ON THE ORTHOGONAL EXPANSION OF THE BOOLEAN POLYNOMIAL AND ITS APPLICATIONS I

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Introduction.

The aim of this paper is twofold: to establish orthogonal expansion as a convenient tool in the theory of Boolean algebra; and to render it useful in discussions concerning the structure of the system of mathematical logic particularly in the intrinsic meaning of quantifiers.

In Chapter I, we deal mainly with the orthogonal expansions of propositional polynomials, which are somewhat different from the conjunctive normalform and the disjunctive normalform of logical formulas[1] and are much more convenient to applications than them. Incidentally, our conclusion will be that any (generalized) truth function can be constructed by five operations: logical sum, logical product, negation, universal quantifier and existensive quantifier. Though our discussion is conducted with propositions, yet it should be made clear that the same procedure can be followed with Boolean algebra.

In Chapter II, we consider the structure of the system of mathematical logic. To understand this, let us observe the following fact.

Define the universal quantifier and the existensive quantifier by the axioms

e) $$(\forall x)F(x) \supset F(y),$$

f) $$F(y) \supset (\exists x)F(x),$$

as in HILBERT and ACKERMANN's[2].

Replace

1) $$(\forall x)F(x) \quad \text{by} \quad F(1) \vee F(2)$$

and

(1) Cf. HILBERT and ACKERMANN: Grundzüge der theoretischen Logik, 2-te Aufl., 1938, p. 14.

(2) Cf. HILBERT and ACKERMANN, loc. cit., p. 56.

$$2) \qquad (\exists x)F(x) \text{ by } F(1)\cdot F(2),$$

and eliminate the quantifiers of those axioms, following the pattern given in HILBERT and ACKERMANN[1]. Then we get

$$(\forall y)[(\forall x)F(x) \supset F(y)]$$

$$((\forall x)F(x) \supset F(1)) \vee ((\forall x)F(x) \supset F(2))$$

$$((F(1) \vee F(2)) \supset F(1)) \vee ((F(1) \vee F(2)) \supset F(2))$$

$$((A \vee B) \supset A) \vee ((A \vee B) \supset B)$$

and

$$(\forall y)[F(y) \supset (\exists x)F(x)]$$

$$(F(1) \supset (\exists x)F(x)) \vee (F(2) \supset (\exists x)F(x))$$

$$(F(1) \supset (F(1)\cdot F(2))) \vee (F(2) \supset (F(1)\cdot F(2)))$$

$$(A \supset A\cdot B) \vee (B \supset A\cdot B).$$

Clearly, these two propositions thus obtained are always true while, oddly enough, in 1) and 2), the universal quantifier and the existensive quantifier have replaced their intrinsic meanings with each other.

This shows that *the axioms* e) *and* f) *can not determine the charactaristic properties of quantifiers completely.* And so we ask: *what are the axioms which determine them?* The answer will be given in Theorem II 1.

The notations are the same as the ones of my note of this volume[2], except the following: a) the negation of the formula $\mathfrak{A}$ is denoted by $\mathfrak{A}^{-1}$, b) the propositional function $F(x)$ is denoted by F, and c) the universal quantifier and the existensive quantifier are denoted by ( ) and (E) respectively, and the notations $(\forall)$ and $(\exists)$ are used to denote the formally defined quantifiers (cf. Definition II 1). The set of all mappings of the set X to the set Y is denoted by Y^X. If we deal with the range of infinite objects, we must assume part of set theory and need to introduce the definitions whose numbers are marked by a $*$. Of course, one's acquaintance of mathematical logic is taken for granted in this matter.

(1) Cf. HILBERT and ACKERMANN, loc. cit., pp. 74 75.
(2) Cf. my note: Some remarks concerning identity.

CHAPTER I. ORTHOGONAL EXPANSIONS OF PROPOSITIONAL POLYNOMIALS.

§ 1. Propositions and propositional functions.

Let R be a non-empty set of objects, and B the set of two integers 1 and -1. The value of a function $f \in \Phi = B^R$ at x is denoted by f_x and called *x-component* of f. Consider a set $\mathfrak{P}$ with the property

1. *If* $f \in \Phi$ *and* $x \in R$, *then* $(f_x = 1) \in \mathfrak{P}$.

2. *If* $P \in \mathfrak{P}$, *then* $P^{-1} \in \mathfrak{P}$.

3. *If* $P, Q \in \mathfrak{P}$, *then* $P \vee Q \in \mathfrak{P}$ *and* $P \cdot Q \in \mathfrak{P}$.

3*. *If* $P^{(\lambda)} \in \mathfrak{P}$, $\lambda \in \Lambda$, *then* $\sum_{\lambda \in \Lambda} P^{(\lambda)} \in \mathfrak{P}$ *and* $\prod_{\lambda \in \Lambda} P^{(\lambda)} \in \mathfrak{P}$.

An element of $\mathfrak{P}$ is called *proposition*. The proposition $f_x = 1$ is called *primitive*. For the sake of brevity, $f_x = 1$ and $(f_x = 1)^{-1}$ are denoted by f_x^+ and f_x^-. P^{-1} is the *negation* of P, and $P \vee Q$ and $P \cdot Q$ are the *sum* and the *product* of P, Q. $\sum_{\lambda \in \Lambda} P^{(\lambda)}$ and $\prod_{\lambda \in \Lambda} P^{(\lambda)}$ are called the *generalized sum* (abbreviated gs) and the *generalized product* (abb. gp) of the indexed system $P^{(\lambda)}$, and are defined as follows:

DEFINITION 1*.
$$\sum_{\lambda \in \Lambda} P^{(\lambda)} = (E\lambda)[\lambda \in \Lambda \cdot P^{(\lambda)}]^{(1)},$$
$$\prod_{\lambda \in \Lambda} P^{(\lambda)} = (\lambda)[\lambda \in \Lambda . \supset . P^{(\lambda)}] .$$

An element F of $\mathfrak{F} = \mathfrak{P}^R$ is called *propositional function* (abb. pf) on R. The x-component of F is denoted by F_x. The pf of which x-component is f_x^+ is called *primitive* and denoted by f^+. The set R on which pf are defined is called the *range* of variable. The *negation* of a pf, the *sum* and the *product* of two pf and the gs and the gp of an indexed system of pf are defined by the

DEFINITION 2. $(F^{-1})_x = (F_x)^{-1}$, $(F \vee G)_x = F_x \vee G_x$, $(F \cdot G)_x = F_x \cdot G_x$.

DEFINITION 2*. $(\sum_{\lambda \in \Lambda} F^{(\lambda)})_x = \sum_{\lambda \in \Lambda} F_x^{(\lambda)}$, $(\prod_{\lambda \in \Lambda} F^{(\lambda)})_x = \prod_{\lambda \in \Lambda} F_x^{(\lambda)}$.

Again we can introduce the notion $\supset$, $\equiv$, etc., in $\mathfrak{F}$, and then may consider it as the theory of propositions.

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(1) $Q = P$ means that the two propositions P and Q are the same.

Let P° denote the pf whose components are all the same proposition P, then by the correspondence $P \rightarrow P^\circ$, $\mathfrak{P}$ is imbeded isomorphically in $\mathfrak{P}^R$. Thus, identifying these notions, we denote P° merely P, and call it the proposition on R.

Moreover we can define the pf on $R_1 \times R_2 \times \cdots \times R_k$. $(x\,y \cdots z)$-component of these pf are denoted by $F_{xy\ldots z}$, $G_{xy\ldots z}$, $\cdots$.

§ 2. Propositional polynomials and truth functions.

We define recursively the notion of *propositional polynomial* (abb, pp) of the symbol X:

DEFINITION 3. 1. *Any proposition is a pp.*

2. *If $x \in R$, then X_x is a pp.*
3. *If $\mathfrak{A}(X)$ is a pp previously defined, then $(\mathfrak{A}(X))^{-1}$, the negation of $\mathfrak{A}(X)$, is a pp.*
4. *If $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$ are pp previously defined, then $\mathfrak{A}(X) \vee \mathfrak{B}(X)$, the sum of $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$, and $\mathfrak{A}(X) \cdot \mathfrak{B}(X)$, the product of $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$, are pp.*

4*. *If $\mathfrak{A}^{(\lambda)}(X)$, $\lambda \in \Lambda$ are pp previously defined, then $\sum_{\lambda \in \Lambda} (\mathfrak{A}^{(\lambda)}(X))$, the gs of $\mathfrak{A}^{(\lambda)}(X)$, and $\prod_{\lambda \in \Lambda} (\mathfrak{A}^{(\lambda)}(X)$, the gp of $\mathfrak{A}^{(\lambda)}(X)$, are pp.*

The X of a pp $\mathfrak{A}(X)$ is called the *variable* of $\mathfrak{A}(X)$, we regard this as the *free pf* on R. Let $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$ be two pp, then

DEFINITION 4. $\mathfrak{A}(X) = \mathfrak{B}(X) \;.\equiv.\; (F)\; [\mathfrak{A}(F) \;.\equiv.\; \mathfrak{B}(F)]$.

The set of all pp of variable X is denoted by $\mathfrak{P}[X]$.

LEMMA 1. *If $\mathfrak{A}(X) \in \mathfrak{P}[X]$, then*

(I) $\mathfrak{A}(F) \;.\equiv.\; \mathfrak{B}(G)$ *whenever* $F \;.\equiv.\; G$.

Proof. If $\mathfrak{A}(X)$ is a proposition or a X_x, then the assertion is obvious. If $\mathfrak{A}(X) = (\mathfrak{B}(X))^{-1}$ and $\mathfrak{B}(F) .\equiv. \mathfrak{B}(G)$ whenever $F .\equiv. G$, then $(\mathfrak{B}(F))^{-1} \;.\equiv.\; (\mathfrak{B}(G))^{-1}$. And so on.

The notion of pp can be generalized as follows:

DEFINITION 5. *Every element $\mathfrak{A} \in \mathfrak{P}^{\mathfrak{F}}$ with the property* (I) *is called truth function* (abb. tf).

$\mathfrak{A}_F$ is the F-component of $\mathfrak{A}$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two tf, then

DEFINITION 6. $\mathfrak{A} = \mathfrak{B} \,.\!\equiv\!.\, (F)[\mathfrak{A}_F \,.\!\equiv\!.\, \mathfrak{B}_F]\,.$

The definition of the *negation* of a tf, the *sum* of two tf, $\cdots$ are the same as definitions 2 and 2*. The set of all tf is denoted by $\mathfrak{T}$.

Let $\mathfrak{A}(X)$ be a pp and $\mathfrak{A}$ a tf, then we define a relation $\sim$:

DEFINITION 7. $\mathfrak{A} \sim \mathfrak{A}(X) \,.\!\equiv\!.\, (F)[\mathfrak{A}_F \,.\!\equiv\!.\, \mathfrak{A}(F)]\,.$

By definition, $\mathfrak{A}(X)$ is *equivalent to* $\mathfrak{A}$, whenever $\mathfrak{A} \sim \mathfrak{A}(X)$.

LEMMA 2. *For any pp $\mathfrak{A}(X)$, there corresponds a tf $\mathfrak{A}$ such that $\mathfrak{A} \sim \mathfrak{A}(X)$, and this correspondence is one-to-one.*

Proof. By definition 3, $\mathfrak{A}(F) \in \mathfrak{P}$ for every $F \in \mathfrak{F}$. Define a function $\mathfrak{A}$ on $\mathfrak{F}$ to $\mathfrak{P}$ by $\mathfrak{A}_F \,.\!\equiv\!.\, \mathfrak{A}(F)$, then manifestly $\mathfrak{A} \in \mathfrak{F}$ and $\mathfrak{A} \sim \mathfrak{A}(X)$. The uniqueness and the univalence of this correspondence is the consequence of definitions 6 and 4.

Though a pp or a tf, say a tf $\mathfrak{A}$, is not a proposition, yet sometimes we let $\mathfrak{A}$ denote the assertion that $\mathfrak{A} = 1$ for brevity's sake. For instance, the $\mathfrak{A}$ in the formula $\mathfrak{A} \vee \mathfrak{A} = \mathfrak{A}$ is a tf, and the $\mathfrak{A}$ in the formula $\mathfrak{A} \vee \mathfrak{A} \,.\!\equiv\!.\, \mathfrak{A}$ is the assertion $\mathfrak{A} = 1$. Thus, we can treat pp and tf like the proposition. For example, the definition of "implication" of pp is as follows:

$$\mathfrak{A}(X) \,.\!\supset\!.\, \mathfrak{B}(X) :\equiv: (\mathfrak{A}(X))^{-1} \vee \mathfrak{B}(X)\,.$$

Moreover, we can define the tf on $\mathfrak{F}_1 \times \mathfrak{F}_2 \times \cdots \times \mathfrak{F}_k$.

§ 3. Orthogonal expansion of pp.

LEMMA 3. *For any formula $\mathfrak{A}_x$, $x = y \cdot \mathfrak{A}_y \,.\!\equiv\!.\, x = y \cdot \mathfrak{A}_x$.*

LEMMA 4. *For any formula $\mathfrak{A}_x$, $(Ey)[x = y \cdot \mathfrak{A}_y] \,.\!\equiv\!.\, \mathfrak{A}_x$.*

LEMMA 5. $(Ey)[x = y]\,.$

These lemmas(1) are often used in the following.

LEMMA 6. $\underset{x\in R}{\Pi}(g^+_x)^{f_x} \,.\!\equiv\!.\, f = g$, *for any $f \in \Phi$ and primitive pf g^+.*

Proof. Using lemma 3

(1) Cf. my note loc. cit..

$(g_x^+)^{f_x} \rightleftarrows (f_x=1 \,.\mathrm{v}.\, f_x \neq 1) \cdot (g_x^+)^{f_x} \rightleftarrows f_x=1 \cdot (g_x^+)^{f_x} \,.\mathrm{v}.\, (f_x=-1) \cdot (g_x^+)^{f_x} \rightleftarrows f_x=1 \cdot g_x^+ \,.\mathrm{v}.\, f_x=-1 \cdot (g_x^+)^{-1} \rightleftarrows f_x=1 \cdot g_x=1 \,.\mathrm{v}.\, f_x=-1 \cdot g_x=-1 \rightleftarrows f_x=1 \cdot f_x=g_x \,.\mathrm{v}.\, f_x=-1 \cdot f_x=g_x \rightleftarrows (f_x=1 \,.\mathrm{v}.\, f_x=-1) \cdot f_x=g_x \rightleftarrows f_x=g_x.$

Then $\prod_{x\in R}(g_x^+)^{f_x} \rightleftarrows \prod_{x\in R}(f_x=g_x) \rightleftarrows (x)[x\in R \,.\supset.\, f_x=g_x] \rightleftarrows f=g.$

In $\mathfrak{F}$, logical equivalence is a congruence relation. $\overline{\mathfrak{F}}$ denotes the quotient algebra of $\mathfrak{F}$ by this relation, and $\overline{F}$ the residue class of which representative is F.

LEMMA 7. *$\Phi^+ \cong \overline{\mathfrak{F}}$, where Φ^+ is the set of all primitive pf.*

Proof. For any $F \in \mathfrak{F}$, choose the function $f \in \Phi$ such as $f_x = 1 \,.\equiv.\, F_x$ for every $x \in R$, and define the mapping $\varphi : \mathfrak{F} \to \Phi^+$ by $\varphi(F) = f^+$. Then $F \equiv G \,.\equiv.\, \varphi(F) = \varphi(G)$, hence we can define the mapping $\bar{\varphi} : \overline{\mathfrak{F}} \to \Phi^+$ by $\varphi(\overline{F}) = \varphi(F)$. It is clear that the mapping $\bar{\varphi}$ is an isomorphism from $\overline{\mathfrak{F}}$ to Φ^+.

LEMMA 8. *$\mathfrak{A}(X) = \mathfrak{B}(X)$ whenever $\mathfrak{A}(f^+) \,.\equiv.\, \mathfrak{B}(f^+)$ for all $f^+ \in \Phi^+$, and $\mathfrak{A} = \mathfrak{B}$ whenever $\mathfrak{A}_{f^+} \,.\equiv.\, \mathfrak{B}_{f^+}$ for all $f^+ \in \Phi^+$.*

Proof. This is a combination of the lemma 7, definitions 4 and 6.

LEMMA 9. $\sum_{f\in\Phi}\prod_{x\in R} X_x^{f_x} = 1.$

Proof. For any $g^+ \in \Phi^+$

$\sum_f \prod_x (g_x^+)^{f_x} \rightleftarrows \sum_f (f=g)$ **(lemma 6)** $\rightleftarrows (Ef)\,[f=g]$ **(definition 1*)** $\rightleftarrows 1$ **(lemma 5).**

Hence, by lemma 8, $\sum_f \prod_x X_x^{f_x} = 1.$

LEMMA 10. *For any pp $\mathfrak{A}(X)$, $\mathfrak{A}(X) \cdot \prod_{x\in R} X_x^{f_x} = \mathfrak{A}(f^+) \cdot \prod_{x\in R} X_x^{f_x}.$*

Proof. For any $g^+ \in \Phi^+$

$\mathfrak{A}(g^+) \cdot \prod_x (g_x^+)^{f_x} \rightleftarrows \mathfrak{A}(g^+) \cdot f=g$ **(lemma 6)** $\rightleftarrows \mathfrak{A}(f^+) \cdot f=g$ **lemma 3)** $\rightleftarrows \mathfrak{A}(f^+) \cdot \prod (g_x^+)^{f_x}$ **(again by lemma 6).**

Hence, the assertion is the consequence of lemma 8.

THEOREM 1. *For any pp* $\mathfrak{A}(X)$

(*) $$\mathfrak{A}(X) = \sum_{f\in\Phi} [A_f \cdot \prod_{x\in R} X_x^{f_x}] = \prod_{f\in\Phi} [A_f \text{ v } \sum_{x\in R} X_x^{-f_x}]$$

where A *is a pf on* Φ *such that* $A_f \;.\equiv.\; \mathfrak{A}(f^+)$.

Proof. Using lemmas 9 and 10, we get

$$\mathfrak{A}(X) = \mathfrak{A}(X) \cdot 1 = \mathfrak{A}(X) \cdot \sum_f \prod_x X_x^{f_x} = \sum_f [\mathfrak{A}(X) \cdot \prod_x X_x^{f_x}]$$
$$= \sum_f [\mathfrak{A}(f^+) \cdot \prod_x X_x^{f_x}].$$

Similarly $(\mathfrak{A}(X))^{-1} = \sum [(\mathfrak{A}(f^+))^{-1} \cdot \prod X_x^{f_x}]$.

Negating both sides of this identity, we get $\mathfrak{A}(X) = \prod_f [\mathfrak{A}(f^+) \text{ v } \sum_x X_x^{-f_x}]$.

Hence defining A by $A_f \,.\equiv.\, \mathfrak{A}(f^+)$,

$$\mathfrak{A}(X) = \sum_f [A_f \cdot \prod_x X_x^{f_x}] = \prod_f [A_f \text{ v } \sum X_x^{-f_x}].$$

These representations are called the *orthogonal expansions* of $\mathfrak{A}(X)$, or precisely, "$\sum\Pi$-*expansion*" and "$\Pi\sum$-*expansion*" respectively. The A_f, f-component of A, is called the *f-component* of $\mathfrak{A}(X)$ and A the *associated* pf with $\mathfrak{A}(X)$.

THEOREM 2. *The correspondence* $\bar{A} \rightarrow \mathfrak{A}(X)$ *is an isomorphism from* $\overline{\mathfrak{P}^{\Phi}}$ *to* $\mathfrak{P}[X]$. *Hence, to within equivalence, the orthogonal expansions and the components of a pp are unique.*

Proof. The uniqueness of this correspondence is obvious. Replacing X by g^+ in the expansions(*), we get $Ag \;.\equiv.\; \mathfrak{A}(g^+)$. This shows the univalence of the correspondence. Let A and B be associated pf with $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$ and $\mathfrak{A}(X) = (\mathfrak{B}(X))^{-1}$, then, replacing X by f^+, we get $A_f \,.\equiv.\, B_f^{-1}$. The proof of the remainder is the same.

COROLLARY. $\mathfrak{A}(X) \,.\equiv.\, \prod_{f\in\Phi} A_f$, *where* A *is the associated pf with* $\mathfrak{A}(X)$.

Proof. $\mathfrak{A}(X) \rightleftarrows \mathfrak{A}(X) = 1 \rightleftarrows \mathfrak{A}(X) = \sum_f \prod_x X_x^{f_x}$ (by lemma 9)

$\rightleftarrows \mathfrak{A}(X) = \sum_f [1 \cdot \prod_x X_x^{f_x}] \rightleftarrows \prod_f A_f \,.\equiv.\, 1$ (by uniqueness of components) $\rightleftarrows$

$\prod_f A_f$.

THEOREM 3. *For any tf* $\mathfrak{A}$

$$\mathfrak{A} \sim \sum_{f} [A_f \cdot \prod_{x} X_x^{f_x}] = \prod_{f} [A_f \cdot \sum_{x} X_x^{-f_x}]$$

where A *is a pf on* Φ *such that* $A_f . \equiv . \mathfrak{A}_{f^+}$ *for every* $f \in \Phi$. *Hence, the correspondence* $\bar{A} \rightarrow \mathfrak{A}$ *is an isomorphism from* $\overline{\mathfrak{P}^{\Phi}}$ *to* $\mathfrak{T}$, *and* $\mathfrak{A} . \equiv . \prod_{f \in \Phi} A_f$.

Proof. Define A, as designated in this theorem, then

$$\mathfrak{A}_{g^+} \rightleftarrows A_g \rightleftarrows \sum_{f} [A_f \cdot \prod_{x} (g_x^+)^{f_x}] .$$

Hence by the definition 7 $\mathfrak{A} \sim \sum_{f} [A_f \cdot \prod_{x} X_x^{f_x}]$.

We also use the terminology "orthogonal expansion", "associated pf", etc. for this case.

REMARK. Roughly speaking, this theorem shows that one can build up any tf from the five basic operations: sum, product, negation, gs and gp. The last two can be replaced by existensive quantifier and universal quantifier.

COROLLARY. $\mathfrak{T} \cong \mathfrak{P}[X]$.

Proof. This is a combination of this theorem and the theorem 2.

THEOREM 4. *Every pp or tf with many variables, say a tf* $\mathfrak{A}$ *defined on* $\mathfrak{P}^R \times \cdots \times \mathfrak{P}^S$, *can be expanded orthogonally*:

$$\mathfrak{A} \sim \sum_{f \in B^R} \cdots \sum_{g \in B^S} [A_{f \ldots g} \cdot \prod_{x \in R} X_x^{f_x} \cdot \cdots \cdot \prod_{y \in S} Y_y^{g_y}] ,$$
$$= \prod_{f \in B^R} \cdots \prod_{g \in B^S} [A_{f \ldots g} . \mathrm{v} . \sum_{x \in R} X_x^{-f_x} . \mathrm{v} . \cdots . \mathrm{v} . \sum_{y \in S} Y_y^{-g_y}] ,$$

where $A_{f \ldots g} . \equiv . \mathfrak{A}_{f^+ \ldots g^+}$. *Hence* $\bar{A} \rightarrow \mathfrak{A}$ *is an isomorphism and* $\mathfrak{A} . \equiv . \prod_{f \in B^R} \cdots \prod_{g \in B^S} A_{f \ldots g}$.

§ 4. The characteristic lattice and the character.

We introduce the ordering relation $-1 < 1$ in B. Then, not only B but also B^R becomes Boolean algebra.

LEMMA 11. 1) $f_x^- . \equiv . (f')_x^+$ (where f' is the lattice complement of f)

2) $f_x^+ \mathrm{~v~} g_x^+ . \equiv . (f \smile g)_x^+$,

3) $f_x^+ \cdot g_x^+ . \equiv . (f \frown g)_x^+$.

Now we shall establish the relation between Φ, $\mathfrak{P}[X]$ and $\mathfrak{T}$. First, we define a subset A of Φ corresponding to a pp $\mathfrak{A}(X)$ or a tf $\mathfrak{A}$ by the

DEFINITION 8. *$f \in A \,.\!\equiv\!.\, A_f$, where the A of A_f is the associated pf with $\mathfrak{A}(X)$ or $\mathfrak{A}$.*

Then, from theorem 2 or 3

(II) *The correspondence $\mathfrak{A}(X) \rightarrow A$ or $\mathfrak{A} \rightarrow A$ is one-to-one.*

For this reason we call Φ the *characteristic lattice* of $\mathfrak{P}[X]$ or $\mathfrak{T}$, and A the *character* of $\mathfrak{A}(X)$ or $\mathfrak{A}$.

Into 2^{Φ}, we introduce the Boolean operations join, meet and complement by set union, intersection and set complement, and denote them by $+$, $\cdot$ and $^{-1}$ respectively.

THEOREM 5. *The correspondence $\mathfrak{A}(X) \rightarrow A$ or $\mathfrak{A} \rightarrow A$ is an isomorphism of $\mathfrak{P}[X]$ to 2^{Φ} or of $\mathfrak{T}$ to 2^{Φ}. Hence $\mathfrak{P}[X]$ or $\mathfrak{T}$ is a Boolean algebra whose cardinal is 2^{2^R}.*

Proof. Let A and B be characters of $\mathfrak{A}(X)$ and $\mathfrak{B}(X)$, then by the corollary of theorem 2

$$\mathfrak{A}(X) \supset \mathfrak{B}(X) \rightleftarrows (\mathfrak{A}(X))^{-1} \vee \mathfrak{B}(X) \rightleftarrows \prod_f (A_f^{-1} \,.\mathrm{v}.\, B_f)$$

$$\rightleftarrows \prod_f (f \in A \,.\!\supset\!.\, f \in B) \rightleftarrows A \subseteqq B\,.$$

The assertion of this theorem is an easy consequence of this fact and (II).

CHAPTER II. APPLICATION TO LOGIC.

In this chapter, we discuss the structure of the system of "Prädikatenkalkül" of HILBERT and ACKERMANN[1].

§ 1. Quantifiers.

First, we shall consider the notion of deducibility. Take a formula $\mathfrak{A}_x$ which has no free variable other than x. It is natural to consider that the deducibility of this formula is independent of the

(1) Cf. HILBERT and ACKERMANN, loc. cit., pp. 55-57.

variable x. Hence let the formula $(\text{Ɑ}x)\,\mathfrak{A}_x$ denote that $\mathfrak{A}_x$ is deducible, then we can consider that the former contains no free variables. We consider the symbol (Ɑ) as a kind of quantifier. Thus we have three quantifiers $(\forall)$, $(\exists)$ and (Ɑ). Moreover, these quantifiers have the property (I) of lemma I 1. Thus, we introduce the notion of formally defined quantifiers:

DEFINITION 1. *The formally defined quantifiers* $(\forall)$, $(\exists)$ *and* (Ɑ) *are tf on* $\mathfrak{F}$, *defined by the*

POSTULATE [1] 1. $(\text{Ɑ}y)\,[\,(\forall x)F_x\ .\supset.\ F_y\,]$,

2. $(\text{Ɑ}y)\,[F_y\ .\supset.\ (\exists x)F_x\,]$,

3. $(\text{Ɑ}x)F_x \cdot (\text{Ɑ}x)\,[F_x \supset G_x]\ .\supset.\ (\text{Ɑ}x)G_x$,

4. $(\text{Ɑ}x)\,[P \supset F_x]\ .\supset.\ P \supset (\forall x)F_x$,

5. $(\text{Ɑ}x)\,[F_x \supset P]\ .\supset.\ (\exists x)F_x \supset P$,

where $F,\ G \in \mathfrak{F}$ *and* P *is a proposition.*

§ 2. Relations between characters of quantifiers.

Let A, E and D be the characters of $(\forall)$, $(\exists)$ and (Ɑ). Then we get

LEMMA 1. i) $(\text{Ɑ}y)\,[(\forall x)\,F_x \supset F_y]\ :\equiv:\ A \subseteqq D \cdot (A = \Phi\ .\text{v}.\ \mathrm{I} \in D)$,

ii) $(\text{Ɑ}y)\,[F_y \supset (\exists x)\,F_x]\ :\equiv:\ (D^{-1})' \subseteqq E \cdot (E = \phi\ .\text{v}.\ \mathrm{I} \in D)$,

iii) $(\text{Ɑ}x)F_x \cdot (\text{Ɑ}x)\ [F_x \supset G_x]\ .\supset.\ (\text{Ɑ}x)\ G_x\ :\equiv:\ D' \smile D^{-1} \subseteqq D^{-1}$,

iv) $(\text{Ɑ}x)\,[P \supset F_x]\ .\supset.\ P \supset (\forall x)\ F_x\ :\equiv:\ D \subseteqq A$,

v) $(\text{Ɑ}x)\,[F_x \supset P]\ .\supset.\ (\exists x)\ F_x \supset P\ :\equiv:\ E \subseteqq (D^{-1})'$,

where I *is the greatest element of* Φ, ϕ *is the null set,* $A' = \{f'\,;\, f \in A\}$, *and* $A \smile B = \{f \smile g\,;\, f \in A \cdot g \in B\}$.

Proof. By the corollary of theorem I 2

$(\text{Ɑ}y)\,[(\forall x)\,F_x \supset F_y] \rightleftarrows \prod_f \{(\text{Ɑ}y)\ [(\forall x)\,f^+_x \supset f^+_y]\} \rightleftarrows \prod_f \{(\text{Ɑ}y)\,[A_f \supset f^+_y]\}$
$\rightleftarrows \prod_f\ \{(\text{Ɑ}y)\,[\langle A_f^{-1}\rangle^+_y\ .\text{v}.\ f^+_y]\}$ **(where $\langle P\rangle$ is the greatest or the least element of**

(1) Postulates 1 and 2 are from " Axiom für 'alle' und 'es gibt' ", postulate 3 is from "Schlussschema", and postulates 4 and 5 are from " Schema für 'alle' und 'er gibt' ". Cf. the book loc. cit., pp. 55 57.

Φ according as P .$\equiv$. 1 or 0) $\rightleftarrows \prod_f D_{\langle A_f^{-1}\rangle \smile f} \rightleftarrows \prod_f [\langle A_f^{-1}\rangle \smile f \in D] \rightleftarrows \prod_f$ $\{(A_f \ .\mathrm{v}.\ \mathrm{I} \smile f \in D) \cdot (A_f^{-1} \ .\mathrm{v}.\ 0 \smile f \in D)\}$ (notice that $(A_f .\mathrm{v}. \mathrm{I} \smile f \in D) \cdot (A_f^{-1}$ $.\mathrm{v}.\ 0 \smile f \in D)$ is the orthogonal expansion of $\langle A_f^{-1}\rangle \smile f \in D) \rightleftarrows \prod_f \{(f \in A \ .\mathrm{v}.$ $\mathrm{I} \in D) \cdot (f \notin A \ .\mathrm{v}.\ f \in D)\} \rightleftarrows \prod_f \ \{f \in A \ .\supset.\ f \in D\} \cdot \{\prod_f (f \in A) \ .\mathrm{v}.$ $\mathrm{I} \in D\} \rightleftarrows A \subseteq D \cdot (A = \Phi \ .\mathrm{v}.\ \mathrm{I} \in D)$.

Thus i) is proved. The proof of ii) is the same.

By the theorem I 4

$$(\mathit{\Pi} x) F_x \cdot (\mathit{\Pi} x) [F_x \supset G_x] \ .\supset.\ (\mathit{\Pi} x) G_x \rightleftarrows \prod_{f,g} \{(\mathit{\Pi} x) f_x^+ \cdot (\mathit{\Pi} x) [f_x^+ \supset g_x^+] \ .\supset.\ (\mathit{\Pi} x) g_x^+\} \rightleftarrows \prod_{f,g} \{D_f \cdot (\mathit{\Pi} x) (f' \smile g)_x^+ \ .\supset.\ D_g\} \rightleftarrows \prod_{f,g} (D_f \cdot D_{f' \smile g} \ .\supset.\ D_g) \rightleftarrows \prod_{f,g} (f \in D \cdot g \in D^{-1} \ .\supset.\ f' \smile g \in D^{-1}) \rightleftarrows D' \smile D^{-1} \subseteq D^{-1}.$$

Thus we obtain iii).

Regarding the left hand member of iv) as a tf with variable P and F, and using the theorem I 4, we get

$$(\mathit{\Pi} x)[P \supset F_x] \ .\supset.\ P \supset (\mathit{V} x) F_x \rightleftarrows \prod_f \ [\{(\mathit{\Pi} x)[1 \supset f_x^+] \ .\supset.\ 1 \supset (\mathit{V} x) f_x^+\} \cdot \{(\mathit{\Pi} x)[0 \supset f_x^+] \ .\supset.\ 0 \supset (\mathit{V} x) f_x^+\}] \ \rightleftarrows \ \prod_f \{(\mathit{\Pi} x) f_x^+ \supset (\mathit{V} x) f_x^+\} \rightleftarrows \prod_f (D_f \supset A_f) \rightleftarrows \prod_f (f \in D \ .\supset.\ f \in A) \rightleftarrows D \subseteq A.$$

We can prove v) similarly.

§ 3. Relations of quantifiers.

LEMMA 2. *If postulates* 1 *and* 4 *hold, then* $(\mathit{V} x) F_x = (\mathit{\Pi} x) F_x$.

Proof. From i) of lemma 1 $A \subseteq D$, and from iv) $D \subseteq A$. Hence $A = D$.

LEMMA 3. *If postulates* 2 *and* 5 *hold, then* $(\exists x) F_x = \{(\mathit{\Pi} x) F_x^{-1}\}^{-1}$.

Proof. From ii) and v) of lemma 1 $(D^{-1})' = E$, then

$$(\exists x) F_x = \prod_f [E_f \ \mathrm{v} \sum_x F_x^{-f_x}] = \prod_f [(D^{-1})'_f \ \mathrm{v} \sum_x F_f^{-f_x}] = \prod_f [D_f^{-1} \ \mathrm{v} \sum_x (F^{-1})_x^{-f_x}] = \{\prod_f [D_f \ \mathrm{v} \ \sum_x (F^{-1})_x^{-f_x}]\}^{-1} = \{(\mathit{\Pi} x) F_x^{-1}\}^{-1}.$$

LEMMA 4. $\mathrm{I} \in A$ *and* $A' \smile A^{-1} \subseteq A^{-1}$ *if and only if* A *is a non-void dual ideal.*

Proof. *Necessity.* Take $f \in A$. If there exists some $g \geqq f$ such that $g \notin A$, then $g \in A^{-1}$. Hence $f' \smile g \in A' \smile A^{-1} \subseteqq A^{-1}$, that is $f' \smile g \in A^{-1}$. While $\mathrm{I} \geqq f' \smile g \geqq f' \smile f = \mathrm{I}$, therefore $\mathrm{I} \in A^{-1}$. Since $\mathrm{I} \in A$, this is a contradiction. Thus we obtain that A is J-closed.

Take f, $g \in A$, and assume that $f \frown g \notin A$. Then $f \frown g \in A^{-1}$. Hence $f' \smile g = f' \smile (f \frown g) \in A' \smile A^{-1} \subseteqq A^{-1}$, that is $f' \smile g \in A^{-1}$. While $g \in A$ and $f' \smile g \geqq g$. Since A is J-closed, this is a contradiction. Thus, if f, $g \in A$, then $f \frown g \in A$.

Sufficiency. From $A \neq \phi$, there exists a $f \in A$. Then, since A is J-closed and $\mathrm{I} \geqq f$, $\mathrm{I} \in A$.

Assume that there exist two elements $f \in A'$ and $g \in A^{-1}$ such that $f \smile g \notin A^{-1}$. Then $f \smile g \in A$. While A is a dual ideal, hence $f' \frown g = f' \frown (f \smile g) \in A$. Then $g \in A$, for A is J-closed and $g \geqq f' \frown g$. This is contrary to the assumption $g \in A^{-1}$.

LEMMA 5. *If R is finite, say $R = \{1, 2, \cdots, n\}$, and if postulates* 1, 3 *and* 4 *hold, then there are $i_1, i_2, \cdots, i_m$, $m \leqq n$, $i_k \leqq n$, $i_k \neq i_l\ (k \neq l)$ and $(\forall x) F_x = F_{i_1} \cdot F_{i_2} \cdot \cdots \cdot F_{i_m}$.*

Proof. From lemma 2 $D = A$. Then from postulate 3 $A' \smile A^{-1} \subseteqq A^{-1}$, and from postulate 1 $A = \Phi . \mathrm{v}. \mathrm{I} \in A \Longrightarrow \mathrm{I} \in A . \mathrm{v}. \mathrm{I} \in A \Longrightarrow \mathrm{I} \in A$.

Then by lemma 4, A is a non-void dual ideal. Since A is finite, there exists an element g of A and $A = J(g)$, J-closure of g. Let $g_1 = g_2 = \cdots = g_m = 1$ and $g_{m+1} = \cdots = g_n = -1$ for instance, then

$$A_f \rightleftarrows f \in A \rightleftarrows f \in J(g) \rightleftarrows f \geqq g \rightleftarrows f_1 = \cdots = f_m = 1 .$$

Therefore

$$(\forall x) F_x = \sum_f [A_f \cdot \prod_{x=1}^{n} F_x^{f_x}] = \sum_{f_1} \cdots \sum_{f_m} \sum_{f'} [f_1 = \cdots = f_m = 1 \cdot \prod_{x=1}^{n} F_x^{f_x}]$$

(where $f' = (f_{m+1}, \cdots, f_n)$) $$= \sum_{f'} [F_1 \cdot \cdots \cdot F_m \cdot \prod_{x=m+1}^{n} F_x^{f_x}] = F_1 \cdot \cdots \cdot F_m .$$

Generally, let $g_x = 1$ for $x = i_1, i_2, \cdots, i_m$ and $= -1$ otherwise, then $(\forall x) F_x = F_{i_1} \cdot F_{i_2} \cdot \cdots \cdot F_{i_m}$.

THEOREM 1. *Let $R = \{1, 2, \cdots, n\}$, then postulates* 1, 2, $\cdots$, 5 *hold if and only if*

i) $(\mathrm{Q}x) F_x = (\forall x) F_x$,

ii) $(\exists x) F_x = \{ (\forall x) F_x^{-1} \}^{-1}$,

iii) $(\forall x) F_x = F_{i_1} \cdot F_{i_2} \cdot \cdots \cdot F_{i_m}$.

Proof. If postulates 1, $\cdots$, 5 hold, then i) ii) and iii) are given by lemmas 2, 3 and 5.

Conversely, if i), ii) and iii) hold, then by direct calculation, it is easily seen that the quantifiers thus defined fulfil the postulates 1, $\cdots$, 5.