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On the Jacobson radical of the center of an infinite group algebra

Dedicated to Professor Goro Azumaya on the occasion of his 60th birthday

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Throughout K will represent an algebraically closed field of characteristic $p>0$, and G a group. We denote by G', $Z(G)$ and P the commutator subgroup, the center and a Sylow p-subgroup of G respectively. For $x\in G$, C_x is the conjugacy class of G containing x. Given a finite subset S of G, we denote by $\hat{S}$ the element $\sum_{x\in S}x$ of the group algebra KG. If R is a ring (with identity), then $Z(R)$ and $J(R)$ denote the center and the (Jacobson) radical of R respectively, and $N(R)$ is the sum of all the nilpotent ideals of R.

In case G is a finite p-solvable group, R. J. Clarke [1] gave a necessary and sufficient condition for $J(Z(KG))$ to be an ideal of KG. Recently, S. Koshitani [2] proved that if G is finite and $J(Z(KG))$ is an ideal of KG then G is p-solvable. Hence, in case G is finite, the problem to find a necessary and sufficient condition for $J(Z(KG))$ to be an ideal of KG has been solved completely. In this paper, we consider this problem for infinite groups, and give an answer for poly-$\{p, p'\}$ groups.

At first we recall the following

THEOREM 1 (Passman [5, Lemma 4. 1. 11]). $J(KG)\cap Z(KG)=J(Z(KG))$.

Now, by making use of the same argument as in the proof of [1, Lemma 4], we shall prove the next

LEMMA 1. *Suppose that $J(Z(KG))$ is an ideal of KG. Then the following statements hold*:

(1) *If G' is an infinite group, then $J(Z(KG))=0$.*

(2) *If G' is a finite group with $p\nmid|G'|$, then $J(Z(KG))=\hat{G}'J(KG)$.*

(3) *If G' is a finite group with $p||G'|$, then $J(Z(KG))=\hat{G}'KG$.*

PROOF. Since $J(Z(KG))$ is an ideal of KG, for x, $y\in G$ and $a\in J(Z(KG))$ we have

$$(x^{-1}y^{-1}xy)\,a=x^{-1}y^{-1}(ya)\,x=x^{-1}ax=a\,.$$

Hence $ga=a$ for all $g\in G'$. Therefore it is easily seen that if G' is infinite then $J(Z(KG))=0$, and that if G' is finite then $J(Z(KG))\subset\hat{G}'KG$. Now, we assume that G' is finite. If $p\nmid|G'|$, then $e=|G'|^{-1}\hat{G}'$ is a central idempotent of KG and we have $J(Z(KG))\subset eJ(KG)$. Since $eKG\subset Z(KG)$, by Theorem 1 we have $eJ(KG)=J(eKG)\subset J(Z(KG))$. Hence it holds that $J(Z(KG))=eJ(KG)=\hat{G}'J(KG)$. Next, if $p\mid|G'|$, then $\hat{G}'$ is a central nilpotent element of KG, and so $\hat{G}'\in J(Z(KG))$. Thus, we have $J(Z(KG))=\hat{G}'KG$.

We call a group H a p'-group if H has no elements of order p. Now, we put

$\Delta(G)=\{x\in G\mid[G: C_G(x)]$ is finite$\}$.

$\Delta^+(G)=\{x\in\Delta(G)\mid x$ is of finite order$\}$.

$\Delta^p(G)=\langle x\in\Delta(G)\mid x$ is of order a power of $p\rangle$.

These are characteristic subgroups of G, and have the following properties ([5, Lemma 8.1.6]).

(i) $\Delta(G)/\Delta^+(G)$ is torsion free abelian.

(ii) $\Delta^+(G)/\Delta^p(G)$ is a locally finite p'-group.

A group G is said to be an FC (finite conjugate) group if $G=\Delta(G)$. The following theorem plays an important role in our subsequent study.

THEOREM 2 (Passman [5, Theorem 4.2.13]). *The following statements are equivalent*:

(1) *KG is semi-prime.*

(2) *$Z(KG)$ is semi-prime.*

(3) *$Z(KG)$ is semi-simple.*

(4) *G has no finite normal subgroups H with $p\mid|H|$.*

(5) *$\Delta(G)$ is a p'-group.*

Combining Theorem 2 with Lemma 1, we can now obtain the following

COROLLARY 1. *Let G be a non-abelian group with G' infinite. Then the following statements are equivalent*:

(1) *$J(Z(KG))$ is an ideal of KG.*

(2) *$J(Z(KG))=0$.*

(3) *G has no finite normal subgroups H with $p\mid|H|$.*

Henceforth, we may therefore restrict our attention to the case that G' is finite. Note that if G' is finite then G is an FC group. Theorem 2 together with Theorem 1 and [5, Lemma 8.1.8] deduces the following

COROLLARY 2. *Let G be an FC group. Then KG is semi-simple if and only if $Z(KG)$ is semi-simple.*

Now, by making use of the same argument as in the proof of [5, Lemma 8.1.8], we shall prove the following lemma, which implies the if part in the above corollary for a twisted group algebra (see Corollary 3 below).

LEMMA 2. *Let G be an FC group, and K^tG a twisted group algebra of G. Then $J(K^tG)=N(K^tG)$.*

PROOF. Since $G/\Delta^+(G)$ is a torsion free abelian group, by [3, Corollary 1.11] we have $J(K^tG)\subset J(K^t\Delta^+(G))K^tG$. Let $a\in J(K^t\Delta^+(G))$, and put $L=\langle\Delta^p(G), \operatorname{Supp} a\rangle$. Since $\Delta^+(G)/\Delta^p(G)$ is a locally finite p'-group, $L/\Delta^p(G)$ is a finite p'-group. Hence, by [3, Proposition 1.5] we see that $a\in J(K^t\Delta^+(G))\cap K^tL\subset J(K^tL)=J(K^t\Delta^p(G))K^tL$. Therefore, by [3, Theorem 3.7], we have $J(K^tG)\subset J(K^t\Delta^+(G))K^tG\subset J(K^t\Delta^p(G))K^tG=N(K^tG)$, namely, $J(K^tG)=N(K^tG)$.

COROLLARY 3. *Let G be an FC group, and K^tG a twisted group algebra of G. If $Z(K^tG)$ is semi-simple, then K^tG is semi-simple.*

PROOF. Suppose $J(Z(K^tG))=0$. Then K^tG is semi-prime by [4, Theorem 2.2]. Hence $N(K^tG)=0$, and so $J(K^tG)=0$ by Lemma 2.

Now, we shall prove the following

LEMMA 3. *Let G be an FC group, and N a finite normal p'-subgroup of G. If $(1-e)J(Z(KG))=0$, then $(1-e)J(KG)=0$, where $e=|N|^{-1}\hat{N}$.*

PROOF. Evidently, $f=1-e$ is a central idempotent of KG. Let $f=f_1+f_2+\cdots+f_n$ be the decomposition of f into the sum of orthogonal central primitive idempotents in KN, and let f_* be an arbitrary one of $\{f_i|1\leqq i\leqq n\}$. Suppose $\operatorname{Supp} f_*=\{x_1, x_2, \cdots, x_k\}$ and set $W=\cap_{i=1}^k C_G(x_i)$. Since G is an FC group, $[G:W]$ is finite. We put $H=\{g\in G|gf_*g^{-1}=f_*\}$. Then H contains W, and so $[G:H]$ is finite. Now, let $G=a_1H\cup a_2H\cup\cdots\cup a_sH$ be the decomposition of G into right cosets with respect to H. Then $a_jf_*a_j^{-1}$ $(1\leqq j\leqq s)$ is some one of $\{f_i|1\leqq i\leqq n\}$. We put $\tilde{f}_*=\sum_{j=1}^s a_jf_*a_j^{-1}$. Then $\tilde{f}_*$ is a central idempotent of KG, and by [5, Themrem 6.1.9], $\tilde{f}_*KG$ is isomorphic to the matrix ring $(K^tH/N)_m$ for some m, where K^tH/N is a suitable twisted group algebra of H/N. Since $fJ(Z(KG))=0$, we see that $J(Z(\tilde{f}_*KG))=\tilde{f}_*J(Z(KG))=0$, and so $J(Z(K^tH/N))=0$. Thus, by Corollary 3 we have $J(K^tH/N)=0$, and so $\tilde{f}_*J(KG)=0$. Hence $fJ(KG)=0$.

The proof of the next lemma is quite similar to that of [1, Lemma 5].

LEMMA 4. *Let N be a finite normal p'-subgroup of G. If $J(Z(KG))$ is an ideal of KG, then $J(Z(KG/N))$ is an ideal of KG/N.*

Now, we shall consider the case that G has a non-trivial normal p-subgroup. In case G is a p-group, it is known that $Z(G)=\{1\}$ if and only if G has no non-trivial finite normal subgroups ([6, Theorem 6.3.1]). In fact, there does exist an infinite p-group Q with $Z(Q)=\{1\}$ (see [6, Example 5, p. 216]).

LEMMA 5. *Let G be a non-abelian group with G' finite. Suppose that G has a non-trivial normal p-subgroup Q. If $J(Z(KG))$ is an ideal of KG, then the following statements hold*:

(1) *$G' \subset Q$, and so P is normal in G and $G' \subset Z(P)$.*

(2) *Let $P = G' \cup (\cup_{i \in I} G's_i)$ be the decomposition of P into left cosets with respect to G'. Then the conjugacy classes of the elements of P in G are $\{1\}$, $G' - \{1\}$ and $\{G's_i \mid i \in I\}$.*

PROOF. Let $s \in Q - \{1\}$. Since Q' is finite, Q is locally finite, and so $\hat{C}_s - |C_s| \in J(KQ) \cap Z(KG)$. Hence $\hat{C}_s - |C_s|$ is a central nilpotent element of KG, so that it is contained in $J(Z(KG))$. Thus, by Lemma 1 we see that $\hat{C}_s - |C_s| \in \hat{G}'KG$, whence it follows that $\hat{C}_s - |C_s| = \sum_{x \in S} k_x \hat{G}' x$, where S is a suitable finite subset of G and $k_x \in K$. Since $C_s \subset G's$, the above equation yields

$$\text{(a)} \qquad \hat{C}_s - |C_s| = \hat{G}' s .$$

Hence we have $\hat{G}' = \hat{C}_s s^{-1} - |C_s| s^{-1}$, which implies that $G' \subset Q$. In partiqular, P is normal in G. Since G' is a finite normal subgroup of P, as was claimed just before Lemma 5, $Z(P)$ is a non-trivial normal subgroup of G, and so $G' \subset Z(P)$, proving (1). Now, since P is normal in G, (a) holds for any element s of $P - \{1\}$. Then (2) readily follows from the last.

REMARK 1. In the above lemma, if G is finite then $G' = Z(P)$ (see [1, Lemma 8]). In fact, if $s \in Z(P)$ then $p \nmid |C_s|$, and hence we have $s \in G'$ from the equation $\hat{C}_s - |C_s| = \hat{G}' s$.

Now, we consider the case that G' is a finite p-solvable group. By making use of Lemmas 4 and 5, we shall prove the following

LEMMA 6. *Let G be a non-abelian group. Assume that G' is a finite p-solvable group. If $J(Z(KG))$ is an ideal of KG, then G' is p-nilpotent.*

PROOF. Suppose that $|G'|$ is divisible by p. We put $N = O_{p'}(G')$ and $\bar{G} = G/N$. Then $J(Z(K\bar{G}))$ is an ideal of $K\bar{G}$ by Lemma 4. Since $O_p(\bar{G}')$ is a nontrivial normal p-subgroup of $\bar{G}$, by Lemma 5 (1) we see that $\bar{G}'$ is a p-group. Hence G' is p-nilpotent.

PROPOSITION 1. *Let G be a non-abelian group with a non-trivial Sylow p-subgroup P. Assume that G' is a finite p-solvable group with $O_{p'}(G') \neq \{1\}$. Then $J(Z(KG))$ is an ideal of KG if and only if the following hold*:

(1) *P is finite.*

(2) *$G'P$ is a Frobenius group with kernel $O_{p'}(G')$ and complement P.*

(3) *$J(Z(KG/O_{p'}(G')))$ is an ideal of $KG/O_{p'}(G')$.*

Proof. We put $N=O_{p'}(G')$ and $e=|N|^{-1}\hat{N}$.

Suppose that $J(Z(KG))$ is an ideal of KG. Then by Lemma 6, G' is a p-nilpotent group. Now, let T be a finite subgroup of G containing G' such that T/N is a p-group. Since T is normal in G, $J(KT)\subset J(KG)$. Moreover, since $J(Z(KG))\subset \hat{G}'KG\subset \hat{N}KG$, by Lemma 3 we have $(1-e)J(KT)\subset(1-e)J(KG)=0$. This implies that $J(KT)=eJ(KT)\cong J(KT/N)\cong J(KQ)$, where Q is a Sylow p-subgroup of T. Then by [7, Theorem 2], T is a Frobenius group with kernel N. Hence, we have $|N|=1+k|Q|$ for some positive integer k, which implies that $|T/G'|\leqq|T/N|=|Q|<|N|$. Thus, the order of any finite subgroup of the abelian p-group PN/G' is not greater than $|N|$. This is only possible if P itself is finite. We see therefore that $G'P$ is a finite Frobenius group with kernel N. Furthermore, (3) follows from Lemma 4.

Conversely, suppose that the conditions (1), (2) and (3) hold. Since $G/G'P$ is abelian and has no elements of order p, we have $J(KG)=J(KG'P)KG$ ([5, Theorem 7.3.1]). Moreover, since $G'P$ is a finite Frobenius group with kernel N, we have $J(KG'P)=eJ(KP)$ ([7, Theorem 2]). Hence, $J(KG)=eJ(KP)KG=eJ(KG)=J(eKG)$. This implies that $J(Z(KG))=eJ(Z(KG))=J(Z(eKG))$, because $J(Z(KG))\subset J(KG)$ (Theorem 1). Since $eKG\cong KG/N$, it holds that $J(Z(eKG))\cong J(Z(KG/N))$, and hence by the condition (3), we see that $J(Z(KG))$ is an ideal of KG.

D. A. R. Wallace [8] gave a necessary and sufficient condition for $J(KG)$ to be contained in $Z(KG)$. The condition (3) in the next corollary is the condition (2) in [8, Theorem 1.2].

Corollary 4. *Let G be a non-abelian group with a non-trivial Sylow p-subgroup P. Assume that G' is a finite p'-group. Then the following are equivalent:*

(1) *$J(Z(KG))$ is an ideal of KG.*

(2) *$J(KG)=J(Z(KG))$.*

(3) *P is finite and $G'P$ is a Frobenius group with kernel G' and complement P.*

Proof. (2)$\Rightarrow$(1)$\Rightarrow$(3) by Proposition 1. If (3) is satisfied, then $J(KG)=J(KG'P)KG=\hat{G}'J(KP)KG=\hat{G}'J(KG)\subset J(Z(KG))$. Hence, by Theorem 1 we have (2).

Now, we consider the case that G has a non-trivial normal p-subgroup, and establish a necessary and sufficient condition for $J(Z(KG))$ to be an ideal of KG. At first, we shall deal with the case that $Z(G)$ has a p-element.

Proposition 2. *Let G be a non-abelian group with G' finite. Assume that $Z(G)$ has a p-element. Then the following are equivalent:*

(1) *$J(Z(KG))$ is an ideal of KG.*

(2) *$p=2$, $|G'|=2$ and $Z(G)\cap P=G'$.*

PROOF. (1)⇒(2): Let s be an arbitrary p-element of $Z(G)$. Then $s-1\in J(Z(KG))$. Since G' is a p-group (Lemma 5), by Lemma 1 we have $s-1\in\hat{G}'KG$. This implies that the order of s is 2 and $G'=\langle s\rangle$. Hence, we have $p=2$ and $Z(G)\cap P=G'$.

(2)⇒(1): If $g\in G-Z(G)$, then $[G: C_G(g)]=2$ by $|G'|=2$, and it is easy to see that G has conjugacy classes $\{z\}_{z\in Z(G)}$ and $\{G'x\}_{x\in S}$, where S is a suitable subset of G. Since each $\hat{G}'x$ $(x\in S)$ is a central nilpotent element of KG, it is contained in $J(Z(KG))$. Now, suppose that $a=\sum_{z\in Z(G)}k_z z$ $(k_z\in K)$ is in $J(Z(KG))$. Then, by Theorem 1 we have $a\in KZ(G)\cap J(KG)$ $\subset J(KZ(G))=J(KG')\,KZ(G)=\hat{G}'KZ(G)$. This implies that $J(Z(KG))=\hat{G}'KG$, which is an ideal of KG.

REMARK 2. Since $Z(G)\cap P\subset Z(P)$, Remark 1 enables us to see that, in the above proposition, if G is finite then the condition (2) may be replaced by the following:

(2)′ *$p=2$, $|G'|=2$ and $Z(P)=G'$* (see [1, Lemma 8]).

COROLLARY 5 (cf. [1, Corollary]). *Let P be a non-abelian p-group. Then $J(Z(KP))$ is an ideal of KP if and only if one of the following conditions holds:*

(1) *$Z(P)=\{1\}$.*

(2) *$p=2$, $|P'|=2$ and $P'=Z(P)$.*

PROOF. Suppose that $J(Z(KP))$ is an ideal of KP. If $J(Z(KP))=0$, then (1) holds by Theorem 2 and the remark stated just before Lemma 5. On the other hand, if $J(Z(KP))\neq 0$, then P' is finite (Lemma 1), and so $Z(P)\neq\{1\}$. Hence (2) holds by Proposition 2. The converse implication is clear by Theorem 2 and Proposition 2.

Next, we consider the case that $Z(G)$ has no elements of order p.

PROPOSITION 3. *Let G be a non-abelian group with G' finite. Assume that G has a non-trivial normal p-subgroup and that $Z(G)$ has no elements of order p. Then the following conditions are equivalent:*

(1) *$J(Z(KG))$ is an ideal of KG.*

(2) *$P=G'$, P is an elementary abelian group of order greater than 2, and it has a complement $H\supset Z(G)$ in G such that $\bar{G}=G/Z(G)=\bar{P}\bar{H}$ is a finite Frobenius group with kernel $\bar{P}$ and complement $\bar{H}$ and $|\bar{H}|=|P|-1$.*

In advance of proving the proposition, we state the following

LEMMA 7. *Suppose that G satisfies the assumptions in Proposition 3.*

If $J(Z(KG))$ is an ideal of KG, then the following statements hold:

(1) *P is an abelian group containing at least three elements, $G'\subset P$ and G' has a complement $H\supset Z(G)$ in G.*

(2) *If $h\in H$ and $C_G(h)\cap G'\neq\{1\}$, then $h\in Z(G)$.*

PROOF. (1) Since G has a non-trivial normal p-subgroup, $G'\subset Z(P)$ by Lemma 5 (1), and hence we have $G'\cap Z(G)=\{1\}$, because $Z(G)$ has no elements of order p. Let $s\in G'-\{1\}$. Then by the above, there exists some $x\in G$ with $xsx^{-1}\neq s$. Now, by Lemma 5 (2), for any $t\in G'-\{1\}$ there exists some $g\in G$ with $gsg^{-1}=t$, and hence we have

$$xtx^{-1}=xgsg^{-1}x^{-1}=gx(x^{-1}g^{-1}xg)\,s(g^{-1}x^{-1}gx)\,x^{-1}g^{-1}.$$

Since $x^{-1}g^{-1}xg\in G'\subset Z(P)$, the last implies that

$$xtx^{-1}=gxsx^{-1}g^{-1}\neq gsg^{-1}=t.$$

Thus, we have $G'\cap C_G(x)=\{1\}$. This together with $[G:C_G(x)]\leqq|G'|$ shows that $H=C_G(x)$ is a complement of G' in G and $H\supset Z(G)$. Again by $G'\subset Z(P)$, we see that P is the direct product of G' and $P\cap H$, and hence P is abelian. Finally, if $|P|=2$, then P is contained in $Z(G)$. But this is a contradiction.

(2) Let $h\in H-\{1\}$, and suppose that $C_G(h)\cap G'\neq\{1\}$. Let $s\in(C_G(h)\cap G')-\{1\}$. Then, by Lemma 5 (2), for any $t\in G'-\{1\}$, there exists $g\in G$ with $gsg^{-1}=t$. Since $hth^{-1}=hgsg^{-1}h^{-1}=gh(h^{-1}g^{-1}hg)\,s(g^{-1}h^{-1}gh)\,h^{-1}g^{-1}=ghsh^{-1}g^{-1}=gsg^{-1}=t$, we see that $h\in C_G(G')$. This together with the fact that H is abelian implies that $h\in Z(G)$.

PROOF OF PROPOSITION 3. (1)$\Rightarrow$(2): Suppose that $J(Z(KG))$ is an ideal of KG. Then, by Lemma 7, P is abelian and has at least three elements, and $G'(\subset P)$ has a complement $H\supset Z(G)$ in G. We put $\bar{G}=G/Z(G)$. Let $\bar{s}\in\bar{G}'-\{\bar{1}\}$, and $\bar{h}\in C_{\bar{H}}(\bar{s})$. Then $shs^{-1}h^{-1}\in G'\cap Z(G)=\{1\}$, and hence $s\in C_G(h)$. Thus, we have $\bar{h}=\bar{1}$ by Lemma 7 (2). Since $\bar{G}=\bar{G}'\bar{H}$, this implies that $C_{\bar{G}}(\bar{s})=\bar{G}'$, and hence $|\bar{H}|=[\bar{G}:\bar{G}']=[\bar{G}:C_{\bar{G}}(\bar{s})]<\infty$. We conclude therefore that $\bar{G}$ is a finite Frobenius group with kernel $\bar{G}'$ and complement $\bar{H}$, which implies also $G'=P$. Now, let $s\in P-\{1\}$. Since $\bar{P}-\{\bar{1}\}$ is a conjugacy class in $\bar{G}$ (Lemma 5 (2)), we have $\{\bar{h}\bar{s}\bar{h}^{-1}|\bar{h}\in\bar{H}\}=\bar{P}-\{\bar{1}\}$. Furtheremore, since $\bar{G}$ is a Frobenius group, we have $|\bar{H}|=|\{\bar{h}\bar{s}\bar{h}^{-1}|\bar{h}\in\bar{H}\}|=|\bar{P}|-1=|P|-1$. Finally, it is clear that P is elementary abelian, because $P-\{1\}$ is a conjugacy class in G.

(2)$\Rightarrow$(1): let g be an arbitary element of $G-Z(G)$. Firstly, assume that $g\in Z(G)P$, and put $g=zs$ with $z\in Z(G)$ and $s\in P-\{1\}$. Since $\bar{G}$ is a Frobenius group and $\bar{P}$ is abelian, there holds that $\bar{P}\subset\overline{C_G(s)}\subset C_{\bar{G}}(\bar{s})=\bar{P}$.

Hence $C_G(s)=Z(G)\,P$, and so $[G:C_G(s)]=[\bar{G}:\bar{P}]=|\bar{H}|=|P|-1$. Noting that $C_G(zs)=C_G(s)$, we have $C_g=(P-\{1\})\,z$. Secondly, assume $g\notin Z(G)\,P$. Then $g=zas$ with some $z\in Z(G)$, $a\in H-Z(G)$ and $s\in P$. Since $\bar{G}$ is a Frobenius group and $\bar{H}$ is abelian, $\overline{as}$ is contained in $\overline{H^u}$ for some $\bar{u}\in\bar{G}$ and there holds that $\overline{H^u}\subset\overline{C_G(as)}\subset C_{\bar{G}}(\overline{as})=\overline{H^u}$. Hence $C_G(as)=H^u$, which implies that $[G:C_G(as)]=|P|$. Noting that $C_G(zas)=C_G(as)$, we have $C_g=Pg$. Thus, we have seen that G has conjugacy classes $\{z\}_{z\in Z(G)}$, $\{(P-\{1\})\,z\}_{z\in Z(G)}$ and $\{Px\}_{x\in S}$, where S is a suitable subset of G. Since each $\hat{P}x$ $(x\in S)$ is a central nilpotent element of KG, it is contained in $J(Z(KG))$. Now, suppose that $a=\sum_{z\in Z(G)}k_z z+\sum_{z\in Z(G)}l_z(\hat{P}-1)\,z\,(k_z,\,l_z\in K)$ is in $J(Z(KG))$. Since $a=\sum_{z\in Z(G)}(k_z-l_z)\,z+\sum_{z\in Z(G)}l_z\hat{P}z$ and $\hat{P}z\in J(Z(KG))$ for all $z\in Z(G)$, by Theorem 1 we have $\sum_{z\in Z(G)}(k_z-l_z)\,z\in KZ(G)\cap J(KG)\subset J(KZ(G))=0$, which implies $J(Z(KG))\subset\hat{P}KG$. Hence $J(Z(KG))=\hat{P}KG=\hat{G'}KG$, which is an ideal of KG.

We call G a *poly-*$\{p,p'\}$ group, if G has a finite normal series

$$G=G_n\supset\cdots\supset G_1\supset G_0=\{1\}$$

such that each quotient G_{i+1}/G_i is a p-group or a p'-group. Now, we can state our principal theorem as follows:

THEOREM 3. *Let G be a non-abelian poly-$\{p,p'\}$ group. Then $J(Z(KG))$ is an ideal of KG if and only if one of the following statements holds:*

(1) *G has no finite normal subgroups H with $p\,|\,|H|$.*

(2) *$p=2$, $|G'|=2$ and $Z(G)\cap P=G'$.*

(3) *$P=G'$, P is a finite elementary abelian group of order greater than 2, and it has a complement $H\supset Z(G)$ in G such that $\bar{G}=G/Z(G)$ is a finite Frobenius group with kernel $\bar{P}(\cong P)$ and complement $\bar{H}$ and $|\bar{H}|=|P|-1$.*

(4) *G' is a finite p'-group, P is finite, and $G'P$ is a Frobenius group with kernel G' and complement P.*

(5) *$p=2$, G' is a finite group of order $2m$ (m is odd), P is finite, and $G'P$ is a Frobenius group with kernel G'' and complement P such that $Z(\bar{G})\cap\bar{P}=\bar{G}'$, where $\bar{G}=G/G''$.*

(6) *p is odd, $|P|=p$ and G' is a Frobenius group of order pn (n is prime to p) with kernel G'' and complement P such that $\bar{G}'$ has a complement $\bar{H}\supset Z(\bar{G})$ in $\bar{G}=G/G''$. Further, $\tilde{G}=\bar{G}/Z(\bar{G})$ is a finite Frobenius group with kernel $\tilde{P}$ $(\cong P)$ and complement $\tilde{H}$ and $|\tilde{H}|=p-1$.*

PROOF. Assume that $J(Z(KG))$ is an ideal of KG. If $J(Z(KG))=0$, then (1) holds by Theorem 2. From now on, we restrict our attention to the case that $J(Z(KG))\neq 0$. Then G' is finite by Lemma 1. If G has a normal p-subgroup, then (2) (resp. (3)) follows from Proposition 2 (resp. Proposition 3). Accordingly, henceforth we may assume that G' is finite and G has no normal p-subgroups. Firstly, if G' is a p'-group, then (4) holds by Corollary 4. Secondly, assume that $|G'|$ is divisible by p. Since G is a poly-$\{p,p'\}$ group, G' is a finite p-solvable group. We have $O_{p'}(G')\neq\{1\}$, because $O_{p'}(G')=\{1\}$ implies a contradiction that $O_p(G')$ is a non-trivial normal p-subgroup of G. We put $N=O_{p'}(G')$ and $\bar{G}=G/N$. Then $\bar{G}$ has a non-trivial normal p-subgroup. By Proposition 1, it holds that $G'P$ is a finite Frobenius group with kernel N and complement P, and $J(Z(K\bar{G}))$ is an ideal of $K\bar{G}$. Now, we shall distinguish between two cases.

Case 1. $Z(\bar{G})$ has a p-element. By Proposition 2, $p=2$ and $Z(\bar{G})\cap\bar{P}=\bar{G}'$ is of order 2. Since $G'P$ is a Frobenius group, G' is also a Frobenius group with kernel N, and hence $N\subset G''$. Noting that G'/N is abelian, we have $N=G''$, and therefore (5).

Case 2. $Z(\bar{G})$ has no elements of order p. By Proposition 3, $\bar{G}'=\bar{P}$ is an elementary abelian group of order greater than 2, $\bar{P}$ has a complement $\bar{H}\supset Z(\bar{G})$ in $\bar{G}$, and $\tilde{G}=\bar{G}/Z(\bar{G})$ is a finite Frobenius group with kernel $\tilde{P}$ ($\cong P$) and complement $\tilde{H}$ of order $|P|-1$. Since G' ($=G'P$) is a Frobenius group with complement P elementary abelian, we see that P is a cyclic group of order $p>2$. Furtheremore, as in Case 1, we have $N=G''$, and therefore (6).

The converse implication follows from Theorem 2, Propositions 1, 2 and 3, and Corollary 4.

COROLLARY 6. *Let G be a non-abelian poly-$\{p,p'\}$ group. If $J(Z(KG))$ is a non-trivial ideal of KG, then P is one of the following groups*:

(1) *a finite elementary abelian group.*
(2) *a finite cyclic group.*
(3) *a finite generalized quaternion group.*
(4) *a 2-group whose commutator subgroup is of order 2.*

REMARK 3. If G satisfies the condition (2) or (5) in Theorem 3 and $|P|=2$, then G is a group cited in [8, Theorem 1.2(1)].

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