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On the annihilators of formal local cohomology modules

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Abstract. Let $\mathfrak{a}$ denote an ideal in a commutative Noetherian local ring $(R, \mathfrak{m})$ and M a non-zero finitely generated R-module of dimension d. Let $d := \dim(M/\mathfrak{a}M)$. In this paper we calculate the annihilator of the top formal local cohomology module $\mathfrak{F}_{\mathfrak{a}}^{d}(M)$. In fact, we prove that $\mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \mathrm{Ann}_R(M/U_R(\mathfrak{a}, M))$, where

$$U_R(\mathfrak{a}, M) := \cup\{N : N \leqslant M \text{ and } \dim(N/\mathfrak{a}N) < \dim(M/\mathfrak{a}M)\}.$$

We give a description of $U_R(\mathfrak{a}, M)$ and we will show that

$$\mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(M)) = \mathrm{Ann}_R(M/\cap_{\mathfrak{p}_j \in \mathrm{Assh}_R M \cap \mathrm{V}(\mathfrak{a})} N_j),$$

where $0 = \bigcap_{j=1}^{n} N_j$ denotes a reduced primary decomposition of the zero submodule 0 in M and N_j is a $\mathfrak{p}_j$-primary submodule of M, for all $j = 1, \ldots, n$.

Also, we determine the radical of the annihilator of $\mathfrak{F}_{\mathfrak{a}}^{d}(M)$. We will prove that

$$\sqrt{\mathrm{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^{d}(M))} = \mathrm{Ann}_R(M/G_R(\mathfrak{a}, M)),$$

where $G_R(\mathfrak{a}, M)$ denotes the largest submodule of M such that $\mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Ass}_R(M/G_R(\mathfrak{a}, M))$ and $\mathrm{Assh}_R(M)$ denotes the set $\{\mathfrak{p} \in \mathrm{Ass} M : \dim R/\mathfrak{p} = \dim M\}$.

Key words: attached primes, local cohomology, annihilator.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ is an ideal of R and M is a non-zero finitely generated R-module. Recall that the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$\mathrm{H}_{\mathfrak{a}}^{i}(M) := \varinjlim_{n \geq 1} \mathrm{Ext}_R^i(R/\mathfrak{a}^n, M).$$

For basic facts about commutative algebra see [7] and [11]; for local cohomology refer to [6].

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Let $\mathfrak{a}$ be an ideal of a commutative Noetherian local ring $(R, \mathfrak{m})$ and M a non-zero finitely generated R-module. For each $i \geq 0$; $\mathfrak{F}^i_{\mathfrak{a}}(M) := \varprojlim_n \mathrm{H}^i_m(M/\mathfrak{a}^n M)$ is called the i-th formal local cohomology of M with respect to $\mathfrak{a}$. The basic properties of formal local cohomology modules are found in [1], [5], [9], [12] and [14].

In [14] Schenzel investigated the structure of formal local cohomology modules and gave the upper and lower vanishing and non-vanishing to these modules. In particular, he proved that $\mathrm{Sup}\{i \in \mathbb{Z} : \mathfrak{F}^i_{\mathfrak{a}}(M) \neq 0\} = \dim(M/\mathfrak{a}M)$. Thus $\mathfrak{F}^{\dim(M)}_{\mathfrak{a}}(M) \neq 0$ if and only if $\dim(M/\mathfrak{a}M) = \dim M$ (cf. [14, 4.5]).

For an R-module M and an ideal $\mathfrak{a}$, the cohomological dimension of M with respect to $\mathfrak{a}$ is defined as $\mathrm{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}$. For more details see [8]. For any ideal $\mathfrak{a}$ of R, the radical of $\mathfrak{a}$, denoted by $\sqrt{\mathfrak{a}}$, is defined to be the set $\{x \in R : x^n \in \mathfrak{a} \text{ for some } n \in \mathbb{N}\}$.

A non-zero R-module M is called secondary if its multiplication map by any element a of R is either surjective or nilpotent. A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \cdots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$ (see [10]).

Recall that $\mathrm{Assh}_R(M)$ denotes the set $\{\mathfrak{p} \in \mathrm{Ass}\, M : \dim R/\mathfrak{p} = \dim M\}$. It is well known that $\mathrm{Att}_R \mathfrak{F}^{\dim M}_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \mathrm{Assh}_R(M) : \mathfrak{p} \supseteq \mathfrak{a}\}$ (cf. [5, Theorem 3.1]).

There are many results about annihilators of local cohomology modules. For example see [2], [3] and [4]. The following theorem is a main result of [2] about the annihilators of the top local cohomology modules.

Theorem 1.1 ([2, Theorem 2.3]) *Let R be a Noetherian ring and $\mathfrak{a}$ an ideal of R. Let M be a non-zero finitely generated R-module such that* $\mathrm{cd}(\mathfrak{a}, M) = \dim M$. *Then* $\mathrm{Ann}_R \mathrm{H}^{\dim M}_{\mathfrak{a}}(M) = \mathrm{Ann}_R(M/T_R(\mathfrak{a}, M))$, *where*

$$T_R(\mathfrak{a}, M) := \cup\{N : N \leqslant M \text{ and } \mathrm{cd}(\mathfrak{a}, N) < \mathrm{cd}(\mathfrak{a}, M)\}.$$

Note that, for a local ring $(R, \mathfrak{m})$, we have $\mathrm{cd}(\mathfrak{m}, M) = \dim M$ (cf. [8]). Thus

$$T_R(\mathfrak{m}, M) := \cup\{N : N \leqslant M \text{ and } \dim N < \dim M\},$$

which is the largest submodule of M such that $\dim(T_R(\mathfrak{m}, M)) < \dim(M)$.

Here, by using the above main result, we obtain some results about annihilators of top formal local cohomology modules. In Section 2, at first we define a new notation $U_R(\mathfrak{a}, M)$ and we prove the following Theorem which is a main result of this paper.

Theorem 1.2 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that $\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Ann}_R M/U_R(\mathfrak{a}, M),$$

where $U_R(\mathfrak{a}, M) := \cup\{N : N \leqslant M$ and $\dim(N/\mathfrak{a}N) < \dim(M/\mathfrak{a}M)\}$.

In Section 3, we obtain the radical of the annihilator of top formal local cohomology module $\mathfrak{F}^{\dim M}_{\mathfrak{a}}(M)$. For this we define notation $G_R(\mathfrak{a}, M)$ and we obtain the following main result.

Theorem 1.3 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that $\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Then*

$$\sqrt{\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M)} = \operatorname{Ann}_R M/G_R(\mathfrak{a}, M),$$

where $G_R(\mathfrak{a}, M)$ denotes the largest submodule of M such that $\operatorname{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a}) \subseteq \operatorname{Ass}_R(M/G_R(\mathfrak{a}, M))$.

2. Annihilators of the top formal local cohomology modules

Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that $\dim(M/\mathfrak{a}M) = d$. In this section, we will calculate the annihilator of the formal local cohomology module $\mathfrak{F}^d_{\mathfrak{a}}(M)$. Note that the assumption $\dim(M/\mathfrak{a}M) = d$ implies that $\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$ by (cf. [14, 4.5]).

Definition 2.1 Let $\mathfrak{a}$ be an ideal of R and M be a non-zero finitely generated R-module. We denote by $U_R(\mathfrak{a}, M)$ the largest submodule of M such that $\dim(U_R(\mathfrak{a}, M)/\mathfrak{a}U_R(\mathfrak{a}, M)) < \dim(M/\mathfrak{a}M)$. One can check that

$$U_R(\mathfrak{a}, M) := \cup\{N : N \leqslant M \text{ and } \dim(N/\mathfrak{a}N) < \dim(M/\mathfrak{a}M)\}.$$

The following lemma is needed in this section.

Lemma 2.2 *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

i) *$M/U_R(\mathfrak{a}, M)$ has no non-zero submodule of dimension less than d;*
ii) $\operatorname{Ass}_R(M/U_R(\mathfrak{a}, M)) = \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M)$;
iii) $\operatorname{Ass}_R U_R(\mathfrak{a}, M) = \operatorname{Ass}_R M - \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M)$;
iv) $\mathfrak{F}^d_{\mathfrak{a}}(M) \simeq \mathfrak{F}^d_{\mathfrak{a}}(M/U_R(\mathfrak{a}, M)) \simeq \mathrm{H}^d_m(M/U(\mathfrak{a}, M))$.

Proof. Let $U := U_R(\mathfrak{a}, M)$.

i) Suppose that L is a submodule of M such that $U \subseteq L \subseteq M$ and $\dim(L/U) < d$. We will show that $U = L$. By [14, Theorem 1.1] and [14, Theorem 3.11], the short exact sequence

$$0 \to U \to L \to L/U \to 0$$

induces an exact sequence

$$\cdots \to \mathfrak{F}^d_{\mathfrak{a}}(U) \to \mathfrak{F}^d_{\mathfrak{a}}(L) \to \mathfrak{F}^d_{\mathfrak{a}}(L/U) \to 0.$$

Since $\dim(L/U) < d$ we have $\mathfrak{F}^d_{\mathfrak{a}}(L/U) = 0$. On the other hand, by Definition 2.1 $\dim(U/\mathfrak{a}U) < d$ and so $\mathfrak{F}^d_{\mathfrak{a}}(U) = 0$. Thus the above long exact sequence implies that $\mathfrak{F}^d_{\mathfrak{a}}(L) = 0$. Hence $\dim(L/\mathfrak{a}L) < d$. Since $U \subseteq L$, it follows from the maximality of U that $U = L$.

ii) The short exact sequence

$$0 \to U \to M \to M/U \to 0$$

induces an exact sequence

$$\cdots \to \mathfrak{F}^d_{\mathfrak{a}}(U) \to \mathfrak{F}^d_{\mathfrak{a}}(M) \to \mathfrak{F}^d_{\mathfrak{a}}(M/U) \to 0.$$

Since $\dim(U/\mathfrak{a}U) < d$, by definition 2.1 we have $\mathfrak{F}^d_{\mathfrak{a}}(U) = 0$. So by using the above long exact sequence we conclude that $\mathfrak{F}^d_{\mathfrak{a}}(M) \cong \mathfrak{F}^d_{\mathfrak{a}}(M/U)$. Therefore $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M/U) \subseteq \operatorname{Ass} M/U$ by [5, Theorem 3.1].

Now we show that $\operatorname{Ass} M/U \subseteq \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M/U)$. Note that by (i) $\dim M/U = d$ and by [5, Theorem 3.1] $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M/U) = \{\mathfrak{p} \in$

$\mathrm{Ass}_R\, M/U : \dim R/\mathfrak{p} = d$ and $\mathfrak{p} \supseteq \mathfrak{a}\}$.

If $\mathfrak{p} \in \mathrm{Ass}\, M/U$ then there exists a submodule K of M such that $U \subsetneq K \leqslant M$ and $R/\mathfrak{p} \simeq K/U \leqslant M/U$. By (i) $\dim R/\mathfrak{p} = d$ and so it suffices to show that $\mathfrak{a} \subseteq \mathfrak{p}$. If not, $\dim R/(\mathfrak{a}+\mathfrak{p}) < \dim R/\mathfrak{p} = d$. Thus $\dim((K/U)/\mathfrak{a}(K/U)) = \dim((R/\mathfrak{p})/\mathfrak{a}(R/\mathfrak{p})) = \dim(R/(\mathfrak{a}+\mathfrak{p})) < d$. Hence $\mathfrak{F}^d_{\mathfrak{a}}(K/U) = 0$. But the exact sequence

$$0 \to U \to K \to K/U \to 0$$

induces an exact sequence

$$\cdots \to \mathfrak{F}^d_{\mathfrak{a}}(U) \to \mathfrak{F}^d_{\mathfrak{a}}(K) \to \mathfrak{F}^d_{\mathfrak{a}}(K/U) \to 0.$$

Since $\mathfrak{F}^d_{\mathfrak{a}}(U) = \mathfrak{F}^d_{\mathfrak{a}}(K/U) = 0$ by the above long exact sequence we have $\mathfrak{F}^d_{\mathfrak{a}}(K) = 0$. Thus $\dim(K/\mathfrak{a}K) < d$. But $U \subsetneq K$ and so from the maximality of U we get a contradiction. Therefore $\mathfrak{a} \subseteq \mathfrak{p}$ and the proof is complete.

iii) Let $\mathfrak{p} \in \mathrm{Ass}_R\, U$. Then there exists a submodule L of U such that $R/\mathfrak{p} \simeq L \leq U$. Thus

$$\dim R/(\mathfrak{a}+\mathfrak{p}) = \dim((R/\mathfrak{p})/\mathfrak{a}(R/\mathfrak{p})) \leq \dim(U/\mathfrak{a}U) < \dim(M/\mathfrak{a}M) = d.$$

Now, if $\mathfrak{p} \in \mathrm{Att}_R\, \mathfrak{F}^d_{\mathfrak{a}}(M)$ then $\mathfrak{a} \subseteq \mathfrak{p}$ and $\dim R/\mathfrak{p} = d$. Hence $\dim R/(\mathfrak{a}+\mathfrak{p}) = d$ which is a contradiction. Therefore $\mathrm{Ass}_R\, U \subseteq \mathrm{Ass}_R\, M - \mathrm{Att}_R\, \mathfrak{F}^d_{\mathfrak{a}}(M)$. On the other hand,

$$\mathrm{Ass}_R\, M - \mathrm{Att}_R\, \mathfrak{F}^d_{\mathfrak{a}}(M) \subseteq \mathrm{Ass}_R\, M \subseteq \mathrm{Ass}_R\, U \cup \mathrm{Ass}_R\, M/U.$$

But by (ii) $\mathrm{Ass}_R\, M/U = \mathrm{Att}_R\, \mathfrak{F}^d_{\mathfrak{a}}(M)$. Thus $\mathrm{Ass}_R\, M - \mathrm{Att}_R\, \mathfrak{F}^d_{\mathfrak{a}}(M) \subseteq \mathrm{Ass}_R\, U$. Therefore $\mathrm{Ass}_R\, M - \mathrm{Att}_R\, \mathfrak{F}^d_{\mathfrak{a}}(M) = \mathrm{Ass}_R\, U$.

iv) Since $\mathrm{Att}_R\, \mathfrak{F}^d_{\mathfrak{a}}(M) \subseteq \mathrm{V}(\mathfrak{a})$, it follows that $\mathrm{Ass}(M/U) \subseteq \mathrm{V}(\mathfrak{a})$ by (ii). Thus $\mathfrak{a} \subseteq \cap_{\mathfrak{p}\in \mathrm{Ass}(M/U)}\mathfrak{p} = \sqrt{(0 : (M/U))}$. This yields that M/U is an $\mathfrak{a}$-torsion R-module. Hence by [5, Lemma 2.1], $\mathfrak{F}^d_{\mathfrak{a}}(M/U) \cong \mathrm{H}^d_{\mathfrak{m}}(M/U)$. But in the proof of (ii) we saw that $\mathfrak{F}^d_{\mathfrak{a}}(M/U) \cong \mathfrak{F}^d_{\mathfrak{a}}(M)$. Therefore $\mathfrak{F}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{m}}(M/U)$. □

Now we can prove the following main result.

Theorem 2.3 *Let $\mathfrak{a}$ be an ideal of a local ring $(R,\mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Ann}_R M/U_R(\mathfrak{a}, M).$$

Proof. Let $U := U_R(\mathfrak{a}, M)$. By Lemma 2.2 (iv), $\mathfrak{F}^d_{\mathfrak{a}}(M) \cong \mathrm{H}^d_{\mathfrak{m}}(M/U)$. Thus $\operatorname{Ann}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(M/U))$. But by Theorem 1.1 we have

$$\operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(M/U)) = \operatorname{Ann}_R((M/U)/T_R(\mathfrak{m}, M/U)).$$

Since $T_R(\mathfrak{m}, M/U) = 0$ by Lemma 2.2 (i), we conclude that

$$\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Ann}_R(\mathrm{H}^d_{\mathfrak{m}}(M/U)) = \operatorname{Ann}_R M/U_R(\mathfrak{a}, M),$$

as required. □

Proposition 2.4 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$\mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Supp}_R(M/U_R(\mathfrak{a}, M)).$$

Proof. By Theorem 2.3,

$$\mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{V}(\operatorname{Ann}_R M/U_R(\mathfrak{a}, M)) = \operatorname{Supp}_R(M/U_R(\mathfrak{a}, M)),$$

as required. □

Theorem 2.5 *Let $\mathfrak{a}$ be an ideal of a complete local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Min}\operatorname{Supp}_R(M/U_R(\mathfrak{a}, M)) = \operatorname{Ass}_R M/U_R(\mathfrak{a}, M).$$

Proof. By [13, Theorem 2.11 (ii)] $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Min}\mathrm{V}(\operatorname{Ann}_R \mathfrak{F}^d_{\mathfrak{a}}(M))$. Now the result follows by Proposition 2.4 and Lemma 2.2 (ii). □

The next Theorem gives us a description of $U_R(\mathfrak{a}, M)$.

Theorem 2.6 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$U_R(\mathfrak{a}, M) = \cap_{\mathfrak{p}_j \in \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a})} N_j,$$

where $0 = \bigcap_{j=1}^{n} N_j$ *denotes a reduced primary decomposition of the zero submodule* 0 *in* M *and* N_j *is a* $\mathfrak{p}_j$*-primary submodule of* M*, for all* $j = 1, \dots, n$.

Proof. Set $N := \cap_{\mathfrak{p}_j \in \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a})} N_j$. At first we show that $\dim(N/\mathfrak{a}N) < d$. By [14, Lemma 2.7] $\operatorname{Ass}_R M/N = \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a})$ and $\operatorname{Ass}_R N = \operatorname{Ass}_R M - \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a})$. If $\dim N/\mathfrak{a}N = d$ then there exists a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R N \cap \mathrm{V}(\mathfrak{a})$ such that $\dim R/\mathfrak{p} = d$. Thus $\mathfrak{p} \in \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a})$ and so $\mathfrak{p} \notin \operatorname{Ass}_R N$. Since $\mathfrak{p} \in \operatorname{Supp}_R N$ and $\dim R/\mathfrak{p} = d$ we have $\mathfrak{p} \in \operatorname{Ass}_R N$ which is a contradiction. Therefore $\dim(N/\mathfrak{a}N) < d$ and so $N \subseteq U_R(\mathfrak{a}, M)$ by Definition 2.1.

Now we prove the reverse inclusion. To do this, suppose that there exists $x \in U$ such that $x \notin N$. Thus there exists an integer $t \in \{1, \dots, n\}$ such that $x \notin N_t$ and $\mathfrak{p}_t \in \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a})$. On the other hand, there exists an integer k such that $(\sqrt{\operatorname{Ann}_R Rx})^k x = 0$. Thus $(\sqrt{\operatorname{Ann}_R Rx})^k x \subseteq N_t$. Since $x \notin N_t$ and N_t is a $\mathfrak{p}_t$-primary submodule, it follows that $\cap_{\mathfrak{p} \in \operatorname{Ass}_R Rx} \mathfrak{p} = \sqrt{\operatorname{Ann}_R Rx} \subseteq \mathfrak{p}_t$. Thus there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass}_R Rx \subseteq \operatorname{Ass}_R U$ such that $\mathfrak{p} \subseteq \mathfrak{p}_t$. Then, as $\mathfrak{p} \in \operatorname{Ass}_R M$ and $\dim R/\mathfrak{p}_t = \dim M$ it follows that $\mathfrak{p} = \mathfrak{p}_t$. Hence $\mathfrak{p} \in \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a}) = \operatorname{Att} \mathfrak{F}_{\mathfrak{a}}^d(M)$. Now Lemma 2.2 (iii) implies that $\mathfrak{p} \notin \operatorname{Ass}_R U$ which is a contradiction, because of $\mathfrak{p} \in \operatorname{Ass}_R Rx \subseteq \operatorname{Ass}_R U$. This completes the proof. $\square$

Corollary 2.7 *Let* $\mathfrak{a}$ *be an ideal of a local ring* $(R, \mathfrak{m})$ *and* M *a finitely generated* R*-module of dimension* d *such that* $\dim(M/\mathfrak{a}M) = d$*. Then*

$$\operatorname{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^d(M)) = \operatorname{Ann}_R(M/ \cap_{\mathfrak{p}_j \in \operatorname{Assh}_R M \cap \mathrm{V}(\mathfrak{a})} N_j),$$

where $0 = \bigcap_{j=1}^{n} N_j$ *denotes a reduced primary decomposition of the zero submodule* 0 *in* M *and* N_j *is a* $\mathfrak{p}_j$*-primary submodule of* M*, for all* $j = 1, \dots, n$.

Proof. The result follows from Theorems 2.3 and 2.6. $\square$

3. The radical of the annihilators of the top formal local cohomology modules

Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that $\dim(M/\mathfrak{a}M) = d$. The aim of this section will be to determine the radical of $\operatorname{Ann}_R(\mathfrak{F}_{\mathfrak{a}}^d(M))$.

Definition 3.1 Let M be a non-zero finitely generated R-module of finite dimension. We denote by $G_R(\mathfrak{a}, M)$ the largest submodule of M such that $\mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Ass}_R(M/G_R(\mathfrak{a}, M))$.

Lemma 3.2 *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an ideal of R. Let M be a finitely generated R-module of finite dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then* $\dim(M/G_R(\mathfrak{a}, M)) = d$.

Proof. Since $\dim(M/\mathfrak{a}M) = d$ we have $\mathfrak{F}^d_{\mathfrak{a}}(M) \neq 0$. Thus $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Assh}_R M \cap \mathrm{V}(\mathfrak{a}) \neq \phi$.

Let $\mathfrak{p} \in \mathrm{Assh}_R M \cap \mathrm{V}(\mathfrak{a})$. Then $\mathfrak{p} \in \mathrm{Ass}_R(M/G_R(\mathfrak{a}, M))$. Thus $\mathrm{Supp}_R(R/\mathfrak{p}) \subseteq \mathrm{Supp}_R(M/G_R(\mathfrak{a}, M))$ and so $d = \dim(R/\mathfrak{p}) \leq \dim(M/G_R(\mathfrak{a}, M))$. On the other hand, $\dim(M/G_R(\mathfrak{a}, M)) \leq \dim M = d$. Therefore $d = \dim(M/G_R(\mathfrak{a}, M))$, as required. $\square$

Lemma 3.3 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$U_R(\mathfrak{a}, M/G_R(\mathfrak{a}, M)) = 0.$$

Proof. Let $G := G_R(\mathfrak{a}, M)$. It suffices to show that for any non-zero submodule L/G of M/G we have $\dim((L/G)/\mathfrak{a}(L/G)) = \dim((M/G)/\mathfrak{a}(M/G))$. It is easy to see that $\mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Ass}_R(M/G) \subseteq \mathrm{Ass}_R L/G \cup \mathrm{Ass}_R M/L$. If $\mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a}) \subseteq \mathrm{Ass}_R(M/L)$ then since $G \subsetneq L$ from the maximality of G we get a contradiction. Thus there exists a prime ideal $\mathfrak{p} \in \mathrm{Assh}_R(M) \cap \mathrm{V}(\mathfrak{a})$ such that $\mathfrak{p} \in \mathrm{Ass}_R L/G$. Hence

$$\begin{aligned}\dim((R/\mathfrak{p})/\mathfrak{a}(R/\mathfrak{p}) &\leq \dim((L/G)/\mathfrak{a}(L/G)) \leq \dim((M/G)/\mathfrak{a}(M/G)) \\ &\leq \dim(M/\mathfrak{a}M).\end{aligned}$$

Since $\mathfrak{p} \in \mathrm{Assh}_R M$, $\dim(R/\mathfrak{p}) = d$. Also, $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$ and so $\dim((R/\mathfrak{p})/\mathfrak{a}(R/\mathfrak{p}) = \dim(R/\mathfrak{p}) = d$. It follows that

$$d \leq \dim((L/G)/\mathfrak{a}(L/G)) \leq \dim((M/G)/\mathfrak{a}(M/G)) \leq d.$$

Therefore $\dim((L/G)/\mathfrak{a}(L/G)) = \dim((M/G)/\mathfrak{a}(M/G))$, as required. $\square$

Lemma 3.4 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M/G_R(\mathfrak{a}, M)).$$

Proof. Let $G := G_R(\mathfrak{a}, M)$. By definition 3.1 $\operatorname{Assh}_R M \cap \operatorname{V}(\mathfrak{a}) \subseteq \operatorname{Ass}_R(M/G)$. Thus, by using Lemma 3.2 we conclude that

$$\begin{aligned}&\{\mathfrak{p} \in \operatorname{Ass}_R M : \dim R/\mathfrak{p} = \dim M\} \cap \operatorname{V}(\mathfrak{a})\\ &\quad \subseteq \{\mathfrak{p} \in \operatorname{Ass}_R M/G : \dim R/\mathfrak{p} = \dim M/G\} \cap \operatorname{V}(\mathfrak{a})\end{aligned}$$

and so $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) \subseteq \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M/G)$. On the other hand, the exact sequence

$$0 \to G \to M \to M/G \to 0$$

induces an exact sequence

$$\cdots \to \mathfrak{F}^d_{\mathfrak{a}}(G) \to \mathfrak{F}^d_{\mathfrak{a}}(M) \to \mathfrak{F}^d_{\mathfrak{a}}(M/G) \to 0.$$

Thus $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M/G)) \subseteq \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M))$. Therefore $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M/G)$, the proof is complete. □

Lemma 3.5 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$\sqrt{\operatorname{Ann}_R(M/G_R(\mathfrak{a}, M))} = \operatorname{Ann}_R(M/G_R(\mathfrak{a}, M)).$$

Proof. Let $G := G_R(\mathfrak{a}, M)$. Let $x \in \sqrt{\operatorname{Ann}_R(M/G)}$. There exists an integer n such that $x^n M \subseteq G$. Thus Lemma 3.4 implies that

$$\operatorname{Att}_R((\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R((\mathfrak{F}^d_{\mathfrak{a}}(M/G)) = \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M/(x^n M + G))).$$

Since $\operatorname{Supp}_R(M/(x^n M + G)) = \operatorname{Supp}_R(M/(xM + G))$ by [5, Corollary 3.2] we have $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M/(x^n M + G))) = \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M/(xM + G)))$. Hence

$$\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M/(xM + G))).$$

But $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M/(xM + G))) \subseteq \operatorname{Ass}_R(M/(xM + G))$. Thus

$$\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \operatorname{Assh}_R M \cap \operatorname{V}(\mathfrak{a}) \subseteq \operatorname{Ass}_R(M/(xM + G)).$$

By definition of G we conclude that $xM + G \subseteq G$. Therefore $xM \subseteq G$ and $x \in \operatorname{Ann}_R(M/G)$, the proof is complete. $\square$

The following result is the main result of this section.

Theorem 3.6 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = d$. *Then*

$$\sqrt{\operatorname{Ann}_R \mathfrak{F}_{\mathfrak{a}}^d(M)} = \operatorname{Ann}_R M/G_R(\mathfrak{a}, M).$$

Proof. Let $G := G_R(\mathfrak{a}, M)$. By Lemma 3.4 and [6, 7.2.11] we have $\sqrt{\operatorname{Ann}_R \mathfrak{F}_{\mathfrak{a}}^d(M)} = \sqrt{\operatorname{Ann}_R \mathfrak{F}_{\mathfrak{a}}^d(M/G)}$. But by Lemma 3.2 $\dim(M/G) = d$ and so by Theorem 2.3 and Lemma 3.3,

$$\operatorname{Ann}_R \mathfrak{F}_{\mathfrak{a}}^d(M/G) = \operatorname{Ann}_R((M/G)/U_R(\mathfrak{a}, M/G)) = \operatorname{Ann}_R M/G.$$

Now Lemma 3.5 implies that $\sqrt{\operatorname{Ann}_R \mathfrak{F}_{\mathfrak{a}}^d(M/G)} = \sqrt{\operatorname{Ann}_R M/G} = \operatorname{Ann}_R M/G$. Thus $\sqrt{\operatorname{Ann}_R \mathfrak{F}_{\mathfrak{a}}^d(M)} = \operatorname{Ann}_R M/G$, as required. $\square$

Corollary 3.7 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d such that* $\dim(M/\mathfrak{a}M) = \dim M$. *Then*

$$\cap_{\mathfrak{p} \in \operatorname{Att}_R(\mathfrak{F}_{\mathfrak{a}}^d(M))} \mathfrak{p} = \operatorname{Ann}_R M/G_R(\mathfrak{a}, M).$$

Proof. It follows by [6, 7.2.11] and Theorem 3.6. $\square$

In the next result, we obtain a necessary and sufficient condition for the equality of the attached prime sets of the two top formal local cohomology modules.

Proposition 3.8 *Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a}$ an ideal of R. Let M and N be two finitely generated R-modules of dimension d such that* $\dim(M/\mathfrak{a}M) = \dim(N/\mathfrak{a}N) = d$. *Then*

$$\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^d(M) = \operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^d(N) \text{ if and only if}$$
$$\operatorname{Supp}_R(M/G_R(\mathfrak{a}, M)) = \operatorname{Supp}_R(N/G_R(\mathfrak{a}, N)).$$

Proof. If $\operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^d(M) = \operatorname{Att}_R \mathfrak{F}_{\mathfrak{a}}^d(N)$ then $\operatorname{Ann}_R M/G_R(\mathfrak{a}, M) = \operatorname{Ann}_R N/G_R(\mathfrak{a}, N)$ by Corollary 3.7 and so $\mathrm{V}(\operatorname{Ann}_R(M/G_R(\mathfrak{a}, M))) = \mathrm{V}(\operatorname{Ann}_R(N/$

$G_R(\mathfrak{a}, N))$). Thus $\mathrm{Supp}_R(M/G_R(\mathfrak{a}, M)) = \mathrm{Supp}_R(N/G_R(\mathfrak{a}, N))$.

Conversly, if $\mathrm{Supp}_R(M/G_R(\mathfrak{a}, M)) = \mathrm{Supp}_R(N/G_R(\mathfrak{a}, N))$ then by [5, Corollary 3.2] we have $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M/G_R(\mathfrak{a}, M))) = \mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(N/G_R(\mathfrak{a}, N)))$. Therefore Lemma 3.4 implies that $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(M)) = \mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a}}(N))$, as required. □

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References

[1] Asgharzadeh M. and Divaani-Aazar K., *Finiteness properties of formal local cohomology modules and Cohen-Macaulayness.* Comm. Algebra **39** (2011), 1082–1103.

[2] Atazadeh A., Sedghi M. and Naghipour R., *On the annihilators and attached primes of top local cohomology modules.* Archiv der Math. **102** (2014), 225–236.

[3] Bahmanpour K., *Annihilators of local cohomology modules.* Comm. Alg. **43** (2015), 2509–2515.

[4] Bahmanpour K., Azami J. and Ghasemi G., *On the annihilators of local cohomology modules.* J. Alg. **363** (2012), 8–13.

[5] Bijan-Zadeh M. H. and Rezaei Sh., *Artinianness and attached primes of formal local cohomology modules.* Algebra Colloquium (2) **21** (2014), 307–316.

[6] Brodmann M. and Sharp R. Y., *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, United Kingdom, 1998.

[7] Bourbaki N., *Commtative Algebra*, Addison-wesley, 1972.

[8] Divaani-Aazar K., Naghipour R. and Tousi M., *Cohomological dimension of certain algebraic varieties.* Proc. Amer. Math. Soc., **130** (2002), 3537–3544.

[9] Eghbali M., *On Artinianness of formal local cohomology, colocalization and coassociated primes.* Math. Scand. (1) **113** (2013), 5–19.

[10] Macdonald I. G., *Secondary representation of modules over a commutative ring.* Symposia Mathematica **11** (1973), 23–43.

[11] Matsumura H., *Commutative ring theory*, Cambridge University Press, 1986.

[12] Rezaei Sh., *Minimaxness and finiteness properties of formal local cohomology modules.* Kodai Math. J. (2) **38** (2015), 430–436.

[13] Rezaei Sh., *Some results on top local cohomolgy and top formal local cohomology modules.* Comm. in Alg. **45** (2017), 1935–1940.

[14] Schenzel P., *On formal local cohomology and connectedness.* J. Algebra (2) **315** (2007), 894–923.

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