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POINTS OF OPENNESS AND CLOSEDNESS OF SOME MAPPINGS

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Abstract. Let X and Y be topological spaces and $f : X \to Y$ be a continuous function. We are interested in finding points of Y at which f is open or closed. We will show that under certain conditions, the set of points of openness or closedness of f in Y, i. e. points of Y at which f is open (resp. closed) is a G_δ subset of Y. We will extend some results of S. Levi, R. Engelking and I. A. Vaĭnšteĭn.

1. Introduction

Let X and Y be topological spaces. Following I.A. Vaĭnšteĭn [21], a continuous mapping $f : X \to Y$ is called *closed at* $y \in Y$ if for every open subset $W \subseteq X$ containing $f^{-1}(y)$, there is a neighborhood V of y such that $f^{-1}(V) \subseteq W$. We denote by $CL(f)$ the set of all points of Y at which f is closed. Then f is closed if and only if $CL(f) = Y$.

Let us recall that $f : X \to Y$ is *open at* $x \in X$ if it maps neighborhoods of x into neighborhoods of $f(x)$ and f is *open at* $y \in Y$ if for each open A in X, $y \in f(A)$ implies $y \in Int f(A)$. It follows from the definition that $f : X \to Y$ is open at $y \in f(X)$ if and only if it is open at each point of $f^{-1}(y)$.

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The investigation of the set of points of Y at which f is open for a continuous closed mapping $f : X \to Y$ has been studied by S. Levi in [16]. In fact, S. Levi [16] proved the following theorem.

Theorem 1.1. *If $f : X \to Y$ is a continuous closed mapping on a metrizable space X, the set of points of Y at which f is open is a G_δ set in Y.*

The study of the points of closedness of a continuous function and its variants has been already considered (see e.g. [18, 20]). In particular, R. Engelking [10] proved the following.

Theorem 1.2. *If f is a continuous mapping from a completely metrizable space X into a first countable Hausdorff space Y, the set of points of Y at which f is closed is a G_δ set in Y.*

The following result is due to I.A. Vaĭnšteĭn [21].

Theorem 1.3. *Let $f : X \to Y$ be a continuous mapping of a completely metrizable space X to a first countable Hausdorff space Y. Then for every set $A \subseteq X$ such that $f|A : A \to f(A)$ is closed, there exists a G_δ set $B \subset X$ such that $A \subseteq B$ and the restriction $f|B : B \to f(B)$ is closed.*

In this paper, we will generalize Theorem 1.1 for topological spaces with *a base of countable order*. We shall also improve Theorems 1.2 and 1.3. More precisely, we will show that Theorems 1.2 and 1.3 are true when X is completely metrizable and Y is a w-space.

Let us recall that G. Gruenhage (1976) introduced a class of topological spaces, called *W-spaces*. It is known that every first countable space is a W-space but the converse is not true in general [12, 17].

2. Openness of mappings from spaces with a base of countable order

We begin this section by recalling some concepts:

Definition 2.1. Let $f : X \to Y$ be a function. The function f is called *open* (resp. *feebly open*) at $x \in X$ if $f(x) \in Int(f(U))$ (resp. $f(U)$ has a nonempty interior) for each neighborhood U of x. f is called open (resp. feebly open) if it is open (feebly open) at each point of X.

Definition 2.2. A topological space X is said to have a *base of countable order* if there is a sequence $\{\mathcal{B}_n\}$ of bases for X such that if $x \in B_n \in \mathcal{B}_n$ and $B_{n+1} \subset B_n$ for each $n \geq 1$, then $\{B_n\}_n$ is a base at x.

Spaces having bases of countable order have been studied in depth by H. Wicke and J. Worell [22]. It is known that every metrizable space has a base of countable order.

In the following definition we will use the notion of a tree. A tree is a partially ordered set $(T, \leq)$ in which, for every $t \in T$ the set $t^* = \{s \in T : s < t\}$ of predecessors is well-ordered. The αth level of T is the set of t for which t^* has order type α. A branch in a tree is a maximal chain. A tree of the height $\aleph_0$ is a tree all of whose levels have cardinality less than $\aleph_0$.

Definition 2.3. A *sieve* for a topological space (X,τ) is a pair (G,T), where $(T,\leq)$ is an indexing tree of the height $\aleph_0$ and $G:T\to\tau$ is a decreasing function (i.e. $t\leq t'$ implies $G(t)\supseteq G(t')$) such that

(i) $G(T_0)=\{G(t):t\in T_0\}$ covers X, where T_0 is the least level of T,
(ii) for each $t\in T$, $G(t)=\bigcup\{G(t'):t'$ is an immediate successor of $t\}$.

We need the following result.

Theorem 2.4. *The following are equivalent:*

(i) *X has a base of countable order.*
(ii) *X has a sieve (G,T) such that if b is a branch of T and $x\in\bigcap_{t\in b}G(t)$, then $\{G(t):t\in b\}$ is a base at x.*

Proof. See [13, Theorem 6. 3]. □

Let X be a topological space with a base of countable order and Y be a topological space. Let $f:X\to Y$ be a function and (G,T) be a sieve for X from Theorem 2.4. Let $(T,\leq)$ be an indexing tree and for $n\in\omega$, let T_n be the nth level of T. Define

$$A_1=\{x\in X:\exists t_{1,x}\in T_1,\ x\in G(t_{1,x}),\ f(x)\in Int\ f(G(t_{1,x})\}.$$

Suppose that the sets $A_1,\dots,A_{n-1}$ have been defined. Define

$$A_n=\{x\in A_{n-1}:\exists t_{n,x}\geq t_{n-1,x},t_{n,x}\in T_n,x\in G(t_{n,x}),f(x)\in Int\ f(G(t_{n,x}))\}.$$

Now, we define a function $\mathcal{O}_f:X\to[0,\infty]$ by

$$\mathcal{O}_f(x)=\begin{cases}\inf\{\frac{1}{n}:x\in A_n\} & x\in A_n \text{ for some } n\geq 1,\\ +\infty & \text{otherwise.}\end{cases}$$

Lemma 2.5. *Let X be a topological space with a base of countable order and Y be a topological space. A function $f:X\to Y$ is open at $x\in X$ if and only if $\mathcal{O}_f(x)=0$.*

Proof. If $f:X\to Y$ is open at $x\in X$, then $x\in A_n$ for every $n\geq 1$. Thus $\mathcal{O}_f(x)=0$.

Suppose now $\mathcal{O}_f(x)=0$. We want to prove that f is open at x. Note that for each $n\geq 1$, $x\in A_n$. Also, $x\in\bigcap_{n\geq1}G(t_{n,x})$. Since $\{t_{n,x}:n\geq 1\}$ is a branch of T, the family $\{G(t_{n,x}):n\geq 1\}$ is a base of neighborhoods of x. Let U be an open set in X such that $x\in U$. There is $n\in\omega$ such that $G(t_{n,x})\subseteq U$. Since $x\in A_n$, $f(G(t_{n,x}))$ is a neighborhood of $f(x)$. Thus $f(U)$ is also a neighborhood of $f(x)$. □

Lemma 2.6. *Let $f:X\to Y$ be a continuous function, where X is a topological space with a base of countable order and Y is a topological space. Then $\mathcal{O}_f:X\to[0,\infty]$ is upper semicontinuous.*

Proof. For each $n\in\omega$, define $f_n:X\to[0,\infty]$ by $f_n(x)=1/n$, if $x\in A_n$ and $f_n(x)=\infty$, if $x\notin A_n$. Since A_n is open, f_n is upper semicontinuous. Therefore, as $\mathcal{O}_f(x)=inf_{n\in\omega}f_n(x)$ for every $x\in X$, $\mathcal{O}_f$ is upper semicontinuous. □

Corollary 2.7. *Let X be a topological space with a base of countable order, Y be a topological space and $f : X \to Y$ be a continuous function. The set of all points of X at which f is open is a G_δ set in X.*

Proof. In view of Lemma 2.5, the set of all $x \in X$ at which f is open equals to the set $\{x \in X : \mathcal{O}_f(x) = 0\}$. Since

$$\{x \in X : \mathcal{O}_f(x) = 0\} = \bigcap_{n=1}^{\infty} \{x \in X : \mathcal{O}_f(x) < \frac{1}{n}\},$$

so that the result follows from Lemma 2.6. □

Recall that a function f from a topological space X into a topological space Y is quasicontinuous at $x \in X$ [15] if for every open set V in Y with $f(x) \in V$ and for every open set U in X with $x \in U$ there is a nonempty open set $U' \subset U$ such that $f(U') \subset V$. f is quasicontinuous if it is quasicontinuous at every $x \in X$.

Theorem 2.8. *Let X be a Baire space with a base of countable order and Y be a topological space. Let $f : X \to Y$ be a feebly open quasicontinuous function. Then the set of points of X at which f is open is a residual set in X (i.e. it contains a dense G_δ set in X).*

Proof. Let $n \in \omega$. Put $H_n = \{x \in X : \mathcal{O}_f(x) < 1/n\}$. To prove the density of H_n, let U be an open nonempty subset of X. There must exist $k \in \omega$, $k > n$ and $t_k \in T_k$ with $G(t_k) \subset U$. Since f is feebly open, $Intf(G(t_k)) \neq \emptyset$. Put $W = Intf(G(t_k))$. The quasicontinuity of f implies that there is a nonempty open set $L \subset f^{-1}(W) \cap G(t_k)$ such that $L \subset A_k \subset \{x \in X : \mathcal{O}_f(x) \leq 1/k\} \subset H_n$. Also, it is easy to verify that $H_n \subset \overline{IntH_n}$ for every $n \in \omega$. Thus for every $n \in \omega$, $IntH_n$ is dense too. □

Theorem 2.9. *Let X be a Baire space with a base of countable order and Y be a topological space. Let $f : X \to Y$ be a feebly open continuous function. Then the set of points of X at which f is open is a dense G_δ set in X.*

Proof. By Theorem 2.8, we know that the set of points of X at which f is open is dense (it is residual). Corollary 2.7 implies that it is a G_δ set. □

The following example shows that the condition of the feebly openness of f in Theorem 2.9 is essential.

Example 2.10. Put $X = \{(x, 0) : x \in \mathbb{Q}\} \cup \{(p/q, 1/q) : p, q$ are relatively prime, $p/q \in \mathbb{Q}\}$, where $\mathbb{Q}$ is the set of rational numbers. Consider X with the topology inherited from the usual topology of the plane [1]. Let f be the natural projection onto $\{(x, 0) : x \in \mathbb{Q}\}$. Of course, f is continuous, X is a Baire metrizable space, but f is not feebly open. It is easy to verify that the set of points of X at which f is open, is not dense in X.

Definition 2.11. A function $f : X \to Y$ is called *irreducible* if $f(X) = Y$ but for each proper closed subset F of X, $f(F) \neq Y$.

Theorem 2.12. *Let $f : X \to Y$ be continuous irreducible and closed. If X is a Baire space with a base of countable order, then the set of points of X at which f is open is a dense G_δ set in X.*

Proof. By Theorem 4.10 (i) in [14], an irreducible closed function is feebly open. Thus we can use Theorem 2.9. □

Example 2.10 shows that the irreducibility of f in the previous Theorem is essential.

Definition 2.13. Let X be a topological space with a base of countable order and $f : X \to Y$ be a function. Define $\mathfrak{O}_f : X \to [0, \infty]$ by

$$\mathfrak{O}_f(y) = \begin{cases} sup\{\mathcal{O}_f(x) : x \in f^{-1}(y)\} & y \in f(X), \\ 0 & \text{otherwise.} \end{cases}$$

The following lemma follows immediately from the definition.

Lemma 2.14. *Let X be a topological space with a base of countable order and Y be a topological space. Then a continuous function $f : X \to Y$ is open at $y \in Y$ if and only if $\mathfrak{O}_f(y) = 0$.*

Proposition 2.15. *Let X be a topological space with a base of countable order, Y be a topological space and $f : X \to Y$ be a continuous function and closed in $y \in Y$. Then $\mathfrak{O}_f$ is upper semicontinuous at y.*

Proof. If $y \notin f(X)$, then $y \notin \overline{f(X)}$ since f is not closed at each point of $\overline{f(X)} \setminus f(X)$. Hence $Y \setminus \overline{f(X)}$ is a neighborhood of y and $\mathfrak{O}_f(z) = 0$ for each $z \in Y \setminus \overline{f(X)}$. If $y \in f(X)$ and $\mathfrak{O}_f(y) = \infty$, then there is nothing to prove. Suppose that $\mathfrak{O}_f(y) < \varepsilon$ and choose $\varepsilon' > 0$ such that $\mathfrak{O}_f(y) < \varepsilon' < \varepsilon$. Then for each $x \in f^{-1}(y)$, we have $\mathcal{O}_f(x) < \varepsilon'$. Since $\mathcal{O}_f$ is upper semicontinuous, for each $x \in f^{-1}(y)$, we can find a neighborhood V_x of x such that $\mathcal{O}_f(t) < \varepsilon'$ for each $t \in V_x$. Let $V = \bigcup_{x \in f^{-1}(y)} V_x$. Then V is an open set which contains $f^{-1}(y)$. Since f is closed in y, there is a neighborhood W of y such that $f^{-1}(W) \subseteq V$. If $z \in W \cap f(X)$, then

$$\mathfrak{O}_f(z) = \sup\{\mathcal{O}_f(t) : t \in f^{-1}(z)\} \leq \varepsilon' < \varepsilon.$$

If $z \in W \cap (Y \setminus f(X))$, then $\mathfrak{O}_f(z) = 0 < \varepsilon$. Hence $\mathfrak{O}_f$ is upper semicontinuous. □

The following theorem generalizes S. Levi's result proved for a metrizable space X, see [16].

Theorem 2.16. *Let X be a topological space with a base of countable order and Y be a topological space. If a continuous function $f : X \to Y$ is closed, then the set of points of Y at which f is open is a G_δ set.*

Proof. Let E denote the set of points of Y at which f is open. Thanks to Lemma 2.14

$$E = \{y \in Y : \mathfrak{O}_f(y) = 0\} = \bigcap_{n \in \omega} \{y \in Y : \mathfrak{O}_f(y) < \frac{1}{n}\}.$$

According to Proposition 2.15, the latter set is a G_δ subset of Y. □

3. Closedness of continuous mappings

In this section, we will study the set of points of closedness of a continuous function. This study is related to the much studied problem of the existence of Choquet kernels for set-valued mappings.

Let $\Phi : X \to 2^Y$ be a set-valued mapping acting between topological spaces X and Y. We say that Φ is *upper semicontinuous at* $x \in X$ if for each open subset V of Y with $\Phi(x) \subseteq V$ there exists an open neighbourhood U of x such that $\Phi(U) \subseteq V$. The function Φ is called *upper semicontinuous* if it is upper semicontinuous at each point of X. Accordingly, a function f maps closed sets into closed sets, if and only if the mapping $f^{-1} : Y \to X$ is upper semicontinuous everywhere.

We also define the *active boundary of* Φ *at* x by

$$\mathrm{Frac}(\Phi)(x) = \bigcap_{U \in \mathcal{U}(x)} \overline{\Phi(U) \setminus \Phi(x)},$$

where $\mathcal{U}(x)$ denotes the set of all neighbourhoods of x and we define $\Phi_x : X \to 2^Y$ by

$$\Phi_x(y) = \begin{cases} \mathrm{Frac}(\Phi)(x) & if\ y = x, \\ \Phi(y) \setminus \Phi(x) & if\ y \neq x. \end{cases}$$

There has been a considerable effort put into the question of when $\mathrm{Frac}(\Phi)(x)$ is a compact kernel for Φ at x, that is, when $\mathrm{Frac}(\Phi)(x)$ is compact and the mapping Φ_x is upper semicontinuous at x, see [4, 5, 7, 8].

In the case when the mapping Φ is strongly injective, i.e., $\Phi(x) \cap \Phi(y) = \emptyset$ for any distinct x and y, see [3], we get the following general result.

Theorem 3.1. *Let (X, τ) be a T_1 topological space and Y be a Čech-complete space. If $\Phi : X \to 2^Y$ is a strongly injective mapping, then*

$$G = \{x \in X : \mathrm{Frac}(\Phi)(x) \text{ is compact and } \Phi_x \text{ is upper semicontinuous at } x\}$$

is a G_δ subset of X.

Proof. Let $\beta(Y)$ be the Čech-Stone compactification of Y. Since Y is Čech-complete, there exists a sequence of open subsets $\{G_n\}_{n\in\omega}$ of $\beta(Y)$ such that $Y = \bigcap_{n\in\omega} G_n$. For each $n \in \omega$, let

$$O_n = \bigcup\{U \in \tau : \overline{\Phi(U \setminus \{x\})}^{\beta(Y)} \subseteq G_n \text{ for some } x \in U\}.$$

Since Φ is strongly injective, for every $x \in X$ we have $\Phi_x(y) = \Phi(y)$ if $y \neq x$. Clearly, each O_n is open. Let $x \in G$ and $n \in \omega$ and $\mathrm{Frac}(\Phi)(x) \subseteq G_n$. There is an open set V_n in $\beta(Y)$ such that $\mathrm{Frac}(\Phi)(x) \subseteq V_n \subseteq \overline{V_n}^{\beta(Y)} \subseteq G_n$. The upper semicontinuity of Φ_x at x implies that there is $U \in \tau$ such that $x \in U$ and $\Phi_x(U) \subseteq V_n$. Thus $\overline{\Phi(U \setminus \{x\})}^{\beta(Y)} \subseteq G_n$. Hence $G \subseteq \bigcap_{n\in\omega} O_n$.

On the other hand if $y \in \bigcap_{n\in\omega} O_n$ and Φ is strongly injective, so that

$$\mathrm{Frac}(\Phi)(y) = \bigcap_{U\in\mathcal{U}(y)} \overline{\Phi(U \setminus \{y\})},$$

then $y \in G$. Thus, $G = \bigcap_{n\in\omega} O_n$; which is a G_δ set. □

We can now prove the following result.

Proposition 3.2. *Suppose that $f : Y \to X$ is a continuous mapping from a Čech-complete space Y into a Hausdorff space X. If*

$$CL(f) \subseteq \{x \in X : \mathrm{Frac}(f^{-1})(x) \text{ is compact and } f_x^{-1} \text{ is upper semicontinuous at } x\},$$

then $CL(f)$ is a G_δ subset of X.

Proof. Since f is continuous and X is Hausdorff, the graph of f is closed and so $\mathrm{Frac}(f^{-1})(x) \subseteq f^{-1}(x)$ for all $x \in X$. Therefore,

$$\{x \in X : \mathrm{Frac}(f^{-1})(x) \text{ is compact and } f_x^{-1} \text{ is upper semicontinuous at } x\} \subseteq CL(f)$$

and so

$$\{x \in X : \mathrm{Frac}(f^{-1})(x) \text{ is compact and } f_x^{-1} \text{ is upper semicontinuous at } x\} = CL(f).$$

The result now follows from the previous theorem since the mapping f^{-1} is strongly injective. □

Corollary 3.3. *Suppose that $f : Y \to X$ is a continuous mapping from a Čech-complete space Y into a Hausdorff first countable space X. If Y has the property that every relatively countably compact subset of Y has a compact closure then $CL(f)$ is a G_δ subset of X.*

Proof. This follows directly from Theorem 2.3 in [4] and the previous proposition. □

Many spaces satisfy the property that every relatively countably compact subset has a compact closure. For example, by the Eberlien-Šmulian Theorem, for any Banach space $(X, \| \,.\, \|)$, $(X, weak)$ has this property as does every Diedonné complete space, see [4] and [8].

Let us recall that G. Gruenhage [12] introduced a class of topological spaces called W-spaces.

Let X be a topological space and $x_0 \in X$. The topological game $\mathcal{G}(X, x_0)$ is played by two players $\mathcal{O}$ and $\mathcal{P}$ as follows.
In the step $n \geq 1$, the player $\mathcal{O}$ selects a neighborhood H_n of x_0 and then $\mathcal{P}$ answers by choosing a point $x_n \in H_n$. If

$$p_1 = (H_1, x_1),\ \cdots, p_n = (H_1, x_1, \cdots, H_n, x_n)$$

are the first "n" moves of some play (of the game), we call p_n the n^{th} (*partial play*) of the game. We say that $\mathcal{O}$ wins the game $p = (H_n, x_n)_{n\geq 1}$ if $x_n \to x_0$. We say that $\mathcal{P}$ wins the game $p = (H_n, x_n)_{n\geq 1}$ if $(x_n)_n$ does not converge to x_0.

A *strategy* for the player $\mathcal{P}$ is a sequence of functions $s = \{s_n\}$, such that s_n is a function from $(H_1, \ldots, H_n)$ to H_n for each $n \geq 1$. When $s = \{s_n\}$ is a strategy for the player $\mathcal{P}$, a s-play for the player $\mathcal{P}$ is a play $p = (H_n, x_n)_n$ such that $x_n = s_n(H_1, \ldots, H_n)$ for each $n \geq 1$. That is a play in which the player $\mathcal{P}$

select his (or her) choices according to the strategy s. Similarly, a strategy for the player $\mathcal{O}$ can be defined. We refer the reader to [2] for further information about other kinds of topological games and their applications in analysis.

A point $x \in X$ is called a *W-point* (respectively a *w-point*) in X if $\mathcal{O}$ has (respectively $\mathcal{P}$ fails to have) a winning strategy in the game $\mathcal{G}(X, x)$. A space X in which each point of X is a W-point (respectively a w-point) is called a *W-space* (respectively a *w-space*.) It is known that every first countable space is a W-space [12, Theorem 3. 3]. However, the converse is not true in general [17, Example 2. 7].

There are w-spaces which are not W-spaces. For example [11] if X is the one point compactification $T \cup \{\infty\}$ of an Aronszajn tree T with the interval topology, then neither $\mathcal{P}$ nor $\mathcal{O}$ has a winning strategy in $G(X, \infty)$.

In order to prove the main result of this section, we need the following auxiliary results.

Lemma 3.4. *Let Y be a metrizable space and X be a Hausdorff w-space. Let $f : Y \to X$ be continuous. If $x \in CL(f)$, then $\partial f^{-1}(x)$ is compact.*

Proof. Let d be a compatible metric on Y. If $x \in CL(f)$ is an isolated point, then $\partial f^{-1}(x) = \emptyset$. If $x \in CL(f)$ is not an isolated point, we will show that $\partial f^{-1}(x)$ is countably compact. To prove this let $\{y_n\}$ be a sequence in $\partial f^{-1}(x)$. Without loss of generality, we may assume that $\{y_n\}$ is infinite. Let U_1 be a neighborhood of x and the first choice of player $\mathcal{O}$. Then we choose some $y'_1 \in f^{-1}(U_1) \setminus f^{-1}(x)$ such that $d(y_1, y'_1) < 1$. Define $x_1 = f(y'_1)$ as the answer of $\mathcal{P}$ to this movement.

In general, in the step n, when the partial play $(U_1, x_1, \ldots, U_n)$ is specified, we choose a point $y'_n \in f^{-1}(U_n) \setminus f^{-1}(x)$ such that $d(y_n, y'_n) < \frac{1}{n}$. Define $x_n = f(y'_n)$ as the next move of player $\mathcal{P}$. In this way, by the induction on n a strategy for the player $\mathcal{P}$ is defined. Since X is a w-space, there is a play $p = (U_n, x_n)_n$ which is won by $\mathcal{O}$. Hence $x_n \to x$.

Let $A = \{y'_1, y'_2, \ldots\}$ and $W = Y \setminus A$. We claim that W is not open. On the contrary, suppose that W is open. Since $f^{-1}(x) \subset W$ and f is closed in x, there is a neighborhood U of x such that $f^{-1}(U) \subset W$. But then $x_n \in U$ for infinitely many n. Therefore $y'_n \in W$ for infinitely many n. This contradiction proves our claim. Let $y \in \bar{A} \setminus A$. Since $d(y_n, y'_n) < \frac{1}{n}$ for each n, y is a cluster point of $\{y_n : n \geq 1\} \subset \partial f^{-1}(x)$. Therefore $y \in \partial f^{-1}(x)$. This proves our result. □

Lemma 3.5. *Let X be a Hausdorff w-space and Y be a topological space. Let $f : Y \to X$ be a continuous mapping. Then*

$$CL(f) \subseteq \{x \in X : f_x^{-1} \textit{ is upper semicontinuous at } x\}.$$

Proof. Let $x \in CL(f)$. We will prove that f_x^{-1} is upper semicontinuous at x. Suppose that f_x^{-1} is not upper semicontinuous at x. There is an open set V in Y such that $\mathrm{Frac}(f^{-1})(x) \subseteq V$ and for every open neighbourhood U of x there is $x_U \in U$, $x_U \neq x$ and $y_u \in f_x^{-1}(x_u) \setminus V$.

Let U_1 be a neighbourhood of x and the first choice of player $\mathcal{O}$. There is $x_1 \in U_1, x_1 \neq x$ and $y_1 \in f_x^{-1}(x_1) \setminus V$. Define x_1 as the answer of $\mathcal{P}$ to this movement.

In general, in the step n, when the partial play $(U_1, x_1, ..., U_n)$ is specified, there is a point $x_n \in U_n, x_n \neq x$ and a point $y_n \in f_x^{-1}(x_n) \setminus V$. Define x_n as the next move of player $\mathcal{P}$. In this way, by the induction on n a strategy for the player $\mathcal{P}$ is defined. Since X is a w-space, there is a play $p = (U_n, x_n)_{n\in\omega}$, which is won by $\mathcal{O}$. Hence $x_n \to x$.

We claim that the sequence $\{y_n\}_{n\in\omega}$ has a cluster point. Suppose there is no cluster point of the sequence $\{y_n\}_{n\in\mathbb{N}}$. Thus the set $L = \{y_n : n \in \omega\}$ is a closed set in Y and $f^{-1}(x) \subseteq Y \setminus L$. Since $x \in CL(f)$, there is an open neighbourhood G of x such that $f^{-1}(G) \subseteq Y \setminus L$, a contradiction.

Let $y \in Y$ be a cluster point of $\{y_n\}_{n\in\omega}$. Then $y \in Y \setminus V$. It is easy to verify that $y \in \mathrm{Frac}(f^{-1})(x)$, a contradiction. □

Theorem 3.6. *Let Y be a completely metrizable space and X be a Hausdorff w-space. Let $f : Y \to X$ be a continuous mapping. Then the set of all points of X at which f is closed is a G_δ subset of X.*

Proof. Follows from Proposition 3.2, Lemmas 3.4 and 3.5 and the fact that $\mathrm{Frac}(f^{-1})(x) = \partial f^{-1}(x)$.

□

Corollary 3.7. [10, Theorem 1] *For every mapping $f : Y \to X$ from a completely metrizable space Y to a first countable Hausdorff space X, the set of all points of X at which f is closed is a G_δ set.*

I.A. Vaĭnšteĭn [21] proved that if f is a continuous mapping of a completely metrizable space Y to a first-countable Hausdorff space X, then for every set $A \subset Y$ such that the restriction $f|A : A \to f(A)$ is closed, there exists a G_δ set $B \subset Y$ such that $A \subset B$ and the restriction $f|B : B \to f(B)$ is closed. Theorem 3.6 enables us to give the following generalization of this result.

Corollary 3.8. *Let f be a continuous mapping from a completely metrizable space Y to a Hausdorff w-space X. Then for every set $A \subset Y$ such that the restriction $f|A : A \to f(A)$ is closed, there exists a G_δ set $B \subset Y$ such that $A \subset B$ and the restriction $f|B : B \to f(B)$ is closed.*

Proof. Let $A \subset Y$ be such that the restriction $f|A : A \to f(A)$ is closed. The set $\overline{A}$ is a completely metrizable space. According to Theorem 3.6, there is a G_δ subset D of X such that $f : \overline{A} \to X$ is closed at each point of D. Observe that $CL(f|A) \subset CL(f|\overline{A})$ ([9], 4.5.13 (a)). Let $D = \bigcap_{n\geq 1} G_n$, where each G_n is open in X. By our assumption, $f(A) \subset D$. Since f is continuous, $f^{-1}(G_n)$ is open. There is a sequence $\{V_n\}_{n\geq 1}$ of open sets in Y such that $\overline{A} = \bigcap_{n\geq 1} V_n$. Thus $B = \bigcap_{n,m\geq 1} f^{-1}(G_n) \cap V_m$ has the required properties. □

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