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Higher order monodiffric difference equation

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1. Introduction. In [1] Berzsenyi has discussed the general solution of the first order monodiffric difference equation. Let D be the first quadrant of the discrete plane, and suppose $a\in D-\{0\}$ and $g\in M(D)$, then the general solution of the monodiffric equation $f'(z)-af(z)=g(z)$ with initial condition $f(0)=c$ is given by the monodiffric function

$$f(z)=cE(z, a)+\frac{1}{a}\left[g*E(z, a)\right] \qquad \text{for every } z\in D,$$

where $E(z, a)$ is the monodiffric exponential function (c. f. [2])

$$E(z, a)=(1+a)^{x}(1+ia)^{y} \qquad \text{for } z=x+iy .$$

In the sequel, we use the notation $e^{a,z}$ instead of $E(z, a)$.

In this paper, we extend Berzsenyi's results to more general cases, namely, the general solution of the n'th order monodiffric linear homogeneous difference equation

$$F^{(n)}(z)+c_{n-1}F^{(n-1)}(z)+\cdots+c_{1}F'(z)+c_{0}F(z)=0\,, \tag{1. 1}$$

where the coefficients c_i $(i=0, 1, \cdots, n)$ are arbitrary constants.

2. Definition. Let $D=\{z|z=x+iy,\ x \text{ and } y \text{ are integers}\}$ and f be a complex-valued function defined on D. We define the monodiffric residue of f at z to be the value $Mf(z)$ given by

$$Mf(z)=(i-1)f(z)+f(z+i)-if(z+1)\,. \tag{1. 2}$$

We say that f is monodiffric at z if $Mf(z)=0$. And a function which is monodiffric at every point in D is monodiffric on D. In this case, we write $f\in M(D)$. The monodiffric derivative f' of f is defined by

$$f'(z)=\frac{1}{2}\left[(i-1)f(z)+f(z+1)-if(z+i)\right].$$

We also use the symbol $\dfrac{df}{dz}$ or D_z.

3. The monodiffric exponential function $e^{a,z}$.

In [2], Isaacs has introduced the monodiffric exponential function $e^{a,z}$, it has a form $e^{a,z}=(1+a)^{x}(1+ia)^{y}$ for $z=x+iy$ and a is a complex number.

Before discussing the general solution of (1. 1), we study some properties of $e^{a,z}$.

PROPOSITION 1. (a) $\frac{d^n}{dz^n}\, e^{a,z} = a^n e^{a,z}$, *where* $\frac{d^n}{dz^n}$ *means n'th monodiffric derivative*

(b) $\frac{d^n}{dz^n}\, e^{a,z} \in M(D) \qquad$ *for* $n = 0, 1, 2, \cdots$

PROPOSITION 2. (a) $\frac{d}{da}\, e^{a,z} = (1+a)^{x-1}(1+ia)^{y-1}\{z + ia(x+y)\}$ *for* $z = x+iy$

(b) $\frac{d}{da}\, e^{a,z} \in M(D)$

where $\frac{d}{da}\, e^{a,z} = \lim_{h\to 0} \frac{e^{(a+h),z} - e^{a,z}}{h}$ *for fixed point* $z \in D$.

PROOF. A proof of (a) of Proposition 1 and 2 is given by a straightforward calculation.
A proof of (b) of Proposition 1 was shown by Isaacs [3]. Now we prove (b) of Proposition 2. Let $f(z) = \frac{d}{da}\, e^{a,z}$ then

$$Mf(z) = (i-1)f(z) + f(z+i) - if(z+1) = i\big[f(z) - f(z+1)\big] + f(z+i) - f(z).$$

Since

$$f(z) - f(z+1) = (1+a)^{x-1}(1+ia)^{y-1}\{-1 - a(z+1+i) - a^2 i(x+y+1)\},$$

and

$$f(z+i) - f(z) = (1+a)^{x-1}(1+ia)^{y-1}\{i + a(zi+i-1) - a^2(x+y+1)\},$$

therefore we have $Mf(z) = 0$.

PROPOSITION 3. $\frac{d}{da}\, e^{a,z}$ *is a solution of* $(D_z - a)^2 F(z) = 0$, *and is also a solution of* $(D_z - a)^m F(z) = 0$ *for any integer* $m \geqq 2$.

PROOF. Since $\frac{d}{da}\, e^{a,z} \in M(D)$, if we put $F(z) = \frac{d}{da}\, e^{a,z}$ we have

$$\begin{aligned}(D_z - a)^2 F(z) &= F''(z) - 2aF'(z) + a^2 F(z) \\ &= F(z+2) - 2(1+a)\, F(z+1) + (1+a)^2 F(z) \\ &= (1+a)^{x+1}(1+ia)^{y-1}\{z+2+ia(x+y+2) - 2 \\ &\qquad \big[z+1+ia(x+y+1)\big] + z + ia(x+y)\} \\ &= 0.\end{aligned}$$

And

$$(D_z-a)^m F(z)=(D_z-a)^{m-2}(D_z-a)^2 F(z)=0\,.$$

PROPOSITION 4. *Let* $G(z)=\dfrac{d^2}{da^2}\, e^{a,z}$, *then we have*

(a) $G(z)=(1+a)^{x-2}(1+ia)^{y-2}$
$\left\{z^2+y-x+2iz(x+y-1)\,a-(x+y)(x+y-1)\,a^2\right\}$

(b) $G(z)\in M(D)$

(c) $(D_z-a)^3 G(z)=0$

(d) $(D_z-a)^m G(z)=0 \quad$ *for* $m\geqq 3$.

PROOF. Differentiating $\dfrac{d}{da}\, e^{a,z}$ with respect to a directly we will get (a). Now, we shall prove that

$$MG(z)=(i-1)\,G(z)+G(z+i)-iG(z+1)=0\,.$$

Since

$$(i-1)\,G(z)=(1+a)^{x-2}(1+ia)^{y-2}$$
$$\left\{(i-1)(z^2+y-x)+2i(i-1)\,z(x+y-1)\,a-(i-1)(x+y)(x+y-1)\,a^2\right\},$$
$$G(z+i)=(1+a)^{x-2}(1+ia)^{y-1}$$
$$\left\{(z+i)^2+y+1-x+2i(z+i)(x+y)\,a-(x+y+1)(x+y)\,a^2\right\},$$
$$-iG(z+1)=(1+a)^{x-1}(1+ia)^{y-2}$$
$$\left\{(-i)\left[(z+1)^2+y-x-1\right]+2(z+1)(x+y)\,a+i(x+y+1)(x+y)\,a^2\right\},$$

we rewrite $MG(z)$ into the form

$MG(z)=(1+a)^{x-2}(1+ia)^{y-2}\{Ai+B\}$, then we can obtain $A=0$ and $B=0$. Thus we have proved (b). Now,

$$\begin{aligned}(D_z-a)^3 G(z)&=G(z+3)-3(1+a)\,G(z+2)+3(1+a)^2 G(z+1)-(1+a)^3 G(z)\\ &=(1+a)^{x+1}(1+ia)^{y-2}\{Ca^2+Da+E\}\end{aligned}$$

we can also show that $C=0$, $D=0$ and $E=0$ by a straightforward calculation, i. e. (c) is proved. A proof of (d) is similar to the proof of Proposition 3.

PROPOSITION 5. *Let* $F(z)=\dfrac{d}{da}\, e^{a,z}$, *then*

$$F^{(n)}(z)=a^{n-1}(1+a)^{x-1}(1+ia)^{y-1}\left\{n+a\left[z+n(1+i)\right]+a^2(x+y+n)\,i\right\}$$

for $n=0, 1, 2, \cdots$, *and* $F^{(0)}(z)\equiv F(z)$, *where $F^{(n)}$ means n'th monodiffric derivative of F.*

PROOF. Since $F(z)\in M(D)$, we have $F^{(n)}(z)\in M(D)$ and

$$\begin{aligned}F^{(n+1)}(z) &= F^{(n)}(z+1)-F^{(n)}(z)\\ &= a^{n-1}(1+a)^{x}\cdot(1+ia)^{y-1}\Big\{n+a\Big[z+1+n(1+i)\Big]+a^{2}(x+y+n+1)\,i\Big\}\\ &\qquad -a^{n-1}(1+a)^{x-1}(1+ia)^{y-1}\Big\{n+a\Big[z+n(1+i)\Big]+a^{2}(x+y+n)\,i\Big\}\\ &= a^{n-1}(1+a)^{x-1}(1+ia)^{y-1}\Big\{a(n+1)+a^{2}\Big[z+(n+1)(1+i)\Big]\\ &\qquad +a^{3}(x+y+n+1)\,i\Big\}.\end{aligned}$$

By induction, Proposition 5 is proved.

4. Monodiffric homogeneous difference equations.

THEOREM 1. *Let $a_1, a_2, \cdots a_n$ be distinct roots of*

$$a^{n}+c_{n-1}a^{n-1}+\cdots+c_{1}a+c_{0}=0\,, \tag{4. 1}$$

then the general solution to (1. 1) is

$$F(z)=\sum_{i=1}^{n}B_{i}e^{a_{i},z}\,,$$

where the coefficients B_i ($i=1, 2, \cdots, n$) are arbitrary constants.

PROOF. Let $F(z)=e^{a,z}$, then from Proposition 1

$$(a^{n}+c_{n-1}a^{n-1}+\cdots+c_{1}a+c_{0})\,e^{a,z}=0\,.$$

Since $e^{a,z}\neq 0$, we must have

$$a^{n}+c_{n-1}a^{n-1}+\cdots+c_{1}a+c_{0}=0\,.$$ Since

$a_1, a_2, \cdots, a_n$ are distinct roots of (4. 1), we obtain that $e^{a_i,z}$ ($i=1, 2, \cdots, n$) is a solution of (1. 1). The general solution to (1. 1) is

$$F(z)=\sum_{i=1}^{n}B_{i}e^{a_{i},z}\,.$$

For any monodiffric function $f(z)$, we can rewrite $(D_z-a)^n f(z)$ into the form $\sum_{k=0}^{n}(-1)^{k}C_{k}^{n}f(z+n-k)(1+a)^{k}$ where $C_{k}^{n}=\dfrac{n\,!}{(n-k)\,!\;k\,!}$ and from the results of Proposition 3 and 4, we have the following Proposition (6) (a) and (b) respectively.

PROPOSITION 6. (a) *Monodiffric difference equation of the second order*

$\sum_{k=0}^{2}(-1)^k C_k^2 f(z+2-k)(1+a)^k=0$ *has monodiffric general solution of the form* $f(z)=c_0 e^{a,z}+c_1 \frac{d}{da} e^{a,z}$.

(b) *Monodiffric difference equation of the third order*

$\sum_{k=0}^{3}(-1)^k C_k^3 f(z+3-k)(1+a)^k=0$ *has monodiffric general solution* $f(z)$ $=c_0 e^{a,z}+c_1 \frac{d}{da} e^{a,z}+c_2 \frac{d^2}{da^2} e^{a,z}$,

where the coefficients c_i $(i=0, 1, 2)$ *are arbitrary constants.*

In general, we can extend to the n'th order monodiffric homogeneous difference equation. Let $E(a, z)=e^{a,z}$, $E^{(n)}(a, z)=\frac{d^n}{da^n} e^{a,z}$ for $n\in N$, by induction we will obtain $E^{(n)}(a, z)\in M(D)$. Suppose it is true for $n=k$, thus $ME^{(k)}(a, z)=0$ for all $z\in D$, then

$$(i-1)E^{(k)}(a, z)+E^{(k)}(a, z+i)-iE^{(k)}(a, z+1)=0$$

$$(i-1)E^{(k)}(a+h, z)+E^{(k)}(a+h, z+i)-iE^{(k)}(a+h, z+1)=0\,.$$

Subtracting the first from the second of above equalities and dividing by h, we have

$$(i-1)\frac{E^{(k)}(a+h, z)-E^{(k)}(a, z)}{h}+\frac{E^{(k)}(a+h, z+i)-E^{(k)}(a, z+i)}{h}$$
$$-i\frac{E^{(k)}(a+h, z+1)-E^{(k)}(a, z+1)}{h}=0\,.$$

Tending h to 0, we see that

$$(i-1)E^{(k+1)}(a, z)+E^{(k+1)}(a, z+i)-iE^{(k+1)}(a, z+1)=0$$

$$ME^{(k+1)}(a, z)=0 \qquad \text{for all } z\in D.$$

Thus we obtain:

PROPOSITION 7. $E^{(n)}(a, z)$ *is a monodiffric function* *for* $n=0, 1, 2, \cdots$.

PROPOSITION 8. $(D_z-a)^n E^{(n-1)}(a, z)=0$ *for* $n=1, 2, \cdots$.

PROOF. It is true for $n=1, 2$. By induction we suppose it is true for $n=k$, i. e. $(D_z-a)^k E^{(k-1)}(a, z)=0$.

Fixing z and differentiating with respect to a, we have

$$(D_z-a)^k E^{(k)}(a, z)=k(D_z-a)^{k-1}E^{(k-1)}(a, z)\,.$$

Applying D_z-a, we have

$$(D_z-a)^{k+1}E^{(k)}(a, z)=k(D_z-a)^k E^{(k-1)}(a, z)=0\,. \quad \text{q. e. d.}$$

By the same idea, we get the following:

THEOREM 2. *Monodiffric homogeneous difference equation of the n'th order* $\sum_{k=0}^{n}(-1)^k C_k^n f(z+n-k)(1+a)^k=0$ *has monodiffric general solution* $f(z)=\sum_{k=0}^{n-1} c_k \frac{d^k}{da^k} e^{a,z}$ *where the coefficients* $c_k(k=0, 1, \cdots, n-1)$ *are arbitrary constants.*

From Theorem 1 *and Theorem* 2, *we can get the general case which is stated as follows.*

THEOREM 3. *The general solution to the homogeneous monodiffric difference equation of the n'th order*

$$F^{(n)}(z)+c_{n-1}F^{(n-1)}(z)+\cdots+c_1F'(z)+c_0F(z)=0$$

is

$$F(z)=\sum_{k=1}^{p}\sum_{j=0}^{m_k-1} B_{kj}\frac{d^j}{da_k^j}e^{a_k,z},$$

where $a_1, a_2, \cdots, a_p$ *with multiplicities* $m_1, m_2, \cdots m_p$ *respectively are the roots of*
$a^n+c_{n-1}a^{n-1}+\cdots+c_1a+c_0=0$ *and the coefficients* B_{kj} *are arbitrary constants.*

References

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