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ZERO-DILATION INDICES OF KMS MATRICES

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Dedicated to Professor Tsuyoshi Ando with admiration

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ABSTRACT. The zero-dilation index $d(A)$ of an n-by-n complex matrix A is the maximum size of the zero matrix which can be dilated to A. In this paper, we determine the value of this index for the KMS matrix

$$J_n(a) = \begin{bmatrix} 0 & a & a^2 & \cdots & a^{n-1} \\ & 0 & a & \ddots & \vdots \\ & & \ddots & \ddots & a^2 \\ & & & \ddots & a \\ 0 & & & & 0 \end{bmatrix}, \quad a \in \mathbb{C} \text{ and } n \geq 1,$$

by using the Li–Sze characterization of higher-rank numerical ranges of a finite matrix.

1. Introduction and preliminaries

For any n-by-n complex matrix A, let $d(A)$ denote the maximum size of a zero matrix which can be dilated to A, called the *zero-dilation index* of A. Recall that a k-by-k matrix B is said to *dilate* to A if $B = V^*AV$ for some n-by-k matrix V with $V^*V = I_k$, the k-by-k identity matrix, or, equivalently, if A is unitarily similar to a matrix of the form $\begin{bmatrix} B & * \\ * & * \end{bmatrix}$. Hence the zero-dilation index of A can

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also be expressed as

$$d(A) = \max\{k \geq 1 : A \text{ is unitarily similar to } \begin{bmatrix} 0_k & * \\ * & * \end{bmatrix}\},$$

where 0_k denotes the k-by-k zero matrix. The study of $d(A)$ was initiated in [4], in which we established its basic properties and its relations with the eigenvalues of A, and we determined the value of $d(A)$ when A is a normal matrix or a weighted permutation matrix with zero diagonals. The main tool we used there is the Li–Sze characterization of higher-rank numerical ranges of A. Recall that for any integer k, $1 \leq k \leq n$, the *rank-k numerical range* $\Lambda_k(A)$ of A is the subset $\{\lambda \in \mathbb{C} : \lambda I_k \text{ dilates to } A\}$ of the complex plane. Note that $\Lambda_1(A)$ coincides with the classical *numerical range* $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of A, where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the standard inner product and its associated norm in $\mathbb{C}^n$. Li and Sze gave in [9, Theorem 2.2] a specific description of $\Lambda_k(A)$, namely,

$$\Lambda_k(A) = \bigcap_{\theta \in \mathbb{R}} \{\lambda \in \mathbb{C} : \operatorname{Re}(e^{-i\theta}\lambda) \leq \lambda_k(\operatorname{Re}(e^{-i\theta}A))\},$$

where, for a complex number z and a matrix B, $\operatorname{Re} z = (z + \overline{z})/2$ and $\operatorname{Re} B = (B + B^*)/2$ are their *real parts*, and, for an n-by-n Hermitian matrix C, $\lambda_1(C) \geq \cdots \geq \lambda_n(C)$ denote its eigenvalues in decreasing order. In particular, it follows that

$$d(A) = \min\{i_{\geq 0}(\operatorname{Re}(e^{-i\theta}A)) : \theta \in \mathbb{R}\} \tag{1.1}$$

for any matrix A, where $i_{\geq 0}(\operatorname{Re}(e^{-i\theta}A))$ denotes the number of nonnegative eigenvalues of $\operatorname{Re}(e^{-i\theta}A)$ (cf. [4, Theorem 2.2]).

The purpose of this paper is to compute $d(A)$ when A is the *KMS matrix*

$$J_n(a) = \begin{bmatrix} 0 & a & a^2 & \cdots & a^{n-1} \\ & 0 & a & \ddots & \vdots \\ & & \ddots & \ddots & a^2 \\ & & & \ddots & a \\ 0 & & & & 0 \end{bmatrix}, \quad a \in \mathbb{C} \text{ and } n \geq 1.$$

The study of the numerical range of $J_n(a)$ was started by Gaaya in [1, 2] and continued by the present authors in [5]. As a meeting ground of the classes of nilpotent, Toeplitz, nonnegative, S_n- and S_n^{-1}-matrices, $J_n(a)$ has diverse and interesting properties concerning its numerical range. The present paper is a further exploration of such properties. In Section 2 below, we show that

$$d(J_n(a)) = \begin{cases} n & \text{if } a = 0, \\ k & \text{if } a \neq 0 \text{ and } \cos\frac{k\pi}{n-1} < |a| \leq \cos\frac{(k-1)\pi}{n-1}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\ 1 & \text{if } |a| > 1 \end{cases}$$

for any $n \geq 2$. This is proven via, in addition to the Li–Sze result, the congruence of $\mathrm{Re}\,(e^{-i\theta} J_n(a))$ and the n-by-n matrix

$$H_n(a,\theta) = \begin{bmatrix} -2|a|\cos\theta & 1 & & & \\ 1 & -2|a|\cos\theta & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & -2|a|\cos\theta & 1 \\ & & & 1 & 0 \end{bmatrix}, \quad a \in \mathbb{C} \text{ and } \theta \in \mathbb{R}.$$

Here $H_1(a,\theta)$ is understood to be the 1-by-1 zero matrix. In the end of Section 2, we carry over the result for $J_n(a)$ to that for the classes of S_n- and S_n^{-1}-matrices with one single eigenvalue.

In the following, we use $\mathrm{diag}\,(a_1,\ldots,a_n)$ to denote the n-by-n diagonal matrix with diagonals $a_1,\ldots,a_n$. For a subset K of $\mathbb{C}^n$, $\bigvee K$ denotes the subspace of $\mathbb{C}^n$ generated by vectors in K. If t is a real number, then $\lfloor t \rfloor$ (resp., $\lceil t \rceil$) denotes the largest (resp., smallest) integer less than (resp., greater than) or equal to t. Our reference for general properties of numerical ranges of matrices is [8, Chapter 1].

2. Main result

The main result of this paper is the following theorem.

Theorem 2.1. *For a in $\mathbb{C}$ and $n \geq 2$, we have*

$$\begin{aligned} d(J_n(a)) &= i_{\geq 0}(\mathrm{Re}\, J_n(a)) \\ &= \begin{cases} n & \text{if } a = 0, \\ k & \text{if } a \neq 0 \text{ and } \cos\frac{k\pi}{n-1} < |a| \leq \cos\frac{(k-1)\pi}{n-1}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\ 1 & \text{if } |a| > 1. \end{cases} \end{aligned}$$

This will be proven after the next two lemmas, the first of which gives the congruence of $\mathrm{Re}\,(e^{-i\theta} J_n(a))$ and $H_n(a,\theta)$ for any real θ. Recall that two n-by-n matrices A and B are *congruent* if $XAX^* = B$ for some invertible matrix X. By *Sylvester's law of inertia* [7, Theorem 4.5.8], two Hermitian matrices A and B are congruent if and only if they have the same numbers of positive, negative and zero eigenvalues. Thus, for congruent A and B, we have $d(A) = d(B)$ by (1.1).

Lemma 2.2. *If $a \neq 0$ in $\mathbb{C}$ and $n \geq 2$, then $\mathrm{Re}\,(e^{-i\theta} J_n(a))$ is congruent to $H_n(a,\theta)$ for any real θ.*

Proof. Since $J_n(a)$ and $J_n(|a|)$ are unitarily similar by [5, Proposition 2.1 (a)], we may assume that $a > 0$. Let $A = \mathrm{Re}\,(e^{-i\theta} J_n(a))$, $E_j = I_{j-1} \oplus \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix} \oplus I_{n-j-1}$ for $1 \leq j \leq n-1$, and $E = E_{n-1} \cdots E_2 E_1$. Then

$$EAE^* = \frac{1}{2} \begin{bmatrix} -2a^2\cos\theta & e^{-i\theta}a & & & \\ e^{i\theta}a & -2a^2\cos\theta & e^{-i\theta}a & & \\ & e^{i\theta}a & \ddots & \ddots & \\ & & \ddots & -2a^2\cos\theta & e^{-i\theta}a \\ & & & e^{i\theta}a & 0 \end{bmatrix}.$$

If $W = \sqrt{2/a}\ \mathrm{diag}\,(1, e^{-i\theta}, e^{-2i\theta}, \ldots, e^{-i(n-1)\theta})$, then $WEAE^*W^* = H_n(a,\theta)$, which shows the congruence of $\mathrm{Re}\,(e^{-i\theta}J_n(a))$ and $H_n(a,\theta)$. □

For $n \geq 1$, let

$$J_n = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

denote the n-by-n *Jordan block*. It is known that the eigenvalues of $\mathrm{Re}\,J_n$ are $\cos(j\pi/(n+1))$, $1 \leq j \leq n$ (cf. [6, p. 373]). The next lemma relates the two Hermitian matrices $H_n(a,\theta)$ and $\mathrm{Re}\,J_{n-2}$.

Lemma 2.3. *For any complex a, integer $n \geq 3$ and real θ, the following hold:*

(a) $\det H_n(a,\theta) = -2^{n-2}\det((\mathrm{Re}\,J_{n-2}) - (|a|\cos\theta)I_{n-2})$,

(b) 0 *is an eigenvalue of $H_n(a,\theta)$ if and only if $|a|\cos\theta = \cos(j\pi/(n-1))$ for some j, $1 \leq j \leq n-2$,*

(c) $i_{\geq 0}(H_n(a,\theta)) = i_{\geq 0}((\mathrm{Re}\,J_{n-2}) - (|a|\cos\theta)I_{n-2}) + 1$, *and*

(d) $i_{\geq 0}(H_n(a,\theta_1)) \leq i_{\geq 0}(H_n(a,\theta_2))$ *for* $0 \leq \theta_1 \leq \theta_2 \leq \pi$.

Proof. For convenience, let $A = H_n(a,\theta)$ and $B_n = 2((\mathrm{Re}\,J_n) - (|a|\cos\theta)I_n)$.

(a) To evaluate $\det A$, we expand it by minors on the last row of A and then on the last column of the resulting $(n-1)$-by-$(n-1)$ submatrix to obtain

$$\det A = -\det B_{n-2} = -2^{n-2}\det((\mathrm{Re}\,J_{n-2}) - (|a|\cos\theta)I_{n-2}).$$

(b) This follows from (a) and the remark before the statement of this lemma.

(c) Note that A is *cyclic* in the sense that there is a vector $x = [1\ 0 \ldots\ 0]^T$ in $\mathbb{C}^n$ such that $x, Ax, \ldots, A^{n-1}x$ generate $\mathbb{C}^n$. Hence $\mathbb{C}^n = \bigvee\{x, (A-\lambda I_n)x, \ldots, (A-\lambda I_n)^{n-1}x\}$ for any complex λ. If λ is an eigenvalue of A, then the range of $A-\lambda I_n$ is not equal to $\mathbb{C}^n$ and thus, from above, x is not in this range. In this case, we deduce that $\mathrm{rank}\,(A-\lambda I_n) = n-1$ or $\dim\ker(A-\lambda I_n) = 1$. In particular, this shows that the eigenvalues of A are all distinct. Let $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ and $\beta_1 > \beta_2 > \cdots > \beta_{n-2}$ be the eigenvalues of A and B_{n-2}, respectively. Since B_{n-2} is a principal submatrix of A, the interlacing property for their eigenvalues [7, Theorem 4.3.8] yields that $\alpha_j \geq \beta_j$ for all j, $1 \leq j \leq n-2$. If $\alpha_{j_0} = \beta_{j_0}$ for some j_0, then apply the interlacing property for A, B_{n-1} and B_{n-2} to infer that β_{j_0} is also an eigenvalue of B_{n-1}. This is impossible since the eigenvalues of B_{n-1} and B_{n-2} are $2(\cos(j\pi/n) - |a|\cos\theta)$, $1 \leq j \leq n-1$, and $2(\cos(k\pi/(n-1)) - |a|\cos\theta)$, $1 \leq k \leq n-2$, respectively, which are distinct from each other. Thus $\alpha_j > \beta_j$ for all j, $1 \leq j \leq n-2$. Similarly, we have $\alpha_j < \beta_{j-2}$ for $3 \leq j \leq n$.

Let $k = i_{\geq 0}(B_{n-2})$. If $|a|\cos\theta$ is an eigenvalue of $\mathrm{Re}\,J_{n-2}$, then 0 is an eigenvalue of B_{n-2} and of A by (b). From $\beta_{k-1} > 0$, $\beta_k = 0$ and $\beta_{k+1} < 0$, we deduce that $\alpha_k > \beta_k = 0$, $\alpha_{k+1} = 0$ and $\alpha_{k+2} < \beta_k = 0$. Therefore, $i_{\geq 0}(A) = k+1$ in this case. On the other hand, if $|a|\cos\theta$ is not an eigenvalue of $\mathrm{Re}\,J_{n-2}$, then the α_j's and β_j's are all nonzero. From the preceding paragraph, we have $\alpha_k > \beta_k > 0$ and $\alpha_{k+3} < \beta_{k+1} < 0$. Since $\prod_{j=1}^{n}\alpha_j = -2^{n-2}\prod_{j=1}^{n-2}\beta_j$ by (a), we deduce that $\alpha_{k+1}\alpha_{k+2} < 0$ and hence $\alpha_{k+1} > 0 > \alpha_{k+2}$. In this case, we again have $i_{\geq 0}(A) = k+1$.

(d) This is an easy consequence of (c). $\square$

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. Assume that $a \neq 0$. If $n = 2$, then a simple computation shows that the eigenvalues of $\mathrm{Re}\,(e^{-i\theta}J_2(a))$ are $\pm|a|/2$, and thus $d(A) = i_{\geq 0}(\mathrm{Re}\,J_2(a)) = 1$ by (1.1). For the remaining part of the proof, we assume that $n \geq 3$. Then a combination of (1.1) and Lemmas 2.2 and 2.3 (d) yields that

$$\begin{aligned} d(J_n(a)) &= \min\{i_{\geq 0}(\mathrm{Re}\,(e^{-i\theta}J_n(a))) : \theta \in \mathbb{R}\} \\ &= \min\{i_{\geq 0}(H_n(a,\theta)) : \theta \in \mathbb{R}\} \\ &= i_{\geq 0}(H_n(a,0)) \\ &= i_{\geq 0}(\mathrm{Re}\,J_n(a)). \end{aligned}$$

Since $(\mathrm{Re}\,J_{n-2}) - |a|I_{n-2}$ has eigenvalues $\cos(j\pi/(n-1)) - |a|$, $1 \leq j \leq n-2$, if $\cos(k\pi/(n-1)) < |a| \leq \cos((k-1)\pi/(n-1))$ for some k, $1 \leq k \leq \lfloor n/2 \rfloor$, then

$$\begin{aligned} d(J_n(a)) &= i_{\geq 0}(H_n(a,0)) = i_{\geq 0}((\mathrm{Re}\,J_{n-2}) - |a|I_{n-2}) + 1 \\ &= (k-1) + 1 = k \end{aligned}$$

by Lemma 2.3 (c). Similarly, if $|a| > 1$, then $d(J_n(a)) = 1$. $\square$

The KMS matrices are closely related to those S_n- and S_n^{-1}-matrices with one single eigenvalue. Recall that an n-by-n matrix A is said to be of *class* S_n if it is a contraction, that is, $\|A\| \equiv \max_{\|x\|=1}\|Ax\| \leq 1$, all its eigenvalues have moduli less than 1, and $\mathrm{rank}\,(I_n - A^*A) = 1$. It is of *class* S_n^{-1} if all its eigenvalues have moduli greater than 1 and $\mathrm{rank}\,(I_n - A^*A) = 1$. These two classes of matrices were first studied in [10] and [3], respectively. They are related to KMS matrices via affine functions: *if* $0 < |a| < 1$ (*resp.*, $|a| > 1$), *then* $((1-|a|^2)/a)J_n(a) - \overline{a}I_n$ *is of class* S_n (*resp., of class* S_n^{-1}) *with the single eigenvalue* $-\overline{a}$ (cf. [5, Lemma 2.4]). Thus Theorem 2.1 may be transferred to one for S_n- and S_n^{-1}-matrices.

Corollary 2.4. *If A is an S_n-matrix (resp., S_n^{-1}-matrix) with the single eigenvalue λ, then $d(A - \lambda I_n) = k$ for $\cos(k\pi/(n-1)) < |\lambda| \leq \cos((k-1)\pi/(n-1))$, $1 \leq k \leq \lfloor n/2 \rfloor$ (resp., $d(A - \lambda I_n) = 1$).*

We remark that, in the preceding corollary, $d(A-\lambda I_n) = 1$ for A an S_n^{-1}-matrix can also be proven by the result in [3]. Indeed, let $\lambda = |\lambda|e^{i\theta}$, where $0 \leq \theta < 2\pi$, and let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $\mathrm{Re}\,(e^{-i\theta}A)$. Since λ is in $W(A)$, we have $\lambda_1 \geq |\lambda| > 1$. On the other hand, by [3, Lemma 2.9 (1)], we also have $\lambda_2 \leq 1$. Thus the eigenvalues $\lambda_j - |\lambda|$, $1 \leq j \leq n$, of $\mathrm{Re}\,(e^{-i\theta}(A - \lambda I_n))$ are such that $\lambda_1 - |\lambda| \geq 0$ and $\lambda_2 - |\lambda| < \lambda_2 - 1 \leq 0$. Therefore, $d(A - \lambda I_n) = 1$ by (1.1).

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