On a characteristic property of Sasakian manifolds with constant φ-holomorphic sectional curvarture

Dedicated to Professor Yoshie Katsurada on her 60th birthday

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Introduction. É. Kosmanek has studied [2] a characteristic property of Kähler manifolds of constant holomorphic sectional curvature. In this paper, we shall show the following theorem making use of an analogous method of in [2].

THEOREM. *Let $M^{2n+'}$, $n\geqq 2$, be a Riemannian manifold with Sasakian structure* (ξ, φ, g). *Assume the property* (P) *to be valid in* M^{2n+1}:

(P); *For each point p of M^{2n+1} and every geodesic $\gamma(t)$ starting from p whose velocity vector at p is orthogonal to ξ_p, there exist functions $f(t)$ and $h(t)$ such that $f\varphi\gamma'+h\xi$ is a Jacobi field along γ and $f(0)\neq 0$.*

Then M is a space of constant φ-holomorphic sectional curvature.

Conversely, a Sasakian space of constant φ-holomorphic sectional curvature satisfies the property (P).

Here and throughout the paper, t means an affine parameter.

§1. Lemmas. Let (M^{2n+1}, g) be a Riemannian space. A unit Killing vector field in M is called a Sasakian structure if it satisfies

$$(\nabla_X\varphi)Y=g(\xi, Y)X-g(X, Y)\xi, \qquad \text{where} \quad \varphi X=\nabla_X\xi. \tag{1.1}$$

A Sasakian manifold is a Riemannian manifold which admits a Sasakian structure. In such a space, we know

$$R(\xi, X)Y=g(X, Y)\xi-g(\xi, Y)X. \tag{1.2}$$

We define the subspace D_p of $T_p(M)$ by $D_p=\{X|g(\xi, X)=0,\ X\in T_p(M)\}$.

LEMMA 1. *Let M be a Sasakian manifold and γ be a geodesic. If the velocity vector γ' of γ at a point p is orthogonal to ξ_p, then γ' is orthogonal to ξ on γ.*

Lemma 1 follows from $\nabla_{\gamma'}(g(\xi, \gamma'))=g(\varphi\gamma', \gamma')=0$.

LEMMA 2. *Assume that a Sasakian space M satisfies (P). Then, for vectors $X, Y\in D_p$ such that $g(\varphi X, Y)=0$,* we have

$$g\big(R(X, \varphi X)X, Y\big)=0.$$

PROOF. Consider any point $p \in M$ and vector $X \in D_p$. Let $\gamma(t)$ be a geodesic such that $\gamma(0)=p$, $\gamma'(0)=X$. By assumption, there exist functions f and h such that $f(0)\neq 0$ and $f\varphi\gamma'+h\xi$ is a Jacobi field along γ. A Jacobi field J along a geodesic γ, by definition, satisfies the Jacobi differential equation

$$\frac{D^2J}{dt^2}-R(\gamma', J)\gamma'=0\,.$$

As t is an affine parameter, $g(\gamma',\gamma')$ is a constant. Putting $a=g(\gamma',\gamma')$ and noticing (1.1) and Lemma 1,

$$fR(\gamma', \varphi\gamma')\gamma'=(f''-af+2h')\varphi\gamma'-(2af'-h'')\xi\,. \tag{1.3}$$

Hence, evaluating at p, we have

$$g\big(R(X, \varphi X)X, Y\big)=0\,, \qquad \text{for} \quad Y\in D_p,\ g(\varphi X, Y)=0\,.$$

LEMMA 3. *Under the assumption* (P), $R(X,\varphi X)X$ *is proportional to* φX *for every vector field* X *such that* $g(X,\xi)=0$.

PROOF. We can denote as

$$R(X,\varphi X)X=\lambda(X)\varphi X+\mu(X)Y+\nu(X)\xi\,,$$

where Y is some non-zero vector field, orthogonal to ξ, φX and $\lambda(X)$, $\mu(X)$, $\nu(X)$ are some functions of M. Since $g(R(X,\varphi X)X, Y)=\mu(X)g(Y, Y)$, by Lemma 2, we have $\mu(X)=0$ for every point. Similarly, noticing (1.2), we have $\nu(X)=0$.

§ 2. Proof of Theorem.

The necessity follows from the following:

THEOREM. (Tanno ([4])) *A Sasakian manifold,* $n\geqq 2$, *is of constant* φ*-holomorphic sectional curvature, if and only if*

$$R(X,\varphi X)X \qquad \textit{is proportional to } \varphi X$$

for every vector field X *such that* $g(X,\xi)=0$.

We prove the converse. Assume that M is of constant φ-holomorphic sectional curvature k. Let $\gamma(t)$ be an arbitrary geodesic such that $\gamma'(0)\in D_{\gamma(0)}$. Consider a function f on γ which is a solution of the differential equation

$$\frac{d^2f}{dt^2}+(k+3)af=0\,, \qquad f(0)\neq 0\,,$$

and put $h=2a\int f\,dt$. So it follows that

$$\begin{aligned}\nabla_{\gamma'}\nabla_{\gamma'}(f\varphi\gamma'+h\xi)&=(f''-2f+2h')\varphi\gamma'-(2af'-h''+ah)\xi\\&=-kaf\varphi\gamma'-ah\xi\,.\end{aligned}$$

On the other hand, the curvature tensor of a Sasakian space with constant φ-holomorphic sectional curvature is represented as

$$4R(X, Y)Z=(k+3)\{g(Y, Z)X-g(X, Z)Y\}+(k-1)\{\eta(X)\eta(Z)Y+\eta(Y)g(X, Z)\xi$$
$$-\eta(Y)\eta(Z)X-\eta(X)g(Y, Z)\xi-g(\varphi X, Z)\varphi Y-g(\varphi Z, Y)\varphi X-2g(\varphi X, Y)\varphi Z\},$$

where we put $\eta(X)=g(\xi, X)$, ([3]). So, $R(\gamma', \varphi\gamma')\gamma'=-k\varphi\gamma'$. Then as

$$R(\gamma', f\varphi\gamma'+h\xi)\gamma'=-fak\varphi\gamma'-ah\xi,$$

we have

$$\nabla_{\gamma'}\nabla_{\gamma'}(f\varphi\gamma'+h\xi)-R(\gamma', f\varphi\gamma'+h\xi)\gamma'=0,$$

i.e. $f\varphi\gamma'+h\xi$ is a Jacobi field. Q.E.D.

We consider the case where $h=0$. We suppose that a Sasakian space M has property (P) with $h=0$. Then M is a space of constant φ-holomorphic sectional curvature by the theorem. Since we can assume that $g(\gamma', \gamma')=1$, (2.2) reduces to

$$fR(\gamma', \varphi\gamma')\gamma'=(f''-f)\varphi\gamma'-2f'\xi.$$

Taking inner product with ξ, we have

$$-2f'=fg\Big(R(\gamma', \varphi\gamma')\gamma', \xi\Big)=0.$$

Therefore $R(\gamma', \varphi\gamma')\gamma'=-\varphi\gamma'$, which implies $k=1$.

Conversely, suppose M to be of constant φ-holomorphic sectional curvature 1. For any geodesic γ, we know

$$R(\gamma', c\varphi\gamma')\gamma'=-cg(\gamma', \gamma')\varphi\gamma', \qquad \nabla_{\gamma'}\nabla_{\gamma'}(c\varphi\gamma')=-cg(\gamma', \gamma')\varphi\gamma',$$

where c is non zero constant. Then $c\varphi\gamma'$ is a Jacobi field along γ. As a space of constant φ-holomorphic sectional curvature 1 is one of constant curvature, thus we have

COROLLARY 1. *Let M^{2n+1}, $n\geqq 2$, be a Sasakian manifold. For any point p and every geodesic γ starting from p ($\gamma'(0)\in D_p$), if there exists a function f such that $f\varphi\gamma'$ is a Jacobi field along γ and $f(0)\neq 0$, then M is of constant curvature* 1. *The converse is ture.*

In this case, f is necessarily constant.

As a corollary we can get.

COROLLARY 2. *If a space with Sasakian* 3*-structure satisfies the property* (*P*) *with respect to one of the three structures, it is of constant curvature.*

In fact, it follows from the fact that if a space with Sasakian 3-structure is of constant φ-holomorphic sectional curvature, then it is of constant curvature, ([1]).

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Bibliography

[1] T. KASHIWADA: A note on a Riemannian space with Sasakian 3-structure, Nat. Sci. Rep. of the Ochanomizu Univ., 22 (1971), 1–2.

[2] È. KOSMANEK: Une propriété caractéristique des varietes kählériennes à coubure holomorphe constante, C. R., 259 (1964), 705–708.

[3] K. OGIUE: On almost contact manifolds admitting axiom of plane or axiom of free mobility, Kōdai Math. Sem. Rep., 16 (1964), 223–232.

[4] S. TANNO: Constancy of holomorphic sectional curvature in almost hermitian manifolds, to appear.

(Received July 21, 1972)