On the commutator of differential operators[1)]

Dedicated to Prof. Yoshie Katsurada celebrating her sixtieth birthday

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§ 0. Introduction. In [3][2)] A. Lichnorowicz has studied the basic properties of differential operators in several kinds of spaces such as the differentiable manifold with a torsionless connection, the reductive homogeneous space with an invariant volume element, and the symmetric space. Basing upon these fundamental researches he has reproved the known theorem due to I. M. Gelfand that the algebra of invariant differential operators on a globally symmetric Riemannian manifold be a commutative one. Such an algebra in fact has a structure of a polynomial ring and there are many applications of this theorem due to A. Selberg, H. Chandra and others in several branches of mathematics such as the theory of numbers, theory of spherical functions and modern physics.

In the present paper we try to study a rather converse problem of Gelfand's theorem basing upon the same foundations and formulas in [3]. To explain our situation more explicitely we propose the following problem: Have a Riemannian homogeneous space with the commutative algebra of invariant differential operators, a parallel Ricci tensor? As an incomplete answer to this problem one of the identities obtained in the present paper contains as a special case the following two identities which are valid under a suitable commutative condition,

$$\nabla_n(R_{ijkl}R_m{}^{jkl})+\nabla_i(R_{mjkl}R_n{}^{jkl})+\nabla_m(R_{njkl}R_i{}^{jkl})=0 \tag{0.1}$$

$$\nabla_k R_{ij}+\nabla_j R_{ki}+\nabla_i R_{jk}=0 \tag{0.2}$$

It is notable that the above identities consist, as a special case, in a weakly symmetric space introduced by A. Selberg [12]. Another remarkable result is that any harmonic vector field be a parallel one in a compact weakly symmetric space.

In § 1 terminologies, fundamental concepts and basic theorems about differential operators are given. In § 2 the commutators of differential operators on Riemannian manifolds are calculated explicitely. In § 3 we obtain

1) A resume of a part of this work is contained in [8].

2) Numbers in brackets refer to the references at the end of the paper.

a main theorem on a certain class of Riemannian homogeneous spaces and its application to the harmonic vectors. Throughout the present paper we use the notations and several conventions which belong to so called "Ricci calculus", for instance Einstenin's convention of dummy indices is frequently used.

§1. On differential operators. Let M be a differentiable manifold, the set of all real valued smooth functions (with compact carrier) will be denoted by $C^\infty(M)$ ($C^\infty_c(M)$) respectively. We introduce into $C^\infty_c(M)$ so called "pseudo topology" in the sense of Laurent Schwartz [7] based on the local uniform convergence of sequences of functions and their successive derivatives. A linear endomorphism D of $C^\infty_c(M)$ into itself is called a differential operator if the following two conditions are satisfied

1) D has a local charactor e. d., the carrier of the function Df is a subset of the carrier af f, where $f \in C^\infty_c(M)$.

2) D is a continuous mapping from $C^\infty_c(M)$ into itself with respect to the above cited pseudo topology.

As an application of the fundamental theorem of the theory of distribution due to L. Schwartz in [7] that any distribution which has only one point as its carrier must be a finite linear combination of real coefficients of Dirac's measure and its successive derivatives in the sense of distribution, we can easily verify that the differential operator in the above sense is nothing but a classically defined one at least locally. More explicitely we have the following local expression of D.

$$(Df)(\mathrm{p}) = \sum_{|p| \leqq N} a_p \partial_p f_*(x^a), \qquad \mathrm{p} \in U \tag{1. 1}$$

where f_* is a composite function $f \circ \varphi$ of f restricted to a coordinate neighbourhood U and a coordinate function $\varphi: R^n \to U$, the coefficient functions in (1. 1) are smooth functions of coordinates.

Any differential operator can be extended uniquely from $C^\infty_c(M)$ onto $C^\infty(M)$ as a linear endomorphism of the latter one. Let Φ be a diffeomorphism of M onto itself, f^Φ denotes the composite function $f \circ \Phi$ where $f \in C^\infty(M)$. f is said to be an invariant function by a diffeomorphism Φ when $f^\Phi = f$. Let D be any linear endomorphism of $C^\infty(M)$ into itself, D^Φ denotes the endomorphism defined by $D^\Phi f = (Df^\Phi)^{\Phi^{-1}}$. When D is a differential operator D^Φ is also easily seen to be a differential operator. A differential operator D is said to be invariant by a diffeomorphism Φ when the identity $D^\Phi = D$ consists. For the sake of simplicity we write f instead of f_* in (1. 1) hereafter.

We define a commutator of two differential operators D_1 and D_2 by

$$[D_1, D_2]f = D_1D_2f - D_2D_1f. \tag{1.2}$$

In the special case of D_i to be a differential operators of first order, the commutator introduced above is nothing but a usual Lie's bracket product of these vector fields. The set of all differential operators on M as well as the set of all invariant differential operators by a diffeomorphism, is easily seen to be a Lie algebra with respect to this commutator, though these two Lie algebras are of infinite dimensional in general.

In a differentiable manifold with an affine connection any differential operator can be expressed in terms of the covariant derivatives as follows. Here we note that a covariant derivative is defined not only locally but may be considered as a globally one, so we obtain a second definition of the differential operator in such a space.

Lemma (Lichnerowicz) *In a differentiable manifold with a torsionless affine connection any differential operator D can be expressed as a linear combination of covariant derivatives of successive orders p with smooth coefficients as follows*

$$Df = \sum_p a^{i_1\cdots i_p}\nabla_{i_1}\nabla_{i_2}\cdots\nabla_{i_p}f \tag{1.3}$$

where each coefficient function is a component of a contravariant symmetric tensor field and moreover this expression is unique.

The modification of the expression (1.1) into (1.3) is a result of an execution of term by term calculations of substituting covariant derivatives in place of ordinary derivatives from the term of the highest order inductively.

§ 2. **From Riemannian geometry.** Let M be a Riemannian manifold with a positive definite metric tensor g_{ij}, ∇_i denote the covariant derivatives with respect to Christoffel symbol. We recall so called Ricci's formula:

$$\begin{aligned} &\nabla_i\nabla_j f = \nabla_j\nabla_i f, \qquad \nabla_i f = \partial_i f = \frac{\partial f}{\partial x^i}, \\ &\nabla_i\nabla_j a_{i_1\cdots i_p} - \nabla_j\nabla_i a_{i_1\cdots i_p} = -\sum_{k=1}^{p} R_{iji_k}{}^{a} a_{i_1\cdots a\cdots i_p}. \end{aligned} \tag{2.1}$$

A diffeomorphism of M onto itself is called an isometrie if it preserves the Riemannian metric tensor g_{ij}. A vector field ξ^i on M generating a local one parameter group of isometries is called a killing one. Any killing vector field is characterized by satisfying the following equations where L_ξ denotes Lie derivative with respect to ξ^i.

$$L_\xi g_{ij} \equiv \nabla_j\xi_i + \nabla_i\xi_j = 0. \tag{2.2}$$

According to the formulas in the theory of Lie derivatives we have a series of equations satisfied by a killing vector field.

$$
\begin{aligned}
L_\xi \Gamma^i_{jk} &\equiv \frac{1}{2} g^{ia}\left\{\nabla_k L_\xi g_{ja} + \nabla_j L_\xi g_{ka} - \nabla_a L_\xi g_{jk}\right\} \\
&\equiv \nabla_j \nabla_k \xi^i + R_{ajk}{}^i \xi^a = 0\,, \\
L_\xi R_{jkl}{}^i &\equiv \nabla_j (L_\xi \Gamma^i_{kl}) - \nabla_k (L_\xi \Gamma^i_{jl}) = 0, \\
L_\xi(\nabla_m R_{jkl}{}^i) &= 0\,, \qquad L_\xi R_{jk} \equiv L_\xi R_{ijk}{}^i = 0\,.
\end{aligned} \tag{2. 3}
$$

Now we are going to investigate the commutators of several pairs of differential operators, but in order to avoid fruitless long calculations in the general case, we are mainly concerned with the differential operators of at most second order.

When $D_1 f = \xi^{ij}\nabla_i \nabla_j f$, $D_2 f = \eta^i \nabla_i f$ where ξ^{ij} are components of a certain symmetric tensor field we have

$$
\begin{aligned}
[D_1, D_2] f &= \xi^{ij}\nabla_i \nabla_j (\eta^k \nabla_k f) - \eta^k \nabla_k (\xi^{ij}\nabla_i \nabla_j f) \\
&= \xi^{ij}\eta^k (\nabla_i \nabla_j \nabla_k - \nabla_k \nabla_i \nabla_j) f + (\xi^{ij}\nabla_i \nabla_j \eta^k)\nabla_k f \\
&\quad + (2\xi^{ij}\nabla_i \eta^k)\nabla_j \nabla_k f - (\eta^k \nabla_k \xi^{ij})\nabla_i \nabla_j f \\
&= \xi^{ij}\eta^k (-R_{ikj}{}^a \nabla_a f) + (\xi^{ij}\nabla_i \nabla_j \eta^k)\nabla_k f \\
&\quad + (2\xi^{ki}\nabla_k \eta^j - \eta^k \nabla_k \xi^{ij})\nabla_i \nabla_j f \\
&= \xi^{ij}(L_\xi \Gamma^k_{ij})\nabla_k f + (2\xi^{ki}\nabla_k \eta^j - \eta^k \nabla_k \xi^{ij})\nabla_i \nabla_j f
\end{aligned} \tag{2. 4}
$$

In a Riemannian manifold a Laplacian operator is defined by Δ:

$$
\Delta f = g^{ij}\nabla_i \nabla_j f = div.\ grad\ f. \tag{2. 5}
$$

From the identity (2. 4) we obtain a new characterization of a killing vector field in terms of its corresponding differential operator.

THEOREM 2. 1. *In order that a differential operator of first order* $D = \eta^i \nabla_i f$ *commute with the Laplacian operator on a Riemannian manifold* M

$$
0 = [\Delta, D] f, \qquad f \in C^\infty(M)
$$

it is necessary and sufficient that the coefficient vector field η^i *be a killing one.*

PROOF. On substituting the metric tensor g^{ij} in place of ξ^{ij} in the identity (2. 4), we obtain the following equation satisfied by η^i as a consequence of the commutativity assumption in this theorem,

$$
0 = [\Delta, D] = g^{ij}(L_\eta \Gamma^k_{ij})\nabla_k f + (\nabla^k \eta^i + \nabla^i \eta^k)\nabla_i \nabla_k f, \quad \nabla^k = g^{ki}\nabla_i\,. \tag{2. 6}
$$

As the variable function f in (2. 6) as well as $\partial_i f$ and $\partial_i \partial_j f$ be arbitrary we obtain from (2. 6) as the vanishing coefficients of $\partial_i \partial_k f$

(2. 7) $$0 = (L_\eta g_{lm}) g^{li} g^{mk} = \nabla^k \eta^i + \nabla^i \eta^k .$$

Now the equation (2. 6) turns into

$$0 = g^{ij} (L_\eta \Gamma^k_{ij}) \nabla_k f .$$

As in the case of (2. 7) arbitrariness of $\partial_k f$ means

(2. 8) $$0 = g^{ij} (L_\eta \Gamma^k_{ij}) .$$

The equation (2. 7) is equivalent to the equation (2. 2) which characterizes the vector field η^i to be a killing one, we have proved the necessity. Conversely a killing vector satisfies (2. 7) and (2. 8), so we have (2. 6). This proves the sufficiency.

Before going into the next case we need a lemma for brevity' sake.

LEMMA. *We have*

(2. 9) $$\begin{aligned} \nabla_i \nabla_j \nabla_l \nabla_m f - \nabla_l \nabla_m \nabla_i \nabla_j f = & -(\nabla_i R_{jlm}{}^k + \nabla_l R_{imj}{}^k) \nabla_k f \\ & -(R_{jlm}{}^k \delta^h_i + R_{imj}{}^k \delta^h_l + R_{ilm}{}^k \delta^h_j + R_{ilj}{}^k \delta^h_m) \nabla_k \nabla_h f . \end{aligned}$$

PROOF. As we have

$$\begin{aligned} \nabla_i \nabla_j \nabla_l \nabla_m f - \nabla_l \nabla_m \nabla_i \nabla_j f = & \nabla_i (\nabla_j \nabla_l \nabla_m f - \nabla_l \nabla_j \nabla_m f) \\ & + \nabla_l (\nabla_i \nabla_m \nabla_j f - \nabla_m \nabla_i \nabla_j f) + (\nabla_i \nabla_l - \nabla_l \nabla_i) \nabla_m \nabla_j f , \end{aligned}$$

substituting from Ricci formula (2. 1) into each term of the right hand member of the above equations

$$\begin{aligned} & = -\nabla_i (R_{jlm}{}^k \nabla_k f) - \nabla_l (R_{imj}{}^k \nabla_k f) - R_{ilm}{}^k \nabla_k \nabla_j f - R_{ilj}{}^k \nabla_m \nabla_k f \\ & = -(\nabla_i R_{jlm}{}^k) \nabla_k f - (\nabla_l R_{imj}{}^k) \nabla_k f - R_{jlm}{}^k \nabla_i \nabla_k f - R_{imj}{}^k \nabla_l \nabla_k f \\ & \quad - R_{ilm}{}^k \nabla_k \nabla_j f - R_{ilj}{}^k \nabla_m \nabla_k f . \end{aligned}$$

From the above, by virtue of $\nabla_i \nabla_j f = \nabla_j \nabla_i f$ in (2. 1), we obtain (2. 9) after gathering of like terms by introducing a Kronecker tensor. q. e. d.

When $D_1 f = \xi^{ij} \nabla_i \nabla_j f$ and $D_2 f = \eta^{ij} \nabla_i \nabla_j f$, the commutator of these operators is scrutinized as follows

(2. 10) $$\begin{aligned} [D_1, D_2] f = & \xi^{ij} \nabla_i \nabla_j (\eta^{lm} \nabla_l \nabla_m f) - \eta^{lm} \nabla_l \nabla_m (\xi^{ij} \nabla_i \nabla_j f) \\ = & \xi^{ij} \eta^{lm} (\nabla_i \nabla_j \nabla_l \nabla_m - \nabla_l \nabla_m \nabla_i \nabla_j) f + 2\xi^{ij} (\nabla_i \eta^{lm}) \nabla_j \nabla_l \nabla_m f \\ & - 2\eta^{lm} (\nabla_l \xi^{ij}) \nabla_m \nabla_i \nabla_j f + (\xi^{ij} \nabla_i \nabla_j \eta^{lm} \nabla_l \nabla_m f - \eta^{lm} \nabla_l \nabla_m \xi^{ij} \nabla_i \nabla_j f) \\ = & -\xi^{ij} \eta^{lm} (\nabla_i R_{jlm}{}^k + \nabla_l R_{imj}{}^k) \nabla_k f - \xi^{ij} \eta^{lm} (R_{jlm}{}^k \delta^h_i + R_{imj}{}^k \delta^h_l \\ & + R_{ilm}{}^k \delta^h_j + R_{ilj}{}^k \delta^h_m) \nabla_h \nabla_k f + (\xi^{lm} \nabla_l \nabla_m \eta^{ij} - \eta^{lm} \nabla_l \nabla_m \xi^{ij}) \nabla_i \nabla_j f \\ & + 2\xi^{ij} (\nabla_i \eta^{lm}) \nabla_j \nabla_l \nabla_m f - 2\eta^{lm} (\nabla_l \xi^{ij}) \nabla_m \nabla_i \nabla_j f . \end{aligned}$$

A formally analogus theorem with the Theorem 2.1 in the case of the differential operators of the second order can be read from (2.10) as follows.

THEOREM 2.2. *In order that a differential operator $\eta^{ij}\nabla_i\nabla_j f$ of the second order commute with the Laplacian operator in a Riemannian manifold, it is necessary and sufficient that the coefficient tensor satisfies the following three equations*

$$
(2.11)\qquad
\begin{aligned}
&i)\ \ \nabla_k\eta_{ij}+\nabla_j\eta_{ki}+\nabla_i\eta_{kj}=0\,,\\
&ii)\ \ \nabla^k\nabla_k\eta_{ij}-2\eta^{lm}R_{ilmj}+R_i^{\ l}\eta_{lj}+R_j^{\ l}\eta_{li}=0\,,\\
&iii)\ \ \eta^{lm}\nabla_a R_{ml}-2\eta^{lm}\nabla_l R_{ma}=0,\\
&\qquad \textit{where}\qquad R_{ij}=R^k{}_{ijk}\,.
\end{aligned}
$$

PROOF. On substituting the metric tensor g^{ij} in place of ξ^{ij} in (2.10) and after taking a symmetric part of each coefficient tensor so that the expression coincides with that of (1.3), we obtain the following equation satisfied by η^{ij}

$$
(2.12)\qquad
\begin{aligned}
0=[\Delta, D_2]=&-\eta^{lm}(\nabla^i R_{ilm}{}^k-\nabla_l R^k{}_m)\nabla_k f+(\nabla^l\nabla_l\eta^{hk}\\
&-2R^h{}_{jl}{}^k\eta^{jl}+2R^h_m\eta^{mk}+R^h_m\eta^{mk})\nabla_h\nabla_k f\\
&+\frac{2}{3}\left(\nabla^i\eta^{lm}+\nabla^m\eta^{il}+\nabla^l\eta^{mi}\right)\nabla_i\nabla_l\nabla_m f.
\end{aligned}
$$

By contracting the metric tensor g^{ik} to the Bianchi's identity:

$$\nabla_m R_{ijk}{}^l+\nabla_j R_{mik}{}^l+\nabla_i R_{jmk}{}^l=0\,,$$

we have a known formula

$$(2.13)\qquad \nabla^i R_{jmi}{}^l=\nabla_m R_j^{\ l}-\nabla_j R_m^{\ l}\,,\qquad R_j^{\ l}=g^{lk}R_{jk}\,.$$

Substituting (2.13) into (2.12) we can observe that the coefficient tensor of $\nabla_k f$ in (2.12) is the same with the left hand member of (2.11) iii). Now making available use of uniqueness assertion of the lemma in § 1 or by the same discussion just as in the proof of the Theorem 2.1, we can deduce (2.11) from (2.12). q. e. d.

LEMMA. *In a Riemannian manifold any symmetric tensor field η_{ij} satisfying the equations $\nabla_k\eta_{ij}+\nabla_i\eta_{kj}+\nabla_j\eta_{ki}=0$ also satisfies the following equations*

$$
(2.14)\qquad
\begin{aligned}
&i)\ \ \nabla^k\nabla_k\eta_{ij}-2\eta^{lm}R_{ilmj}+R_i^{\ l}\eta_{lj}+R_j^{\ l}\eta_{li}-\nabla_i\nabla_j\eta^k{}_k=0\,,\\
&ii)\ \ \nabla^l\nabla_j\eta_{ml}-\nabla^l\nabla_m\eta_{jl}=R^k_m\eta_{kj}-R_j^{\ k}\eta_{km}\,.
\end{aligned}
$$

PROOF. Taking a skew symmetric part of twicefold covariant derivative: $(\nabla_l\nabla_m-\nabla_m\nabla_l)\eta_{ij}$, we have

$$-\nabla_l(\nabla_i\eta_{jm}+\nabla_j\eta_{mi})-\nabla_m\nabla_l\eta_{ij}=-(R_{lmi}{}^k\eta_{kj}+R_{lmj}{}^k\eta_{ki})\,. \tag{2. 15}$$

By contracting g^{li} to (2. 15) we obtain

$$-\nabla^k\nabla_k\eta_{ij}-\nabla^k\nabla_j\eta_{ki}-\nabla_i\nabla^k\eta_{jk}=-\eta^{lm}R_{lijm}+\eta_{jk}R_i^{\ k}\,. \tag{2. 16}$$

On the other hand contracting g^{kj} and ∇^k to (2. 11) i) we obtain

$$2\nabla^k\eta_{ik}+\nabla_i\eta^k{}_k=0\,,$$

$$\nabla^k\nabla_k\eta_{ij}+\nabla^k\nabla_j\eta_{ki}+\nabla^k\nabla_i\eta_{jk}=0\,.$$

Taking a symmetric part of (2. 16) and making use of the above two equations we obtain

$$-2\nabla^k\nabla_k\eta_{ij}-2\nabla^k\nabla_{(j}\eta_{|k|i)}-2\nabla_{(i}\nabla^k\eta_{j)k}=-2\eta^{lm}R_{lijm}+\eta_{jk}R_i^{\ k}+\eta_{ik}R_j^{\ k}\,.$$

This equation is nothing but (2. 14 i). (2. 14) ii) is directly obtained by taking a skew symmetric part of (2. 16) with respect to the indices k and i.

THEOREM 2. 2′. *In order that a differential operator $\eta^{ij}\nabla_i\nabla_jf$ of the second order commute with the Laplacian operator in a Riemannian manifold, it is necessary and sufficient that η^{ij} safisfies the following equations*

$$i)\ \ \nabla_k\eta_{ij}+\nabla_i\eta_{jk}+\nabla_j\eta_{ki}=0\,,\qquad ii)\ \ \nabla_k\nabla_l\eta^i{}_i=0\,,$$

$$iii)\ \ \eta^{lm}\nabla_kR_{lm}-2\eta^{lm}\nabla_lR_{mk}=0\,.$$

From (2. 11) ii) and (2. 13) i) which is a consequence of (2. 11) i), we have the above condition ii). Conversely i) and ii) mean together (2. 11) ii). q. e. d.

THEOREM 2. 3. *If a differential operator $\eta^{ij}\nabla_i\nabla_jf$ of the second order commute with the Laplacian operator Δ, $\eta^i{}_i$ must be constant under the one of following three additional conditions on the Riemannian manifold M 1) M is compact. 2) M is irreducible. 3) The matrix of Ricci tensor is non-singular.*

PROOF. From Theorem 2. 2′ we have the following equation in either case

$$\nabla_k\nabla_l\eta^i{}_i=0\,. \tag{2. 17}$$

In the case of 1) the well known theorem due to E. Hopf can be applied to (2. 17) and $\eta^i{}_i$ must be constant. In the case of irreducible space there is no nonvanishing parallel vector field, so from (2. 17) we conclude that $\nabla_l\eta^i{}_i=0$. In the case os 3) we have no nonvanishing parallel vector field because any parallel vector v_i satisfies

$$R_{ij}v^j=0 \qquad\qquad \text{q. e. d.}$$

By the way we obtain a condition of symmetric tensor η_{ij} to be parallel.

THEOREM 2. 4. *In order that a symmetric tensor η_{ij} be parallel one in a compact Riemannian manifold, it is necessary and snfficient that η_{ij} satisfies the following two conditions.*

$$\text{(2. 18)} \qquad \begin{aligned} &i)\quad \nabla_k\eta_{ij}+\nabla_i\eta_{jk}+\nabla_j\eta_{ki}=0\,,\\ &ii)\quad (\nabla_l\nabla_m-\nabla_m\nabla_l)\eta_{ij}=0\,. \end{aligned}$$

From (2. 18) ii) we obtain as an application of Ricci's formula: (2. 1)

$$\text{(2. 19)} \qquad 0=g^{il}(\eta_{ik}R^k{}_{jlm}+\eta_{kj}R^k{}_{ilm})=-\eta_{kl}R^k{}_{jm}{}^l+\eta_{kj}R^k{}_m\,.$$

On the other hand (2. 14) i) follows to (2. 18) i), so from (2. 19) and (2. 14) i) η_{ij} must satisfy the following equations

$$\text{(2. 20)} \qquad \nabla^k\nabla_k\eta_{ij}-\nabla_i\nabla_j\eta^k{}_k=0\,.$$

As we have the identity

$$\nabla^k\nabla_k(\eta_{ij}\eta^{ij})=2(\nabla^k\eta_{ij})(\nabla_k\eta_{ij})+2(\nabla^k\nabla_k\eta_{ij})\eta^{ij}\,,$$

substituting (2. 20) into this identity we have

$$\begin{aligned}\nabla^k\nabla_k(\eta_{ij}\eta^{ij})&=2(\nabla^k\eta_{ij})(\nabla_k\eta_{ij})+2\eta^{ij}(\nabla_i\nabla_j\eta^k{}_k)\\ &=2(\nabla^k\eta_{ij})(\nabla_k\eta^{ij})+2\left\{\nabla_i(\eta^{ij}\nabla_j\eta^k{}_k)-(\nabla_i\eta^{ij})(\nabla_j\eta^k{}_k)\right\}\\ &=2\nabla_i(\eta^{ij}\nabla_j\eta^k{}_k)+2(\nabla^k\eta^{ij})(\nabla_k\eta_{ij})+(\nabla^j\eta^l{}_l)(\nabla_j\eta^k{}_k)\,.\end{aligned}$$

On applying Green's integral formula:

$$0=\int \text{div. } U\ d\sigma\,,$$

where U is a vector field, we have the following equation

$$\begin{aligned}0&=\int\left\{\nabla^k\nabla_k(\eta_{ij}\eta^{ij})-2\nabla_i(\eta^{ij}\nabla_j\eta^k{}_k)\right\}d\sigma\\ &=\int 2(\nabla^k\eta^{ij})(\nabla_k\eta_{ij})+(\nabla^j\eta^l{}_l)(\nabla_j\eta^k{}_k)\,d\sigma\geqq 0\,.\end{aligned}$$

Then we have

$$0=(\nabla^k\eta^{ij})(\nabla_k\eta_{ij})\,,\qquad 0=(\nabla^j\eta^l{}_l)(\nabla_j\eta^k{}_k)\,.$$

Then we obtain

$$0=\nabla_k\eta_{ij}\,.$$

As a final step in this paragraph we will establish a general form which includes the Theorem 2. 1 and the Theorem 2. 2 as a special case.

THEOREM 2. 5. *In order that a differential operator $\eta^{i_1\cdots i_p}\nabla_{i_1\cdots i_p}f$ of p-th order commute with the Laplacian operator, it is necessary that the* *coefficient symmetric tensor field* $\eta_{i_1 \cdots i_p}$ *satisfies*

$$\nabla_{i_{p+1}}\eta_{i_1 \cdots i_p} + \nabla_{i_1}\eta_{i_2 \cdots i_p i_{p+1}} + \cdots\cdots + \nabla_{i_p}\eta_{i_{p+1} i_1 \cdots i_{p-1}} = 0 . \tag{2.21}$$

PROOF. By a straightforward calculation in the same manner as in the proof of Theorem 2.1 and 2.2 we can easily see that the condition is necessary.

§3. Applications to Riemannian homogeneous spaces. Let G be a connected Lie group and H a closed subgroup of G and we assume moreover that the adjoint representation of subgroup H in G is compact and H contains no normal subgroup of G. Such a homogeneous space G/H is called an (effective) Riemannian homogeneous space because of the existnence of a certain Riemannian metric tensor being invariant by the action of each element of G on G/H. Henceforth in the present paper "a Riemannian homogeneous space" is understood as a stronger form of choicing and fixing an invariant metric. A Riemannian homogeneous space can be regarded as a reductive homogeneous space in the sense of [6], but as we are mainly concerned with the Levi Civita's connection associated to the choicing Riemannian metric, so we have not any interest to the canonical connections in [5].

Let $I(G/H)$ denote the set of all isometries of G/H. In a Riemannian homogeneous space the action of any element of G is of isometric one, so G can be regarded as a subgroup of $I(G/H)$. In this space the known scalars such as the scalar curvature: $R = R_{ij}g^{ij}$, the mass scalar of Ricci tensor: $R_{ij}R^{ij}$, and the mass scalar of curvature tensor: $R_{ijkl}R^{ijkl}$ are all constant.

Now we introduce the concept of known three kinds of Riemannian homogeneous spaces.

(1) A Riemannian manifold M is called a globally (locally) symmetric space if at each point of M the geodesic symmetry at this point can be extendes to the isometrie of M (the geodesic symmetry is a local isometry).

(2) A Riemannian manifold M is called weakly symmetric space if a subgroup G of $I(M)$ act on M transitively and there is an element μ of $I(M)$ satisfying the following three conditions

i) $\mu G \mu^{-1} = G$, ii) $G \ni \mu^2$,

iii) Let x and y be any pair of points on M, then there exists an element m of G such that $\mu x = my$, $\mu y = mx$.

A globally symmetric space can be regarded as a weakly symmetric space if we put $G = I(M)$, $\mu = I$.

Let $D(G/H)$ denote the set of differential operators which are invariant by the action of each element of G in the sense of § 1. In our space $D(G/H)$ is not a trivial one because of the existence of the Laplacian operator. The invariantness of the Laplacian operator may be obtained directly from its definition. Following to A. Lichnerowicz we give a characterization of invariant differential operators by the invariantness of coefficient tensor fields as follows.

THEOREM 3. 1. (A. Lichnerowicz) *Any invariant differential operator in $D(G/H)$ can be expressed as follows*

$$Df = \sum_p a_p^{i_1\cdots i_p} \nabla_{i_1} \cdots\cdots \nabla_{i_p} f, \tag{3.1}$$

where the coefficient symmetric tensors a_p's are invariant ones by the action of G, that is, Lie derivatives of $a_p^{i_1\cdots i_p}$ with respect to the infinitesimal transformation corresponding to any element of the Lie algebra of G vanish.

As the action of each element of G on G/H is of isometric one, we can deduce the invariant property of the Laplacian operator from this theorem.

Our main concern in the present paper lies on a Riemannian homogeneous space on which $D(G/H)$ be a commutative algebra, so we call such a space "the space which satisfies the condition (c)" for the brevity' sake. We start with aiming at the two object one of which is the problem already mentioned in the introduction, the another is exposing out of latent properties and formulas of symmetric spaces by making available use of the commutativity of $D(G/H)$.

The following is a well known theorem due to K. Yano and S. Bochner being fundamental in order to study a harmonic vector field in a homogemeouse space.

THEOREM 3. 2. *In a compact Riemannian manifold the inner product $\xi_i\eta^i$ of a killing vector field ξ and a harmonic one η is constant.*

As a corollary of this theorem we have

THEOREM 3. 3. *In a compact Riemannian manifold the Lie's bracket product $[\xi, \eta]$ of a killing vector field ξ and a harmonic one η annihilates.*

PROOF. From Theorem 3. 2 we have by taking covariant derivatives

$$0 = \nabla_k(\xi_i\eta^i) = (\nabla_k\xi_i)\eta^i + \xi_i\nabla_k\eta^i . \tag{3.2}$$

From the definition of killing vector field and of harmonic one, these satisfy the following equations respectively

$$\nabla_j\xi_i + \nabla_i\xi_j = 0 , \qquad \nabla_j\eta_i = \nabla_i\eta_j . \tag{3.3}$$

Substituting from (3. 3) into (3. 2) we obtain

$$-(\nabla_i \xi_k)\eta^i + (\nabla_i \eta^k)\xi^i = 0 .$$

The followings are main results in the present paper.

THEOREM 3. 4. *In a compact Riemannian homogeneous space with the condition* (c), *any harmonic vector field must be a parallel oue.*

PROOF. From Theorem 3. 3, any harmonic vector field must be an invariant one by the group G. From the assumption of the condition (c) and Theorem 2. 1 it must be a killing one. Consequently a harmonic vector field in such a space must be a parallel one. q. e. d.

As a special case of the above theorem we have

THEOREM 3. 5. *In a compact globally symmetric Riemannian manifold or as a slight generalization, in a compact weakly symmetric Riemannian manifold, any harmonic vector field must be a papallel one.*

PROOF. In a compact globally symmetric Riemannian manifold, the theorem of I. M. Gelfand introduced in § 0 shows the commutativity of $D(G/H)$. In a weakly symmetric case, as Selberg has shown $D(G/H)$ is commutative also. We can apply Theorem 3. 4 to the both cases.

REMARK. The analogous results with Theorem 3. 5 in the case of globally symmetric case have been obtained by M. Matsumoto and T. Nagano and others [4] [5]. In a globally symmetric case our result coincides with Nagano's. He shows that with a suitable exchange of the Riemannian metric any harmonic vector field can be regarded as a parallel one in a compact homogenous space. Our situation is a little different from Nagano's because we are fixing our invariant metric. Theorem 3. 5 in fact can be obtained directly from Theorem 3. 3, for it is well known that in a symmetric homogeneous space any invariant tensor be a parallel one with respect to the first canonical connection in the sense of Nomizu [6].

From Theorem 3. 4 we have

THEOREM 3. 6. *In a compact Riemannian homogeneous space with the condition* (c), *one of the following conditions means the vanishing of the first Betti number.*

(i) The determiant of the matrix of Ricci tensor is non zero.

(ii) The space is irreducible.

(iii) The determinant of the matrix of the tensor field $R_{ijkl}R_h^{\ jkl}$ is non zero.

PROOF. Each one of these three conditions means non existence of non trivial parallel vector field. On the other hand from Theorem 3. 4 any harmonic vector field must be a parallel one in such a space, so we have proved the theorem.

REMARK. There are many other conditions of assuring the non existence of nontrivial parallel vector field, for instance det. $(R_{ijkl}R^{jk}) \neq 0$. Without any assumption of compactness we have

THEOREM 3.7. *In a Riemannian homogeneous space with the condition* (*c*) (*especially in a globally or weakly symmetric Riemannian space*) *the following conditions* (*i*) *and* (*ii*) *are equivalent and from these the condition* (*iii*) *follows.*

(i) There is no invariant differential operator of the first order.

(ii) The centralizer of the Lie algebra of G in the algebra of killing vector fields is trivial.

(iii) G semi simple.

PROOF. The equivalence of (i) and (ii) follows directly from Theorem 2.1 (ii) means (iii) evidently.

For the differential operators of the second order we have

THEOREM 3.8. *In a Riemannian homogeneous space satisfying the condition* (*c*) *any invariant symmetric tensor field* η_{ih} *must satisfy the following conditions*

$$\nabla_k \eta_{ij} + \nabla_i \eta_{jk} + \nabla_j \eta_{ki} = 0 . \tag{3.4}$$

PROOF. From the necessity part of Theorem 2.2 we have (3.4). Note that the conditions ii) and iii) of Theorem 2.2' is identical if (3.4) is satisfied and η_{ih} be the one of the following tensor fields R_{ij}, $R_{ijkl}R_h{}^{jkl}$ etc.

As a corollary of this theorem we have

THEOREM 3.9. *In a Riemannian homogeneous space satisfying the condition* (*c*) *Ricci tensor* R_{ij} *and the tensor* $R_{ijkl}R_h{}^{jkl}$ *satisfy the equation* (0.2) *and* (0.1) *respectively.*

We have a general form of this type of theorems

THEOREM 3.10. *In a Riemannian homogeneous space satisfying the condition* (*c*) *any symmetric tensor which is an invariant one must satisfies the following condition*

$$\nabla_{i_{p+1}} \eta_{i_1 \cdots i_p} + \nabla_i \eta_{i_2 \cdots i_{-1} i_p} + \cdots\cdots + \nabla_{i_1} \eta_{i_{1+1} i_1 \cdots i_{p-1}} = 0 . \tag{3.5}$$

It may be a surprising matter that in a weakly symmetric space (0.1) and (0.2) consist. But we have no application of these formulas till nowadays. There may be some meaning in the case of hermitian space [9].

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