SOME REMARKS CONCERNING IDENTITY

By

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In mathematics, the concept of identity is most fundamental, and in all mathematical deductions, it plays an important rôle. But hitherto, as far as I am aware, the treatment of this notion has not been carried on to its ultimate stage, so one is liable to lose sight of the delicate functions it fulfils in mathematical deductions.

In this note, I will try to treat this notion thoroughly, and show that, by doing so, mathematical deductions become much more formal and accurate.

After having explained the notations in § 1, I will discuss in § 2 some formulas concerning identity and their relations, and then in § 3, some of their applications.

§ 1. Notations.

(1) Propositions: $P, Q, \cdots$; Propositional functions: $F(x)$, $G(x, y)$, $\cdots$.

(2) Logical sum (or): v; logical product (and): $\cdot$; Negation: $\overline{}$; Implication: $\supset$; Equivalence: $\equiv$; Universal quantifier: (V) ; Existensive quantifier: $(\exists)$.

For example, PQ v R means that $(P \cdot Q)$ v R.

(3) Formulas: $\mathfrak{A}$, $\mathfrak{B}(x)$, $\cdots$.

(4) In mathematical deductions, we use two auxiliary notations $\rightarrow$ and $\Longrightarrow$. $\mathfrak{A} \rightarrow \mathfrak{B}$ if and only if $\mathfrak{A} \supset \mathfrak{B}$ is valid. $\mathfrak{A} \Longrightarrow \mathfrak{B}$ means that $\mathfrak{A}$ is valid and $\mathfrak{B}$ is deducible from $\mathfrak{A}$. If $\mathfrak{A}$ is valid and $\mathfrak{A} \rightarrow \mathfrak{B}$, then $\mathfrak{B}$ is valid. If $\mathfrak{A} \Longrightarrow \mathfrak{B}$, then $\mathfrak{B}$ is valid.

For instance, if $\mathfrak{A}(x)$ is valid, then we get $(Vx)\,\mathfrak{A}(x)$, that is $\mathfrak{A}(x) \Longrightarrow (Vx)\,\mathfrak{A}(x)$.

(5) We use the notations 1 and 0 for the true formulas and the false formulas respectively. $\mathfrak{A} . \equiv . 1$ if and only if $\mathfrak{A}$ is required as one of the axioms of some system with which we are concerned or can be deduced from these axioms. $\mathfrak{A} . \equiv . 0$ if and only if $\overline{\mathfrak{A}} . \equiv . 1$.

$1 \vee \mathfrak{A} . \equiv . 1, \quad 0 \vee \mathfrak{A} . \equiv . 1 \cdot \mathfrak{A} . \equiv . \mathfrak{A},$ etc..

(6) Usual axioms and rules of logic are assumed.[1]

§ 2. Identity.

The axioms of identity are as follows:

(A_1) $x = x$.

(A_2) For any formula $\mathfrak{A}(x)$, $x = y . \supset . \mathfrak{A}(y) \supset \mathfrak{A}(x)$.

As usual, we obtain from these axioms that

2.1 $\quad x = y . \supset . y = x$.

2.2 $\quad x = y \cdot y = z . \supset . x = z$.

From (A_2) and 2.1, we get the

Lemma 1. *For any formula* $\mathfrak{A}(x)$, $\quad x = y \cdot \mathfrak{A}(x) . \equiv . x = y \cdot \mathfrak{A}(y)$.

Though this lemma is a very simple consequence of axioms, this is more convenient than (A_2), for this enables us to calculate mathematical deductions without disturbing equality. So is the following lemma.

Lemma 2. *For any formula* $\mathfrak{A}(x)$,

$$(\mathrm{A})\quad (\exists y)[x = y \cdot \mathfrak{A}(y)] . \equiv . \mathfrak{A}(x).$$

Proof. From (A_2)

$$x = y . \supset . \mathfrak{A}(y) \supset \mathfrak{A}(x) \Longrightarrow x = y \cdot \mathfrak{A}(y) . \supset . \mathfrak{A}(x)$$
$$\Longrightarrow (\forall y)[x = y \cdot \mathfrak{A}(y) . \supset . \mathfrak{A}(x)] \Longrightarrow (\exists y)[x = y \cdot \mathfrak{A}(y)] . \supset . \mathfrak{A}(x).$$

Conversely by (A_1)

$$\mathfrak{A}(x) \Longrightarrow x = x \cdot \mathfrak{A}(x) \Longrightarrow (\exists y)[x = y \cdot \mathfrak{A}(y)].$$

Hence, $\mathfrak{A}(x) . \supset . (\exists y)[x = y \cdot \mathfrak{A}(y)]$.

This formula can be written moreover in the form

$$(\mathrm{A}')\quad (\forall y)[x \neq y . \vee . \mathfrak{A}(y)] . \equiv . \mathfrak{A}(x),$$

$$(\mathrm{A}'')\quad (\forall y)[x = y . \supset . \mathfrak{A}(y)] . \equiv . \mathfrak{A}(x).$$

(1) For example, HILBERT-ACKERMANN: Grundzüge der theoretischen Logik.

Lemma 3. $(\exists y)\,[x = y]$.

Proof. Let $\mathfrak{A}(x) . \equiv . 1$ in (A), then

$$(\exists y)\,[x = y \cdot \mathfrak{A}(y)] \ . \equiv . \ \mathfrak{A}(x) \Rightarrow (\exists y)\ [x = y \cdot 1] \ . \equiv . \ 1$$
$$\Rightarrow (\exists y)\ [x = y] .$$

Theorem. (A) *is equivalent to* (A_1) *and* (A_2).

Proof. That (A) follows from (A_1) and (A_2) was shown already. Assume (A), then we get (A_2) immediately. By lemma 3 and (A)

$$(\exists y)\,[x = y] \Rightarrow (\exists y)\,[x = y \cdot x = y] \Rightarrow x = x,$$

that is, (A_1) is proved.

§ 3. Applications.

In this §, I will show some examples of the application of above lemmas. The propositions and notions refer to K. Gödel.[1]

3. 1 $\quad \{x\,y\} = \{u\,v\} :\equiv: x = u \cdot y = v \ .\mathrm{v}.\ x = v \cdot y = u .$

Proof. $\{x\,y\} = \{u\,v\} \rightleftarrows (\forall t)\,[t \in \{x\,y\} \ . \equiv . \ t \in \{u\,v\}]$

$\rightleftarrows (\forall t)\,[t \neq x \cdot t \neq y \ .\mathrm{v}.\ t = u \ .\mathrm{v}.\ t = v]\ (\forall t)\,[t \neq u \cdot t \neq v \ .\mathrm{v}.\ t = x \ .\mathrm{v}.\ t = y] \rightleftarrows (\forall t)\,[t \neq x \ .\mathrm{v}.\ t = u \ .\mathrm{v}.\ t = v]\ (\forall t)\,[t \neq y \ .\mathrm{v}.\ t = u \ .\mathrm{v}.\ t = v]\ (\forall t)\,[t \neq u \ .\mathrm{v}.\ t = x \ .\mathrm{v}.\ t = y]\ (\forall t)\,[t \neq v \ .\mathrm{v}.\ t = x \ .\mathrm{v}.\ t = y]$ (by (A′)) $\rightleftarrows (x = u \ .\mathrm{v}.\ x = v)\ (y = u \ .\mathrm{v}.\ y = v)\ (u = x \ .\mathrm{v}.\ u = y)\ (v = x \ .\mathrm{v}.\ v = y) \rightleftarrows \{(x = u \ .\mathrm{v}.\ x = v)\ (y = v \ .\mathrm{v}.\ x = v)\}\ \{(x = u \ .\mathrm{v}.\ y = u)\ (y = v \ .\mathrm{v}.\ y = u)\}$

$\rightleftarrows (x = u \cdot y = v \ .\mathrm{v}.\ x = v)\ (x = u \cdot y = v \ .\mathrm{v}.\ y = u)$

$\rightleftarrows x = u \cdot y = v \ .\mathrm{v}.\ x = v \cdot y = u .$

3. 2 $\quad \langle x\,y \rangle = \langle u\,v \rangle :\equiv: x = u \cdot y = v$.[2]

Proof. $\langle x\,y \rangle = \langle u\,v \rangle \rightleftarrows \{\,\{x\}\,\{x\,y\}\,\} = \{\,\{u\}\,\{u\,v\}\,\}$

$\rightleftarrows \{x\} = \{u\} \cdot \{x\,y\} = \{u\,v\} \ .\mathrm{v}.\ \{x\} = \{u\,v\} \cdot \{x\,y\} = \{u\}$

$\rightleftarrows x = u \cdot (x = u \cdot y = v \ .\mathrm{v}.\ x = v \cdot y = u) \ .\mathrm{v}.\ x = u \cdot x = v \cdot x = u \cdot y = u$

$\rightleftarrows x = u \cdot y = v \ .\mathrm{v}.\ x = u \cdot x = v \cdot y = u$

$\rightleftarrows x = u \cdot y = v$ (by lemma 1 and absorptive law in lattice theory).

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(1) Kurt Gödel; The consistency of the axiom of choice and of generalized continuum-hypothesis with the axioms of set theory, 1940.

(2) K. Gödel, loc. cit., p. 4.

3. 3 $x \notin x$[1].

Proof. In axiom D[2], putting $A = \{x\}$, we get

$$\overline{\mathfrak{Em}(A)} \;.\supset.\; (\exists u)\;[u \in A \cdot \mathfrak{Ex}\;(uA)\,]$$

$$\Longrightarrow \overline{\mathfrak{Em}(\{x\})} \;.\supset.\; (\exists u)\;[u \in \{x\} \cdot \mathfrak{Ex}\,(u\,\{x\})]$$

$$\Longrightarrow \overline{(\forall u)\,[u \notin \{x\}]} \;.\supset.\; (\exists u)\,[u{=}x \cdot (\forall v)\;\overline{(v \in u \cdot v \in \{x\})}\qquad \text{(by (A))}$$

$$\Longrightarrow \overline{(\exists u)\,[u{=}x]} \;.\mathrm{v}.\; (\forall v)\;\overline{[v \in x \cdot v \in \{x\}]}\qquad \text{(by lemma 3)}$$

$$\Longrightarrow \overline{\mathrm{I}} \;.\mathrm{v}.\; (\forall v)\,[v \notin x \;.\mathrm{v}.\; v{\neq}x]\quad \text{(by (A'))} \;\Longrightarrow x \notin x.$$

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(1), (2) K. Gödel, loc. cit., p. 6.