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TWO NEW CLASSES OF SUBALGEBRAS OF $L^1(G)$

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Abstract. In this paper, two new classes of subalgebras of $L^1(G)$ generated by Lorentz–Karamata spaces are investigated and some fundamental properties of these spaces are examined. Also, the multipliers space (homomorphisms space) on $L^1(G) \cap L(p, q; b)(G)$ spaces are characterized.

1. Introduction and preliminaries

A new generalization of Lebesgue, Lorentz, Zygmund, Lorentz-Zygmund and generalized Lorentz-Zygmund spaces was studied by D.E.Edmunds, R.Kerman and L.Pick in [7]. By using Karamata theory, they introduced Lorentz–Karamata spaces $L_{p,q;b}(\Omega)$ and compared corresponding quasinorms on these spaces. Nevertheless, J.S.Neves studied $L_{p,q;b}(R, \mu)$ spaces in [19] where $p, q \in (0, \infty]$, b is a slowly varying function on $[1, \infty)$ and (R, μ) is a measure space. These spaces cover the generalized Lorentz-Zygmund spaces $L_{p,q;\alpha_1,\ldots\alpha_m}(R)$ (introduced in [6]), Lorentz-Zygmund spaces $L^{p,q}(\log L)^{\alpha}(R)$ (introduced in [1]), Zygmund spaces $L^p(\log L)^{\alpha}(R)$ (introduced in [2, 25]), Lorentz spaces $L^{p,q}(R)$ and Lebesgue spaces $L^p(R)$ under convenient choices of slowly varying functions and parameters p, q. In this section, we will give some definitions and properties of slowly varying functions to establish Lorentz–Karamata spaces. Throughout this paper, certain well-known terms such as rearrangement-invariant Banach space, absolutely continuous norm, Segal algebra, Fourier transform, homogeneous Banach space etc. will be used frequently in the sequel. We will not give their definitions, but one can refer to [2, 5, 9, 10, 22, 24] and references therein. For any two

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non-negative expressions A and B, the symbol $A \precsim B$ means that $A \leq cB$, for some positive constant c independent of the variables in the expressions A and B. If $A \precsim B$ and $B \precsim A$, we write $A \approx B$ and say that A and B are equivalent.

Definition 1.1. The family of all extended scalar valued (real or complex) μ-measurable functions on a measure space (X, μ) will be denoted by $M(X, \mu)$ and $M_0(X, \mu)$ will stand for the subset of $M(X, \mu)$ consisting of all those functions which are finite $\mu-$a.e.

Definition 1.2. Let f be a measurable function defined on a measure space (X, μ). We assume that the function f is finite valued almost everywhere and for $y > 0$

$$\mu\{x \in X : \ |f(x)| > y\} < \infty.$$

Then the distribution function λ_f of f is defined by

$$\lambda_f(y) = \mu\{x \in X : \ |f(x)| > y\}.$$

The nonnegative rearrangement of f is given by

$$f^*(t) = \inf\{y > 0 : \lambda_f(y) \leq t\}, \ t \geq 0$$

where $\inf \emptyset = \infty$. Also the average (maximal) function of f on $(0, \infty)$ is given by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\, ds.$$

Note that $\lambda_f(\cdot), f^*(\cdot)$ and $f^{**}(\cdot)$ are non-increasing and right continuous functions. Also, it is obvious that if X has a finite measure, then λ_f is bounded above by $\mu(X)$ and so $f^*(t) = 0$ for all $t \geq \mu(X)$.

A positive measurable function L, defined on some neighborhood of infinity, is said to be slowly varying if, for every $s > 0$,

$$\frac{L(st)}{L(t)} \to 1 \qquad (t \to +\infty).$$

These functions were introduced by Karamata in [13]. Another definition of slowly varying functions can be found in [7] such as:

Definition 1.3. A positive and Lebesgue measurable function b is said to be *slowly varying* (s.v.) on $[1, \infty)$ in the sense of Karamata if, for each $\varepsilon > 0$, $t^\varepsilon b(t)$ is equivalent to a non-decreasing function and $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $[1, \infty)$.

The detailed study of Karamata theory, properties and examples of s.v. functions can be found in [5, 6, 13] and [25]. Let $m \in \mathbb{N}$ and $\alpha = (\alpha_1, ..., \alpha_m) \in \mathbb{R}^m$. If we denote by ϑ_α^m the real function defined by

$$\vartheta_\alpha^m(t) = \prod_{i=1}^m l_i^{\alpha_i}(t) \quad \text{for all } t \in (0, \infty),$$

where $l_1, ..., l_m$ are positive functions defined on $(0, \infty)$ by

$$l_1(t) = 1 + |\log t|, \ l_i(t) = 1 + \log l_{i-1}(t), \ i \geq 2$$

then the following functions are s.v. on $[1, \infty)$:

(1) $b(t) = \vartheta_\alpha^m(t)$ with $m \in \mathbb{N}$ and $\alpha \in \mathbb{R}^m$;
(2) $b(t) = \exp(\log^\alpha t)$ with $0 < \alpha < 1$;
(3) $b(t) = \exp(l_m^\alpha(t))$ with $0 < \alpha < 1$, $m \in \mathbb{N}$;
(4) $b(t) = l_m(t)$ with $m \in \mathbb{N}$.

Given a s.v. function b on $[1,\infty)$, we denote by γ_b the positive function defined by

$$\gamma_b(t) = \begin{cases} b(t), & \text{if } t \geq 1 \\ b\left(\frac{1}{t}\right), & \text{if } 0 < t < 1 \end{cases}.$$

It is known that any s.v. function b on $(0,\infty)$ is equivalent to a s.v. continuous function $\widetilde{b}$ on $(0,\infty)$. Consequently, without loss of generality, we may assume that all s.v. functions in question are continuous functions in $(0,\infty)$ [11].

Definition 1.4. Let $p,q \in (0,\infty]$ and let b be a s.v. function on $[1,\infty)$. Lorentz–Karamata space $L_{p,q;b}(G)$ is defined to be the set of all functions $f \in M_0(G,\mu)$ such that

$$\|f\|_{p,q;b}^* := \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^*(t) \right\|_{q;(0,\infty)} \tag{1.1}$$

is finite. Here $\|\cdot\|_{q;(0,\infty)}$ stands for the usual L_q (quasi-) norm over the interval $(0,\infty)$.

Let $0 < p,q \leq \infty$ and b be a s.v. function on $[1,\infty)$. Let us introduce the functional $\|f\|_{p,q;b}$ defined by

$$\|f\|_{p,q;b} := \left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) f^{**}(t) \right\|_{q;(0,\infty)}; \tag{1.2}$$

this is identical with that defined in (1.1) except that f^* is replaced by f^{**}. However, when $p = \infty$, $L_{p,q;b}(G)$ spaces are different from the trivial spaces if and only if $\left\| t^{\frac{1}{p}-\frac{1}{q}} \gamma_b(t) \right\|_{q;(0,\infty)} < \infty$. It is easy to see that $L_{p,q;b}(G)$ spaces endowed with a convenient norm (1.2), are rearrangement-invariant Banach function spaces with associate spaces $L_{p',q';b^{-1}}(R,\mu)$ if (R,μ) is a resonant measure space and have absolutely continuous norm when $p \in (1,\infty)$ and $q \in [1,\infty)$.

It is clear that, for $0 < p < \infty$, $L_{p,q;b}(G)$ spaces contain the characteristic function of every measurable subset of G with finite measure and hence, by linearity, every $\mu-$simple function. From the definition of $\|\cdot\|_{p,q;b}^*$, it follows that if $f \in L_{p,q;b}(G)$ and $p,q \in (0,\infty)$, then the function $\lambda_f(y)$ is finite valued. In this case, with a little thought, it is possible to construct a sequence of (simple) functions which satisfy Lemma 1.1 in [3]. Therefore, if we use the same method as employed in the proof of Proposition 2.4 in [12], we can show that Lebesgue dominated convergence theorem holds and so the set of simple functions is dense in Lorentz–Karamata spaces for $p \in (1,\infty)$ and $q \in [1,\infty)$. Also, we can see the density of continuous and complex-valued functions with compact support since μ is a Haar measure.

It follows from [19] that $\|f\|_{p,q;b}^* \leq \|f\|_{p,q;b} \precsim \|f\|_{p,q;b}^*$ for all $f \in M_0(G,\mu)$ where $1 < p \leq \infty$, $1 \leq q \leq \infty$ and b is a s.v. function on $[1,\infty)$. In particular, $L_{p,q;b}(G)$ spaces consist of all those functions f for which $\|f\|_{p,q;b}$ is finite. Since the function $f \to f^{**}$ is subadditive, it is obvious that $\|\cdot\|_{p,q;b}$ is a norm if $q \geq 1$.

For more information on Lorentz–Karamata spaces, one can refer to [2, 5, 7, 9, 11, 19] and references therein.

2. Two new classes of subalgebras of $L^1(G)$

Let G and $\widehat{G}$ be locally compact abelian groups in Pontryagin duality with Haar measures μ and η respectively, such that Plancherel theorem holds and $C_c(G)$ denote the space of all continuous, complex-valued functions on G with compact support. The Fourier transform of a function $f \in L^1(G)$ will be denoted by $\widehat{f}$.

For $1 \leq p < \infty$, the spaces

$$A^p(G) = \left\{ f \in L^1(G) : \ \widehat{f} \in L^p\left(\widehat{G}\right) \right\}$$

$$B^p(G) = L^1(G) \cap L^p(G)$$

have been studied in [14, 15, 16, 17, 18, 21, 23] by many authors. They found that $A^p(G)$ and $B^p(G)$ are Banach algebras under the usual convolution product with respect to the norms $\|\cdot\| = \|\cdot\|_1 + \|\widehat{\cdot}\|_p$ and $\|\cdot\| = \|\cdot\|_1 + \|\cdot\|_p$, respectively. Besides this, for $1 < p < \infty$, $1 \leq q < \infty$, the spaces

$$A(p,q)(G) = \left\{ f \in L^1(G) : \ \widehat{f} \in L(p,q)\left(\widehat{G}\right) \right\}$$

$$B(p,q)(G) = L^1(G) \cap L(p,q)(G)$$

are defined and similar results are obtained in [4, 8, 20, 24]. Based on this idea, we will examine two new subalgebras of $L^1(G)$. Let $A_{p,q;b}(G)$ and $B_{p,q;b}(G)$ be the subspaces of $L^1(G)$ such that

$$A_{p,q;b}(G) = \left\{ f \in L^1(G) : \widehat{f} \in L_{p,q;b}\left(\widehat{G}\right) \right\}$$

provided that each function in $L_{p,q;b}\left(\widehat{G}\right)$ is locally integrable and

$$B_{p,q;b}(G) = L^1(G) \cap L_{p,q;b}(G).$$

For every $f \in A_{p,q;b}(G)$ and $g \in B_{p,q;b}(G)$, we can supply two norms by

$$\|f\|_A = \|f\|_1 + \left\|\widehat{f}\right\|_{p,q;b}$$

$$\|g\|_B = \|g\|_1 + \|g\|_{p,q;b},$$

respectively. Some new results for Lorentz–Karamata spaces are obtained in [9]. Since these results will help us to show some properties of $\left(A_{p,q;b}(G), \|\cdot\|_A\right)$ and $\left(B_{p,q;b}(G), \|\cdot\|_B\right)$ spaces, now they will be reminded without their proofs.

Proposition 2.1. *Let f be a scalar valued, measurable functions on (G,μ). If we define the function $L_s f(t) = f(t-s)$ for any $s \in G$, then we have the following:*

1. *$\lambda_{L_s f}(y) = \lambda_f(y)$ for all $y \geq 0$,*
2. *$(L_s f)^*(t) = f^*(t)$ for all $t \geq 0$ and $(L_s f)^{**}(t) = f^{**}(t)$ for all $t > 0$,*
3. *If $p, q \in (0,\infty)$, then $\|L_s f\|^*_{p,q;b} = \|f\|^*_{p,q;b}$, $\|L_s f\|_{p,q;b} = \|f\|_{p,q;b}$.*

Proposition 2.2. *For any $f \in L_{p,q;b}(G)$, $1 < p < \infty$ and $1 \leq q < \infty$, the function $s \to L_s f$ is continuous from G into $L_{p,q;b}(G)$.*

Proposition 2.3. *Let T be a convolution operator like defined in Definition 2.1 of [3] and $h = T(f,g)$. T can be uniquely extended so that if $f \in L_{p,q;b}(G)$, $1 < p,q < \infty$ and $g \in L^1(G)$, then $h \in L_{p,s;b}(G)$, where $q \le s$. Moreover $\|h\|_{p,q;b} \precsim \|f\|_{p,q;b}\|g\|_1$.*

Lemma 2.4. *There is an approximate identity $\{a_\alpha\}_{\alpha\in I}$ of $L^1(G)$ such that $\|a_\alpha\|_1 = 1$ for each $\alpha \in I$ and $f * a_\alpha \to f$ for all $f \in L_{p,q;b}(G)$ where $1 < p,q < \infty$.*

The following lemma is a generalization of Lemma 3.3 of [24].

Lemma 2.5. *If $f \in L_{p_1,s;b}(G) \cap L_{p_2,s;b}(G)$, then $f \in L_{r,s;b}(G)$ for all r such that $p_1 < r < p_2$.*

Proof. Let $\theta = \left(\frac{1}{r} - \frac{1}{p_2}\right) / \left(\frac{1}{p_1} - \frac{1}{p_2}\right)$. Then using Hölder's inequality, we get

$$\begin{aligned}
\left(\|f\|^*_{r,s;b}\right)^s &= \int_0^\infty t^{\frac{s}{r}-1}\gamma_b^s(t)\,(f^*(t))^s\,dt \\
&= \int_0^\infty \left[t^{\theta\left(\frac{s}{p_1}-1\right)}\gamma_b^{\theta s}(t)\,(f^*(t))^{\theta s}\right]\left[t^{\theta\left(\frac{s}{p_2}-1\right)}\gamma_b^{(1-\theta)s}(t)\,(f^*(t))^{(1-\theta)s}\right]dt \\
&\le \left(\|f\|^*_{p_1,s;b}\right)^{\theta s}\left(\|f\|^*_{p_2,s;b}\right)^{(1-\theta)s}.
\end{aligned}$$

□

The spaces $A_{p,q;b}(G)$ and $B_{p,q;b}(G)$ are the generalization of the spaces $A_{p,q}(G)$ and $B_{p,q}(G)$ which are examined in [24]. Therefore, we will not give a detailed study of $A_{p,q;b}(G)$ and $B_{p,q;b}(G)$ spaces and easy proofs of the following. An interested reader can prove them by using Proposition 2.1, Proposition 2.2, Proposition 2.3 and Lemma 2.4.

Theorem 2.6. *(i) There is an approximate identity $\{e_\alpha\}$ of $A_{p,q;b}(G)$ for $1 < p,q < \infty$, which is a bounded approximate identity of $L^1(G)$ such that $\|e_\alpha\|_1 \le 1$ and $\widehat{e_\alpha}$ have compact support for all α.*
(ii) $(A_{p,q;b}(G), \|\cdot\|_A)$ is a Segal Algebra, i.e.
→ $(A_{p,q;b}(G), \|\cdot\|_A)$ is a homogeneous Banach space,
→ $(A_{p,q;b}(G), \|\cdot\|_A)$ is a Banach algebra with its norm $\|\cdot\|_A \ge \|\cdot\|_1$ and
→ $(A_{p,q;b}(G), \|\cdot\|_A)$ is a dense subspace of $L^1(G)$.
(iii) Since $L^1(G)$ and $L_{p,q;b}(G)$ are strongly character invariant, $A_{p,q;b}(G)$ is strongly character invariant and the maps $f \to M_t f$, $t \to M_t f$ are continuous where $M_t f(x) = \langle x,t\rangle f(x)$ for all $f \in A_{p,q;b}(G)$, $x \in G$ and $t \in \widehat{G}$.
(iv) $A_{p,q;b}(G)$ and $L_{p,q;b}(G)$ are essential Banach $L^1(G)-$modules.
(v) $A_{1;1;1}(G) \subset A_{p,q;b}(G)$ for all $1 \le p < \infty$, $1 \le q \le \infty$.
(vi) The Fourier transforms $\widehat{A_1(G)}$ and $\widehat{A_{p,q;b}(G)}$ are dense in $L_{p,q;b}\left(\widehat{G}\right)$ for $1 \le p,q < \infty$.

Theorem 2.7. *(i) There is an approximate identity $\{e_\alpha\}$ of $B_{p,q;b}(G)$ for $1 < p,q < \infty$, which is a bounded approximate identity of $L^1(G)$ such that $\|e_\alpha\|_1 \le 1$.*

(ii) $(B_{p,q;b}(G), \|\cdot\|_B)$ *is a Segal Algebra, i.e.*
$\rightarrow (B_{p,q;b}(G), \|\cdot\|_B)$ *is a homogeneous Banach space,*
$\rightarrow (B_{p,q;b}(G), \|\cdot\|_B)$ *is a Banach algebra with its norm* $\|\cdot\|_B \geq \|\cdot\|_1$ *and*
$\rightarrow (B_{p,q;b}(G), \|\cdot\|_B)$ *is a dense subspace of* $L^1(G)$.
(iii) Since $L^1(G)$ *and* $L_{p,q;b}(G)$ *are strongly character invariant,* $B_{p,q;b}(G)$ *is strongly character invariant and the maps* $f \rightarrow M_t f$, $t \rightarrow M_t f$ *are continuous where* $M_t f(x) = \langle x, t\rangle f(x)$ *for all* $f \in B_{p,q;b}(G)$, $x \in G$ *and* $t \in \widehat{G}$.
(iv) $B_{p,q;b}(G)$ *is an essential Banach* $L^1(G)-$*module.*

Theorem 2.8. *Let* $S(G) = A_{p,q;b}(G)$ *or* $B_{p,q;b}(G)$. *Then*
(i) *The maximal ideal space of* $S(G)$ *can be identified with the dual group* $\widehat{G}$ *of* G*;*
(ii) *the algebra* $S(G)$ *satisfies Ditkin's condition;*
(iii) *Shilov-Wiener Tauberian theorem holds in* $S(G)$.

Proof. The proof can be seen from Theorem 2.6(ii), Theorem 2.7(ii) and the fact that every Segal algebra has properties (i)-(iii). □

The group algebra is known to have the factorization property, but in general $A^p(G)$, $B^p(G)$, $A(p,q)(G)$ and $B(p,q)(G)$ don't satisfy this property [18, 23, 24].

Lemma 2.9. $A_{p,q;b}(G) * A_{p,q;b}(G) \subset A_{\frac{p}{2},\frac{q}{2};b}(G)$.

Proof. By using the techniques of Lemma 3.1 in [24], we get the result. □

As a consequence of this lemma, we can give the following theorem.

Theorem 2.10. *If* G *is non-discrete, then* $A_{p,q;b}(G) * A_{p,q;b}(G) \neq A_{p,q;b}(G)$ *and* $B_{p,q;b}(G) * B_{p,q;b}(G) \neq B_{p,q;b}(G)$ *for* $p \in (1,\infty)$ *and* $q \in [1,\infty)$.

Proof. Is is known that if a Segal algebra $S(G)$ is different from $L^1(G)$, then $S(G) \cdot S(G) \neq S(G)$. Since $A_{p,q;b}(G)$ and $B_{p,q;b}(G)$ are Segal algebras different from $L^1(G)$, we get the result. □

3. Multipliers space on $B_{p,q;b}(G)$

Let us denote the space of all bounded linear operators on $B_{p,q;b}(G)$ as $\mathfrak{M}$ which is a Banach algebra under the usual operator norm. Besides this, let $Hom_{L^1(G)}(B_{p,q;b}(G), B_{p,q;b}(G))$ be the space of all module homomorphisms of $L^1(G)-$module $B_{p,q;b}(G)$, that is, an operator $T \in \mathfrak{M}$ satisfies $T(f * g) = f * T(g)$ for all $f \in L^1(G)$ and $g \in B_{p,q;b}(G)$.The module homomorphisms space, called the *multipliers space*

$$Hom_{L^1(G)}(B_{p,q;b}(G), B_{p,q;b}(G)) = Hom_{L^1(G)}(B_{p,q;b}(G))$$

is a Banach $L^1(G)-$module by $(f \circ T)(g) = f * T(g) = T(f * g)$ for all $g \in B_{p,q;b}(G)$.

Proposition 3.1. *The set*

$$\mathfrak{D} = \overline{span\{W_f \mid f \in L^1(G)\}} = \overline{\{W_f \mid f \in L^1(G)\}}$$

is a complete subalgebra of $\mathfrak{M}$ *and it possesses a minimal approximate identity.*

Proof. It is easy to see by the definition of $\mathfrak{D}$ that $\mathfrak{D}$ is a complete subalgebra of $\mathfrak{M}$ under the operator norm with usual composition. For each $f \in L^1(G)$ and $h \in B_{p,q;b}(G)$ if we define $W_f(h) = f * h$, then we have

$$\|W_f\| = \sup_{\|h\|_B \leq 1} \|W_f(h)\|_B = \sup_{\|h\|_B \leq 1} \|f * h\|_B \leq \|f\|_1, \tag{3.1}$$

and for all $f, g \in L^1(G)$, $h \in B_{p,q;b}(G)$

$$\begin{aligned} (W_f - W_g)(h) &= f * h - g * h = (f - g) * h = W_{f-g}(h) \\ (W_f \circ W_g)(h) &= W_f(g * h) = f * g * h = W_{f*g}(h). \end{aligned} \tag{3.2}$$

Let $f \in L^1(G)$. Using (3.1), (3.2) and the minimal approximate identity of $L^1(G)$, say $\{e_\alpha\}$, we get

$$\begin{aligned} \lim_\alpha \|W_{e_\alpha} \circ W_f - W_f\| &= \lim_\alpha \|W_{e_\alpha * f} - W_f\| \\ &= \lim_\alpha \|W_{e_\alpha * f - f}\| \\ &\leq \lim_\alpha \|e_\alpha * f - f\|_1 = 0. \end{aligned}$$

Consequently, we have $\lim\limits_\alpha \|W_{e_\alpha} \circ T - T\| = 0$ for all $T \in \mathfrak{D}$. □

Proposition 3.2. *The space $\mathfrak{D}$ is a complete subalgebra of $Hom_{L^1(G)}(B_{p,q;b}(G))$.*

Proof. For any $f \in L^1(G)$, we have $W_f \in \mathfrak{M}$. Since $B_{p,q;b}(G)$ is an essential Banach $L^1(G)$ -module, we have

$$W_f(g * h) = f * g * h = g * W_f(h)$$

for all $g \in L^1(G)$ and $h \in B_{p,q;b}(G)$. Thus W_f belongs to $Hom_{L^1(G)}(B_{p,q;b}(G))$. Since $Hom_{L^1(G)}(B_{p,q;b}(G))$ is a Banach space under the usual operator norm, $\mathfrak{D}$ is a complete subalgebra of $Hom_{L^1(G)}(B_{p,q;b}(G))$. □

Proposition 3.3. *The space $\mathfrak{D}$ is an essential Banach $L^1(G)$ -module.*

Proof. For any $g \in L^1(G)$ and $W_f \in \mathfrak{D}$, define $g \circ W_f : B_{p,q;b}(G) \to B_{p,q;b}(G)$ by letting $(g \circ W_f)(h) = W_f(h * g) = W_f(g * h)$ for each $h \in B_{p,q;b}(G)$. In this case

$$\begin{aligned} \|g \circ W_f\| &= \sup_{\|h\|_B \leq 1} \|(g \circ W_f)(h)\|_B = \sup_{\|h\|_B \leq 1} \|W_f(g * h)\|_B \\ &\leq \|W_f\| \sup_{\|h\|_B \leq 1} \|g * h\|_B \leq \|W_f\| \|g\|_1 \end{aligned}$$

can be found and it implies that $\mathfrak{D}$ is a Banach $L^1(G)$ -module. On the other hand, if we consider the bounded approximate identity $\{e_\alpha\}$ of $B_{p,q;b}(G)$ as in Theorem 2.7(i), then we have

$$\begin{aligned} \|e_\alpha \circ W_f - W_f\| &= \sup_{\|u\|_B \leq 1} \|(e_\alpha \circ W_f - W_f)(u)\|_B \\ &= \sup_{\|u\|_B \leq 1} \|f * u * e_\alpha - f * u\|_B \\ &\leq \sup_{\|u\|_B \leq 1} \|f * e_\alpha - f\|_1 \|u\|_B \\ &= \|f * e_\alpha - f\|_1 \to 0 \end{aligned}$$

for any $W_f \in \mathfrak{D}$ by Theorem 2.7(iv). Therefore $\mathfrak{D}$ is an essential Banach $L^1(G)$ -module. $\square$

Also for any $f \in L^1(G)$ and $W_{e_\alpha} \in \mathfrak{D}$, we have

$$\begin{aligned}
\lim_\alpha \|f - f \circ W_{e_\alpha}\| &= \lim_\alpha \left(\sup_{\|u\|_B \le 1} \|(f - f \circ W_{e_\alpha})(u)\|_B \right) \\
&= \lim_\alpha \left(\sup_{\|u\|_B \le 1} \|f * u - e_\alpha * (f * u)\|_B \right) \\
&\le \lim_\alpha \left(\sup_{\|u\|_B \le 1} \|f - e_\alpha * f\|_1 \|u\|_B \right) \\
&\le \lim_\alpha \|f - e_\alpha * f\|_1 = 0.
\end{aligned}$$

So $f \in \overline{L^1(G) \circ \mathfrak{D}}$ and $f \in \mathfrak{D}$ by Proposition 3.3. That is to say $L^1(G) \hookrightarrow \mathfrak{D}$.

Proposition 3.4. *Let $T \in Hom_{L^1(G)}(B_{p,q;b}(G))$. Therefore $T \circ W \in \mathfrak{D}$ for each $W \in \mathfrak{D}$.*

Proof. Since $B_{p,q;b}(G)$ is a Segal algebra, it is easy to see that

$$\mathfrak{D} = \overline{span\{W_f \mid f \in L^1(G)\}} = \overline{span\{W_g \mid g \in B_{p,q;b}(G)\}}.$$

Let us take any $W_g \in \mathfrak{D}$. Then for all $h \in B_{p,q;b}(G)$, we get

$$(T \circ W_g)(h) = T(g * h) = T(g) * h = W_{T(g)}(h)$$

and $T \circ W_g \in \mathfrak{D}$ since $T(g) \in B_{p,q;b}(G)$. Now take any $W \in \mathfrak{D}$. By the definition of $\mathfrak{D}$, for all $\varepsilon > 0$ we can find $g_\varepsilon \in B_{p,q;b}(G)$ such that $\|W - W_{g_\varepsilon}\| < \frac{\varepsilon}{\|T\|}$. Since $T \circ W_g \in \mathfrak{D}$ and T is bounded on $B_{p,q;b}(G)$, we have

$$\begin{aligned}
\|T \circ W - T \circ W_{g_\varepsilon}\| &= \sup_{\|h\|_B \le 1} \|(T \circ W)(h) - (T \circ W_{g_\varepsilon})(h)\|_B \\
&= \sup_{\|h\|_B \le 1} \|T(W(h)) - T(g_\varepsilon * h)\|_B \\
&\le \|T\| \sup_{\|h\|_B \le 1} \|W(h) - g_\varepsilon * h\|_B \\
&= \|T\| \sup_{\|h\|_B \le 1} \|W(h) - W_{g_\varepsilon}(h)\|_B \\
&= \|T\| \|W - W_{g_\varepsilon}\| < \varepsilon.
\end{aligned}$$

Therefore we say that $T \circ W \in \overline{span\{W_g \mid g \in B_{p,q;b}(G)\}} = \mathfrak{D}$. $\square$

Theorem 3.5. *Let G be a locally compact abelian group. Then $M(\mathfrak{D}, \mathfrak{D}) = M(\mathfrak{D})$, the space of multipliers on Banach algebra $\mathfrak{D}$, is isometrically isomorphic to the space $Hom_{L^1(G)}(B_{p,q;b}(G))$.*

Proof. Define a mapping $\Psi : Hom_{L^1(G)}(B_{p,q;b}(G)) \to M(\mathfrak{D})$ by letting $\Psi(T) = \rho_T$ for each $T \in Hom_{L^1(G)}(B_{p,q;b}(G))$, where $\rho_T(S) = T \circ S$ for all $S \in \mathfrak{D}$. Note that Ψ is well-defined by Proposition 3.4 and moreover if $\rho_T(S \circ K) = T \circ S \circ K = \rho_T(S) \circ K$ for all $S, K \in \mathfrak{D}$, then we see that $\Psi(T) = \rho_T \in M(\mathfrak{D})$. It is obvious that the mapping Ψ is linear and injective. Also, for $T \in Hom_{L^1(G)}(B_{p,q;b}(G))$ and any $S \in \mathfrak{D}$, we have

$$\begin{aligned}\|T \circ S\| &= \sup_{\|g\|_B \le 1} \|(T \circ S)(g)\|_B = \sup_{\|g\|_B \le 1} \|T(S(g))\|_B \\ &\le \|T\| \sup_{\|g\|_B \le 1} \|S(g)\|_B = \|T\| \|S\|\end{aligned}$$

and so

$$\|\rho_T\| = \sup_{S \in \mathfrak{D}} \frac{\|\rho_T(S)\|}{\|S\|} = \sup_{S \in \mathfrak{D}} \frac{\|T \circ S\|}{\|S\|} \le \|T\|.$$

On the other hand, since $\{W_{e_\alpha}\}$ is a minimal approximate identity for the space $\mathfrak{D}$, we get

$$\|\rho_T\| = \sup_{S \in \mathfrak{D}} \frac{\|T \circ S\|}{\|S\|} \ge \sup_{\alpha} \frac{\|T \circ W_{e_\alpha}\|}{\|W_{e_\alpha}\|} \ge \sup_{\alpha} \|T \circ W_{e_\alpha}\| \ge \|T\|$$

and $\|\rho_T\| = \|T\|$.

Finally we will show that the mapping $\Psi : Hom_{L^1(G)}(B_{p,q;b}(G)) \to M(\mathfrak{D})$ is onto. Let ρ be an element of $M(\mathfrak{D})$ and $\{e_\alpha\}$ approximate identity for $L_1(G)$. Since $\mathfrak{D} \subset Hom_{L^1(G)}(B_{p,q;b}(G))$ and $\rho e_\alpha \in \mathfrak{D}$, we have

$$\rho e_\alpha (f * g) = (f \circ (\rho e_\alpha))(g) \tag{3.3}$$

for any $f \in L^1(G)$ and $g \in B_{p,q;b}(G)$. Also $M(\mathfrak{D}) \subset Hom_{L^1(G)}(\mathfrak{D})$ implies that

$$\rho(f * e_\alpha)(g) = (f \circ (\rho e_\alpha))(g). \tag{3.4}$$

Therefore by (3.3) and (3.4), we get

$$\rho e_\alpha (f * g) = (f \circ (\rho e_\alpha))(g) = \rho(f * e_\alpha)(g).$$

So for each $f \in L^1(G)$ and $g \in B_{p,q;b}(G)$

$$\begin{aligned}\lim_\alpha \|\rho(f * e_\alpha)(g) - \rho f(g)\|_B &= \lim_\alpha \|(\rho(f * e_\alpha) - \rho f)(g)\|_B \\ &= \lim_\alpha \|\rho(f * e_\alpha - f)(g)\|_B \\ &\le \lim_\alpha \|\rho(f * e_\alpha - f)\| \|g\|_B \\ &\le \|\rho\| \lim_\alpha \|f * e_\alpha - f\|_1 \|g\|_B = 0\end{aligned}$$

is obtained. Thus we get

$$\lim_\alpha (\rho e_\alpha)(f * g) = \lim_\alpha (f \circ (\rho e_\alpha))(g) = \lim_\alpha \rho(f * e_\alpha)(g) = \rho f(g).$$

Since the space $B_{p,q;b}(G)$ is an essential Banach $L^1(G)-$module by Theorem 2.7(iv), the limit of $(\rho e_\alpha)(f * g) = (f \circ (\rho e_\alpha))(g)$ exists and equal to $f * T(g) \in B_{p,q;b}(G)$ while T is an operator in $Hom_{L^1(G)}(B_{p,q;b}(G))$. Therefore, since $\lim_\alpha (\rho e_\alpha)(f * g) = \lim_\alpha (f \circ (\rho e_\alpha))(g) = \rho f(g)$ exists, we can write $f \circ T = \rho f$ for all $f \in L^1(G)$. Then $e_\alpha \circ T \circ W = (\rho e_\alpha) \circ W = \rho(e_\alpha \circ W)$ can be written for all $W \in \mathfrak{D}$. By Proposition 3.3, for all $W \in \mathfrak{D}$,we get $T \circ W = \rho(W)$ or $\rho_T(W) = \rho(W)$. Therefore $\rho_T = \rho$. $\square$

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