilde{+} \mu \cdot L, \cdot) = \lambda f(K,\cdot) + \mu f(L,\cdot), \tag{2.6}$$

where the linear combination $\lambda \cdot K \tilde{+} \mu \cdot L$ is called a Blaschke linear combination.

The relationship between Blaschke and Minkowski scalar multiplication is given by

$$\lambda \cdot K = \lambda^{1/(n-1)}K.$$

2.4. **Mixed affine area and mixed bodies.** The affine surface area $\Omega(K)$ of $K \in \kappa^n$ is defined by

$$\Omega(K) := \int_{S^{n-1}} f(K,u)^{n/(n+1)}dS(u).$$

It is well known that this functional is invariant under unimodular affine transformations. For $K, L \in \kappa^n$ and $i \in \mathbb{R}$, the $i$th mixed affine surface area $\Omega_i(K,L)$ of $K$ and $L$ was defined in [5] by

$$\Omega_i(K,L) := \int_{S^{n-1}} f(K,u)^{(n-i)/(n+1)} f(L,u)^{i/(n+1)}dS(u).$$

Now, we define the $i$th affine area $\Omega_i(K)$ of $K \in \kappa^n$ to be $\Omega_i(K,B)$. Since $f(B,\cdot) = 1$, one has

$$\Omega_i(K) = \int_{S^{n-1}} f(K,u)^{(n-i)/(n+1)}dS(u),\ i \in \mathbb{R}. \tag{2.7}$$

Lutwak [8] defined mixed bodies of convex bodies $K_1, \ldots, K_{n-1}$ as $[K_1, \ldots, K_{n-1}]$. The following property will be used later:

$$[K_1 + K_2, K_3, \ldots, K_n] = [K_1, K_3, \ldots, K_n]\tilde{+}[K_2, K_3, \ldots, K_n]. \tag{2.8}$$

2.5. **Mixed projection bodies and their polars.** If $K$ is a convex body that contains the origin in its interior, we define the polar body $K^*$ of $K$ by

$$K^* := \{x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K\}.$$

If $K_i (i = 1, 2, \ldots, n-1) \in K^n$, then the mixed projection body of $K_i (i = 1, 2, \ldots, n-1)$ is denoted by $\Pi(K_1, \ldots, K_{n-1})$, and whose support function is given, for $u \in S^{n-1}$, by [7]

$$h\left(\Pi(K_1, \ldots, K_{n-1}), u\right) := v(K_1^u, \ldots, K_{n-1}^u).$$

It is easy to see that $\Pi(K_1, \ldots, K_{n-1})$ is centered.

We use $\Pi^*(K_1, \ldots, K_{n-1})$ to denote the polar body of $\Pi(K_1, \ldots, K_{n-1})$. It is also called the polar of mixed projection body of $K_i (i = 1, 2, \ldots, n-1)$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_1, \ldots, K_{n-1})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, B)$ is called the $i$th projection body of $K$ and is denoted by $\Pi_i K$. We write $\Pi_0 K$ as $\Pi K$. We will simply write $\Pi_i^* K$ and $\Pi^* K$ rather than $(\Pi_i K)^*$ and $(\Pi K)^*$, respectively.

The following property will be used:

$$\Pi(K_3, \ldots, K_n, K_1 + K_2) = \Pi(K_3, \ldots, K_n, K_1) + \Pi(K_3, \ldots, K_n, K_2). \tag{2.9}$$

## 3. Main results

Our main results are the following Theorems.

**Theorem 3.1.** *Let $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ and $C = (K_3, \ldots, K_n)$.*
(i) *For $i < n-1$,*

$$B_i(\Pi(C, K_1 + K_2))^{1/(n-i)} \leq B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2)^{1/(n-i)}, \tag{3.1}$$

*with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.*
(ii) *For $i > n$,*

$$B_i(\Pi(C, K_1 + K_2))^{1/(n-i)} \geq B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2)^{1/(n-i)}, \tag{3.2}$$

*with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.*

*Proof.* Here, we only give a proof of (ii).

From (2.1), (2.3), (2.4), (2.9) and applying the Minkowski inequality for integrals [14, p. 147], we obtain

$$\begin{aligned}
&B_i(\Pi(C, K_1 + K_2))^{1/(n-i)} \\
= &\left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C, K_1 + K_2), u)^{n-i} dS(u)\right)^{1/(n-i)} \\
= &\left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C, K_1) + \Pi(C, K_2), u)^{n-i} dS(u)\right)^{1/(n-i)} \\
= &\left(\frac{1}{n}\int_{S^{n-1}} (b(\Pi(C, K_1), u) + b(\Pi(C, K_2), u))^{n-i} dS(u)\right)^{1/(n-i)} \\
\geq &\left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C, K_1), u)^{n-i} dS(u)\right)^{1/(n-i)} \\
&+ \left(\frac{1}{n}\int_{S^{n-1}} b(\Pi(C, K_1), u)^{n-i} dS(u)\right)^{1/(n-i)} \\
= &B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2))^{1/(n-i)},
\end{aligned}$$

with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ have similar width. In view of $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are centered (centrally symmetric with respect to origin), we conclude that the equality holds if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic. □

Taking $i = 0$, inequality (3.1) reduces to the following result:

**Corollary 3.2.** *If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$, and $C = (K_3, \ldots, K_n)$, then*

$$B(\Pi(C, K_1 + K_2))^{1/n} \leq B(\Pi(C, K_1))^{1/n} + B(\Pi(C, K_2)^{1/n},$$

*with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.*

Taking $i = 2n$, inequality (3.2) reduces to the following result:

**Corollary 3.3.** *If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$, and $C = (K_3, \ldots, K_n)$, then*

$$B_{2n}(\Pi(C, K_1 + K_2))^{-1/n} \geq B_{2n}(\Pi(C, K_1))^{-1/n} + B_{2n}(\Pi(C, K_2)^{-1/n}, \tag{3.3}$$

*with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.*

From (2.5), (3.3), and noting that the projection body is centered(centrally symmetric with respect to origin), we obtain the following Brunn–Minkowski inequality of polars of mixed projection bodies.

**Corollary 3.4.** *If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ and $C = (K_3, \ldots, K_n)$, then*

$$V(\Pi^*(C, K_1 + K_2))^{-1/n} \geq V(\Pi^*(C, K_1)^{-1/n} + V(\Pi^*(C, K_2))^{-1/n},$$

*with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.*

**Theorem 3.5.** *Suppose $K_1, K_2, \dots, K_n \in \mathcal{K}^n$ and all mixed bodies of $K_1, \dots, K_n$ have positive continuous curvature functions.*
(i) *For $i < -1$,*

$$\begin{aligned} &\Omega_i([K_1 + K_2, K_3, \dots, K_n])^{(n+1)/(n-i)} \\ &\leq \Omega_i([K_1, K_3, K_4 \dots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \dots, K_n])^{(n+1)/(n-i)}, \end{aligned}$$

*with equality if and only if $[K_1, K_3, K_4, \dots, K_n]$ and $[K_2, K_3 \dots, K_n]$ are homothetic.*
(ii) *For $i > -1$,*

$$\begin{aligned} &\Omega_i([K_1 + K_2, K_3, \dots, K_n])^{(n+1)/(n-i)} \\ &\geq \Omega_i([K_1, K_3, K_4, \dots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \dots, K_n])^{(n+1)/(n-i)}, \end{aligned}$$

*with equality if and only if $[K_1, K_3, K_4, \dots, K_n]$ and $[K_2, K_3 \dots, K_n]$ are homothetic.*

*Proof.* (i) From (2.6), (2.7), (2.8) and in view of the Minkowski inequality for integrals [14, p. 147], we obtain

$$\begin{aligned} &\Omega_i([K_1 + K_2, K_3, K_4, \dots, K_n])^{(n+1)/(n-i)} \\ = &\left(\int_{S^{n-1}} f([K_1 + K_2, K_3, K_4, \dots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\ = &\left(\int_{S^{n-1}} f([K_1, K_3, \dots, K_n] \tilde{+} [K_2, K_3, \dots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\ = &\left(\int_{S^{n-1}} (f([K_1, \dots, K_n], u) + f([K_2, \dots, K_n], u))^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\ \leq &\left(\int_{S^{n-1}} f([K_1, K_3, K_4, \dots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\ &+ \left(\int_{S^{n-1}} f([K_2, K_3, \dots, K_n], u)^{(n-i)/(n+1)} dS(u)\right)^{(n+1)/(n-i)} \\ = &\ \Omega_i([K_1, K_3, K_4, \dots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \dots, K_n])^{(n+1)/(n-i)}, \end{aligned}$$

with equality if and only if $[K_1, K_3, K_4, \dots, K_n]$ and $[K_2, K_3, \dots, K_n]$ are homothetic.
(ii) Similarly, from (2.6), (2.7), (2.8) and in view of inverse Minkowski inequality [14, p. 147], we can obtain (3.5). □

Taking $i = 0$ in (3.5), we have

**Corollary 3.6.** *If $K_1, K_2, \dots, K_n \in \mathcal{K}^n$ and all mixed bodies of $K_1, K_2, \dots, K_n$ have positive continuous curvature functions, then*

$$\begin{aligned} &\Omega([K_1 + K_2, K_3, \dots, K_n])^{(n+1)/n} \\ \geq &\ \Omega([K_1, K_3, K_4, \dots, K_n])^{(n+1)/n} + \Omega([K_2, K_3, \dots, K_n])^{(n+1)/n}, \end{aligned}$$

*with equality if and only if $[K_1, K_3, K_4, \dots, K_n]$ and $[K_2, K_3 \dots, K_n]$ are homothetic.*

Taking $i = 2n$ in (3.5), we have

**Corollary 3.7.** *If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ and all mixed bodies of $K_1, K_2, \ldots, K_n$ have positive continuous curvature functions, then*

$$\begin{aligned} & \Omega_{2n}([K_1 + K_2, K_3, \ldots, K_n])^{-(n+1)/n} \\ \geq \; & \Omega_{2n}([K_1, K_3, K_4 \ldots, K_n])^{-(n+1)/n} + \Omega_{2n}([K_2, K_3, \ldots, K_n])^{-(n+1)/n}, \end{aligned}$$

*with equality if and only if $[K_1, K_3, K_4, \ldots, K_n]$ and $[K_2, K_3 \ldots, K_n]$ are homothetic.*

Taking $i = -n$ in (3.5), we have

**Corollary 3.8.** *If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ and all mixed bodies of $K_1, K_2, \ldots, K_n$ have positive continuous curvature functions, then*

$$\begin{aligned} & \Omega_{-n}([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/2n} \\ \leq \; & \Omega_{-n}([K_1, K_3, K_4 \ldots, K_n])^{(n+1)/2n} + \Omega_{-n}([K_2, K_3, \ldots, K_n])^{(n+1)/2n}, \end{aligned}$$

*with equality if and only if $[K_1, K_3, K_4, \ldots, K_n]$ and $[K_2, K_3 \ldots, K_n]$ are homothetic.*

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