ON FINITE MODULARS

By

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The modulared semi-ordered linear space[1] R is a universally continuous semi-ordered linear space associated with a functional $m(a)$ $(a \in R)$ satisfying the following conditions:

1) $0 \leqq m(a) \leqq +\infty$ for every $a \in R$;

2) if $m(\xi a) = 0$ for every positive number $\xi > 0$, then we have $a = 0$;

3) for any $a \in R$ we can find a positive number ξ such that $m(\xi a) < +\infty$;

4) $m(\xi a)$ is a convex function of $\xi > 0$, that is, $\alpha, \beta > 0$ implies $m\left(\frac{\alpha+\beta}{2} a\right) \leqq \frac{1}{2} \{m(\alpha a) + m(\beta a)\}$ for every $a \in R$;

5) $|a| \leqq |b|$ implies $m(a) \leqq m(b)$;

6) $a \frown b = 0$ implies $m(a+b) = m(a) + m(b)$;

7) $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda} a$ implies $m(a) = \sup_{\lambda \in \Lambda} m(a_\lambda)$.

This functional $m(a)$ $(a \in R)$ is called a *modular* on this modulared semi-ordered linear space R.

If $m(\xi a) < +\infty$ for every $\xi > 0$, then a is called a *finite element.* When every element is finite, then the modular is called a *finite modular.* If there exists a number $\gamma > 0$ such that

$$m(2\xi a) \leqq \gamma m(\xi a) \qquad \text{for every} \quad \xi > 0,$$

then a is called an *upper bounded element.* When the γ is uniquely determined for every $a \in R$, then the modular is called an *upper bounded modular*.

It is easily seen that the upper boundedness implies finiteness, but the converse is not always true. For example, the following one is interesting: A collection of sequences $x = (\xi_\nu)$ such that

$$m(\alpha x) = \sum_{\nu=1}^{\infty} (e^{\alpha |\xi_\nu|} - 1) < +\infty \qquad \text{for some} \quad \alpha > 0$$

1) H. NAKANO, Modulared semi-ordered linear spaces, Tokyo Mathematical Book Series, Vol. I (1950).

is a modulared semi-ordered linear space. Let

$$\sum_{\nu=1}^{\infty}(e^{|\xi_\nu|}-1)<+\infty .$$

Then we have

$$\sum_{\nu=1}^{\infty}(e^{2|\xi_\nu|}-1)=\sum_{\nu=1}^{\infty}(e^{|\xi_\nu|}+1)(e^{|\xi_\nu|}-1)\leqq\gamma\sum_{\nu=1}^{\infty}(e^{|\xi_\nu|}-1)<+\infty ,$$
$$\gamma=\sup_{\nu\geqq1}(e^{|\xi_\nu|}+1) ,$$

hence this modular is finite. But, it is obvious that this is not upper bounded.

The object of this paper is to investigate the relation between finiteness and upper boundedness.

§ 1. *A historical note.*

Let $M(\xi)$ be a continuous, convex function of $\xi>0$ such that the following conditions are satisfied:

1) $M(\xi)=0$ if and only if $\xi=0$;
2) $\lim_{\xi\to0}\frac{M(\xi)}{\xi}=0$, $\lim_{\xi\to+\infty}\frac{M(\xi)}{\xi}=+\infty$.

A set (L^M) of measurable functions $x(t)\,(0\leqq t\leqq1)$ such that

$$\int_0^1 M(\alpha|x(t)|)dt<+\infty \qquad \text{for some} \quad \alpha>0$$

was firstly defined and investigated by W. ORLICZ.[2] Similarly, he defined the space (l^M) of sequences (ξ_ν) such that

$$\sum_{\nu=1}^{\infty}M(\alpha|\xi_\nu|)<+\infty \qquad \text{for some} \quad \alpha>0 .$$

Concerning this function $M(\xi)$, W. ORLICZ and Z. B. BIRNBAUM[3] have obtained the following proposition:
(L^M)-*case*: *In order that*

$$\int_0^1 M(|x(t)|)\,dt<+\infty \quad \textit{implies} \quad \int_0^1 M(2|x(t)|)dt<+\infty ,$$

it is necessary and sufficient that we can find $\alpha,\gamma>0$ *such that*

2) W. ORLICZ, Ueber eine gewisse Klasse von Räumen vom Typus (B), Bulletine de l'Akademie Polonaise des Science et des Lettres. (1932) pp. 207–220, Ueber Räume (L^M), ibid. (1936) pp. 93–107.

3) Z. B. BIRNBAUM and W. ORLICZ, Ueber die Verallgemeinerung des Begriffes der zueinander konjugierten Potenzen, Studia Math. III (1931) pp. 1–67.

$$M(2\xi) \leqq \gamma M(\xi) \qquad \text{for} \quad \xi > \alpha.$$

(l^M)-*case*: *In order that*

$$\sum_{\nu=1}^{\infty} M(|\xi_\nu|) < +\infty \quad \textit{implies} \quad \sum_{\nu=1}^{\infty} M(2|\xi_\nu|) < +\infty ,$$

it is necessary and sufficient that we can find $\alpha, \gamma > 0$ *such that*

$$M(2\xi) \leqq \gamma M(\xi) \qquad \text{for} \quad 0 < \xi < \alpha.$$

This proposition provides a clue for our problem.

§ 2. *A general property of finite modulars.*

In this section we will show that the notion of finiteness contains that of upper boundedness in a certain extended sense.

Let $[a]R$ be finite and $\xi a (\xi > 0)$ be simple. Since we have

$$m(4\nu a) = \int_{[a]} \omega(4, \nu a, \mathfrak{p})\, m(d\mathfrak{p}\nu a)$$

for the modular spectra

$$\omega(4, \nu a, \mathfrak{p}) = \lim_{[p]\to\mathfrak{p}} \frac{m(4\nu[p]a)}{m(\nu[p]a)} ,$$

$\omega(4, \nu a, \mathfrak{p})$ is almost finite in $U_{[a]}$. Hence, there exists a sequence of projectors $[p_{\mu,\nu}]$ such that

$$[p_{\mu,\nu}] \uparrow_{\mu=1}^{\infty} [a]$$

and

$$\omega(4, \nu a, \mathfrak{p}) \leqq \mu \qquad (\mathfrak{p} \in U_{[p_{\mu,\nu}]}).$$

As $[a]R$ is totally continuous, we can find a sequence of projectors $[p_\rho] (\rho = 1, 2, \cdots)$ such that $[p_\rho] \uparrow_{\rho=1}^{\infty} [a]$ and

$$[p_\rho] \leqq [p_{\mu_{\nu,\rho},\nu}] \qquad (\nu, \rho = 1, 2, \cdots).$$

Therefore, we have

$$\omega(4, \nu a, \mathfrak{p}) \leqq \mu_{\nu,\rho} \qquad (\mathfrak{p} \in U_{[p_\rho]})$$

and hence

$$m(4\nu[p]a) \leqq \mu_{\nu,\rho}\, m(\nu[p]a) \qquad ([p] \leqq [p_\rho]).$$

Hence it appears that

$$m(2\xi[p]a) \leqq \mu_{\nu,\rho}\, m(\xi[p]a) \qquad \text{for } [p] \leqq [p_\rho] \text{ and } 1 \leqq \xi \leqq \nu .$$

Applying the same argument to $\frac{1}{\nu} a$, we can find a sequence of projectors $[p_{\rho'}]$ $(\rho'=1, 2, \cdots)$ such that

$$[p_{\rho'}] \uparrow_{\rho'=1}^{\infty} [a] ,$$

and for $[p] \leqq [p_{\rho'}]$ and $\frac{1}{\nu} \leqq \xi \leqq 1$ we have

$$m(2\xi[p]a) \leqq \mu_{\nu,\rho'} m(\xi[p]a) .$$

Hence we have

$$[p_\rho][p_{\rho'}] \uparrow [a]$$

and for $[p] \leqq [p_\rho][p_{\rho'}]$ and $\frac{1}{\nu} \leqq \xi \leqq \nu$ we have

$$m(2\xi[p]a) \leqq \max.(\mu_{\nu,\rho}, \mu_{\nu,\rho'}) \cdot m(\xi[p]a) .$$

That is:

Theorem. *Whenever $[a]R$ is finite and simple, we can find a sequence of projectors $[p_\nu]$ and numbers $\gamma_\nu \uparrow_{\nu=1}^{\infty}$ such that*

$$[p_\nu] \uparrow_{\nu=1}^{\infty} [a] ,$$

and for $[p] \leqq [p_\nu]$ and $\frac{1}{\nu} \leqq \xi \leqq \nu$ we have

$$m(2\xi[p]a) \leqq \gamma_\nu \cdot m(\xi[p]a) .$$

N. B. 1. Let a be itself *atomic*, namely $0 \neq [p] \leqq [a]$ implies $[p]=[a]$. In this case, we have

$$\omega(4, \nu a, \mathfrak{p}) = m(4\nu a)/m(\nu a) \equiv \alpha_\nu < +\infty \qquad (\mathfrak{p} \in U_{[a]}) ,$$

and hence,

$$m(4\nu a)/m(\nu a) \leqq \mathrm{Max.}(\alpha_1, \cdots, \alpha_\nu) \qquad (\nu = 1, 2, \cdots, \nu) .$$

Similarly we have

$$m\left(4\frac{1}{\nu}a\right)\Big/m\left(\frac{1}{\nu}a\right) \leqq \mathrm{Max.}(\beta_1, \cdots, \beta_\nu) \qquad (\nu = 1, 2, \cdots, \nu)$$

for $\beta_\nu = \omega\left(4, \frac{1}{\nu}a, \mathfrak{p}\right)$. Therefore we have

$$m(2\xi a) \leqq \gamma_\nu m(\xi a) ,$$

where $\gamma_\nu \uparrow_{\nu=1}^{\infty} +\infty$ and $\frac{1}{\nu} \leqq \xi \leqq \nu$.

N. B. 2. Let a be *finite dimensional*, that is,

$$[a]=[p_1]+[p_2]+\cdots+[p_\kappa], \qquad [p_\nu][p_\mu]=0 \qquad (\nu \neq \mu),$$

and every $[p_\nu]$ is atomic. Then there exist sequences of numbers $\gamma_{\mu,\nu}\uparrow_{\nu=1}^{\infty}$ such that

$$m(2\xi[p_\mu]a) \leqq \gamma_{\mu,\nu}\, m(\xi[p_\mu]a) \qquad \text{for } \frac{1}{\nu} \leqq \xi \leqq \nu .$$

Hence, summing up the both sides, we have

$$m(2\xi a) \leqq (\gamma_{\nu,1}+\gamma_{\nu,2}+\cdots+\gamma_{\nu,\kappa})\, m(\xi a) = \gamma_\nu\, m(\xi a),$$

where

$$\gamma_\nu = \sum_{\mu=1}^{\kappa} \gamma_{\mu,\nu} \uparrow_{\nu=1}^{\infty} \quad \text{and} \quad \frac{1}{\nu} \leqq \xi \leqq \nu .$$

N. B. 3. Even if a one dimensional modular $f(\xi)$ is finite and simple, we can not always find such numbers $\alpha, \gamma > 0$ that

1) $\qquad f(2\xi) \leqq \gamma \cdot f(\xi) \qquad (0 < \xi < \alpha)$,

2) $\qquad f(2\xi) \leqq \gamma \cdot f(\xi) \qquad (\xi > \alpha)$.

For instance,

$$f(\xi) = e^{\xi} - 1$$

satisfies 1) for some $\alpha, \gamma > 0$ but not 2), and

$$f(\xi) = \begin{cases} 1/\nu\,! & \left(\xi = \frac{1}{\nu} (\nu = 2, 3, \cdots)\right) \\ \textit{linear} & (\textit{elsewhere}) \end{cases}$$

satisfies 2) but not 1).

§ 3. *Modulared sequence spaces.*

In this section, we will settle the problem in the case when R is a *modulared sequence space* $l(f_\nu)$.[4] It may be regarded as the most general sequence space which satisfies the modular conditions.

When all f_ν are equal, that is, $f=f_\nu$ $(\nu=1, 2, \cdots)$ for some f, then $l(f)$ is an ORLICZ space. In this case, if $l(f)$ is finite then $l(\alpha_\nu f)$ is also finite for any integer $\alpha_\nu \geqq 1$. In fact, if

$$\sum_{\nu=1}^{\infty} \alpha_\nu f(\xi_\nu) < +\infty ,$$

then, putting

4) S. YAMAMURO, Modulared sequence spaces, this journal.

$$\eta_1 = \xi_1, \cdots\cdots\cdots\cdots, \eta_{\alpha_1} = \xi_1,$$
$$\eta_{\alpha_1+1} = \xi_2, \cdots\cdots\cdots\cdots, \eta_{\alpha_1+\alpha_2} = \xi_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,$$

we obtain a sequence (η_ν) such that

$$\sum_{\nu=1}^{\infty} f(\eta_\nu) = \sum_{\nu=1}^{\infty} \alpha_\nu f(\xi_\nu) < +\infty .$$

However we have

$$\sum_{\nu=1}^{\infty} f(2\eta_\nu) = \sum_{\nu=1}^{\infty} \alpha_\nu f(2\xi_\nu) < +\infty ,$$

because $(\xi_\nu) \in l(f)$ and $l(f)$ is finite by the assumption. Therefore $l(\alpha_\nu f)$ is finite.

When the functions f_ν are power functions, that is, when $f_\nu(\xi) = \xi^{p_\nu}$ for $p_\nu \geqq 1$, its modular is *of unique spectra*. In this case, we can find the same property. Namely, if $l(\xi^{p_\nu})$ is finite, then $l(\alpha_\nu \xi^{p_\nu})$ also is finite for any real numbers $\alpha_\nu > 0$. Because, if $l(\xi^{p_\nu})$ is finite, then the sequence p_ν $(\nu = 1, 2, \cdots)$ is bounded, that is, $l(\xi^{p_\nu})$ is upper bounded, and, even, in the general modulared sequence space $l(f_\nu)$, if $l(f_\nu)$ is upper bounded, then $l(\alpha_\nu f_\nu)$ is upper bounded for every sequence $\alpha_\nu > 0$.

But, in general cases, there exists a modulared sequence space $l(f_\nu)$ which is finite, but not $l(\alpha_\nu f_\nu)$ for $\alpha_\nu = \nu$ $(\nu = 1, 2, \cdots)$. For example, the following will satisfy our curiosity.

$$f_\nu(\xi) = \begin{cases} 0 \quad \text{if} \quad \xi \leqq \dfrac{\nu^2-2}{2(\nu^2-1)}, \\ 2\dfrac{\nu^2-1}{\nu^4}\xi + \dfrac{2-\nu^2}{\nu^4} \quad \text{otherweise} . \end{cases}$$

Since $f_\nu(\xi)$ are linear for large $\xi > 0$, $f_\nu(\xi)$ are finite, but not upper bounded. As it is clear that $(\nu^2-2)/2(\nu^2-1) \leqq \dfrac{1}{2}$, we have

$$m\left(\left(\frac{1}{2}\right)\right) = \sum_{\nu=1}^{\infty} f_\nu\left(\frac{1}{2}\right) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^4},$$

$$m((1)) = \sum_{\nu=1}^{\infty} f_\nu(1) = \sum_{\nu=1}^{\infty} \frac{1}{\nu^5}.$$

Putting $\alpha_\nu = \nu$, in the space $l(\nu f_\nu)$, we have

$$m\left(\left(\frac{1}{2}\right)\right)=\sum_{\nu=1}^{\infty}\nu\cdot f_\nu\left(\frac{1}{2}\right)=\sum_{\nu=1}^{\infty}\frac{1}{\nu^3}<+\infty\ ,$$

$$m((1))=\sum_{\nu=1}^{\infty}\nu\cdot f_\nu(1)=\sum_{\nu=1}^{\infty}\frac{1}{\nu}=+\infty\ .$$

which shows that $l(\nu f_\nu)$ is not finite.

Now, we will state our theorem:

Theorem. *If* $l\subset l(f_\nu)$ *and* $l(\alpha_\nu f_\nu)$ *is finite for every sequence of real numbers* $\alpha_\nu\geqq 1$, *then there exist numbers* $\alpha,\gamma>0$ *such that*

$$(*)\qquad f_\nu(2\xi)\leqq\gamma\cdot f_\nu(\xi)\qquad(0\leqq\xi\leqq\alpha)$$

for almost all ν.

Proof. If we can not find such numbers $\alpha,\gamma>0$, then there exists a sequence (ξ_μ) such that

$$f_{\nu_\mu}(2\xi_\mu)>\mu\cdot f_{\nu_\mu}(\xi_\mu)\qquad(\mu=1,2,\cdots),$$

and then we have

$$\frac{1}{\mu^2}\leqq\alpha_\mu\cdot f_{\nu_\mu}(\xi_\mu)<\frac{2}{\mu^2}$$

for a suitably determined sequence α_μ. Therefore we have

$$\sum_{\mu=1}^{\infty}\alpha_\mu\cdot f_{\nu_\mu}(\xi_\mu)\leqq\sum_{\mu=1}^{\infty}\frac{2}{\mu^2}<+\infty\ .$$

On the other hand, we have

$$\sum_{\mu=1}^{\infty}\alpha_\mu\cdot f_{\nu_\mu}(2\xi_\mu)\geqq\sum_{\mu=1}^{\infty}\alpha_\mu\cdot\mu\cdot f_{\nu_\mu}(\xi_\mu)\geqq\sum_{\mu=1}^{\infty}\frac{\alpha_\mu}{\mu}\ .$$

Here we may determine α_μ as $\alpha_\mu\geqq 1$, because $l\subset l(f_\nu)$ implies the existence of such numbers $\alpha>0$ that $\sup_{\nu\geqq 1}f_\nu(\alpha)<+\infty$ and hence we can select ξ_μ as

$$f_{\nu_\mu}(\xi_\mu)\leqq\frac{1}{\mu^2}\qquad(\mu=1,2,\cdots)$$

at the first step. Therefore, we can suppose that $\alpha_\mu\geqq 1$, so that

$$\sum_{\mu=1}^{\infty}\alpha_\mu\cdot f_{\nu_\mu}(2\xi_\mu)=+\infty\ .$$

Putting

$$\eta_\nu=\begin{cases}\xi_\mu & (\nu=\mu)\\ 0 & (\nu\neq\mu)\ ,\end{cases}$$

and

$$\alpha_\nu = \begin{cases} \alpha_\mu & (\nu = \mu) \\ 0 & (\nu \neq \mu), \end{cases}$$

we obtain

$$\sum_{\nu=1}^{\infty} \alpha_\nu f_\nu(\eta_\nu) < +\infty$$

and

$$\sum_{\nu=1}^{\infty} \alpha_\nu \cdot f_\nu(2\eta_\nu) = +\infty .$$

Hence $l(\alpha_\nu f_\nu)$ is not finite, which contradicts the assumption. Therefore, there exist numbers $\alpha, \gamma > 0$ such that

$$f_\nu(2\xi) \leqq \gamma \cdot f_\nu(\xi) \qquad (0 < \xi \leqq \alpha)$$

for almost all ν.

The converse of this theorem is almost evident.

Theorem. *If* $l(f_\nu) \subset m$ *and satisfies* (*), *then* $l(\alpha_\nu f_\nu)$ *is finite for every sequence* $\alpha_\nu > 0$.

§ 4. *General case.*

Let R be a modulared semi-ordered linear space and m be its modular. The modulared semi-ordered linear space can be represented by a function space on its proper space. If all projectors are atomic, then the space may be regarded as the modulared sequence space and the case was considered in the previous section. In this section, we consider the case where no atomic projector exists.

Theorem. *Let* $[a]R$ *be finite. If* $[a]R$ *has no atomic projector and is monotone complete, then for any partition* $[p_\nu]$ *such that*

$$[a] = \sum_{\nu=1}^{\infty} [p_\nu], \quad [p_\nu][p_\mu] = 0 \qquad (\nu \neq \mu),$$

we can find at least one ν *such that*

$$m(2\xi[p_0]a) \leqq \gamma \cdot m(\xi[p_0]a) \qquad (\xi > \alpha)$$

for some real numbers $\alpha, \gamma > 0$ *and a projector* $[p_0] \leqq [p_\nu]$.

Proof. If we can not find such α, γ and ν, then there exist sequences $\xi_\nu \uparrow_{\nu=1}^{\infty} + \infty$ and $[p_\nu] (\nu = 1, 2, \cdots)$ such that

$$[a] = \sum_{\nu=1}^{\infty} [p_\nu] \text{ and } m(2\xi_\nu[p]a) > \nu \cdot m(\xi_\nu[p]a) \quad ([p] \leqq [p_\nu]).$$

Here we can select ξ_ν as $m(\xi_\nu[p_\nu]a) \geqq 1$. Therefore, there exist $[q_\nu]$ $(\nu = 1, 2, \cdots)$ such that

$$[q_\nu] \leqq [p_\nu] \quad (\nu = 1, 2, \cdots) \text{ and } m(\xi_\nu[q_\nu]a) = \frac{1}{\nu^2} .$$

Since $[a]R$ is monotone complete,

$$a_1 = \sum_{\nu=1}^{\infty} \xi_\nu [q_\nu] a$$

converges and

$$m(a_1) = \sum_{\nu=1}^{\infty} m(\xi_\nu [q_\nu] a) < +\infty .$$

On the other hand, we have

$$m(2a_1) = \sum_{\nu=1}^{\infty} m(2\xi_\nu)[q_\nu]a) > \sum_{\nu=1}^{\infty} \nu \cdot m(\xi_\nu[q_\nu]a) = \sum_{\nu=1}^{\infty} \frac{1}{\nu} = +\infty ,$$

which means that a_1 is not finite.