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Notes on totally umbilical submanifolds in constant curvature spaces

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Introduction. Let E^n be an n-dimensional Euclidean space and X_x be a position vector of a point $x \in E^n$ with respect to a fixed origin. Then a mapping $x \to X_x$ defines a position vector field in E^n and it is a homothetic Killing vector field. The position vector field in E^n plays an important role in the investigations of submanifolds in E^n. From this veiwpoint, Y. Katsurada [2][1)] introduced the idea such that a conformal Killing vecotr field is available for the study of hypersurfaces in an n-dimensional Riemannian space M^n. By virtue of this idea, various results for global properties of closed orientable hypersurfaces in E^n have been generalized for those in M^n [4]. In the present paper, a closed submanifold means a compact connected submanifold without boundary.

Now, an odd dimensional sphere S^{2n+1} has a normal contact metric structure [8]. Making use of the properties of this structure, M. Okumura [7] gave a condition for a closed orientable sumbanifold of codimension 2 in S^{2n+1} to be totally umbilic.

A skew symmetric tensor field $T_{i_1 \cdots i_p}$ in M^n is called a conformal Killing tensor field of degree p ([9], [1]), if there exists a skew symmetric tensor field $\rho_{i_1 \cdots i_{p-1}}$ such that

$$\begin{aligned} T_{i_1 \cdots i_p ; i} + T_{i i_2 \cdots i_p ; i_1} &= 2 \rho_{i_2 \cdots i_p} g_{i_1 i} \\ &\quad - \sum_{h=2}^{p} (-1)^h \{ \rho_{i_1 \cdots \hat{i}_h \cdots i_p} g_{i i_h} + \rho_{i \cdots \hat{i}_h \cdots i_p} g_{i_1 i_h} \} , \end{aligned}$$

where the symbol $\wedge$ over i_h indicates the index i_h is to be omitted and the symbol ; means the covariant differentiation with respect to the Christoffel symbols formed with the metric tensor g_{ij} of M^n. Then we can see that the structure tensor field of a normal contact metric space is a conformal Killing tensor field of degree 2. M. Morohashi [5] has shown that S^n admits a conformal Killing tensor field of degree p for any positive integer

1) Numbers in bracketes refer to the references at the end of this paper.

p such that $p \leqq n$. Then he used this tensor field for the study of submanifolds of codimension p in S^n and gave a certain generalization of the theorem due to M. Okumura. This result suggested to us that a conformal Killing tensor field of degree p in M^n may be used effectively for the study of submanifolds of codimension p in M^n. In particular, when an ambient space is a constant curvature space we have

THEOREM (M. Morohashi [6]). *Let $M^{m+p}(c)$ and V^m $(m, p \geqq 2)$ be an $(m+p)$-dimensional Riemannian space of constant curvature c and an m-dimensional closed orientable sumbanifold in $M^{m+p}(c)$ respectively If*

(i) *$M^{m+p}(c)$ admits a conformal Killing tensor field $T_{i_1 \cdots i_p}$*

(ii) *the mean curvature vector field of V^m is parallel with respect to the connection induced on the normal bundle of V^m,*

(iii) *the connection induced on the normal bundle of V^m is trivial,*

(iv) *$T_{i_1 \cdots i_p} N^{i_1}_{m+1} \cdots N^{i_p}_{m+p}$[2] has fixed sign on V^m,*

then V^m is a totally umbilical submanifold.

The purpose of the present paper is to show that a totally umbilical submanifold is characterized by the existence of a certain tensor field along the sumbanifold. § 1 is devoted to give some notations and fundamental formulas in the theory of submanifolds in a Riemannian space. In § 2 we give some lemmas for a submanifold with parallel mean curvature vector field. In § 3 we give a necessary and sufficient condition for a closed orientable submanifold to be totally umbilic.

§ 1. **Preliminaries.** Let $M^{m+p}(c)$ be an $(m+p)$-dimensional Riemannian space of constant curvature c covered by a system of coordinate neighborhoods $\{U; x^i\}$ and denote by g_{ij}, Γ^h_{ij} and R_{ihjk} the metric tensor, the Christoffel symbols formed with g_{ij} and the curvature tensor respectively. Then we have

$$(1.1) \qquad R_{ihjk} = c(g_{ik} g_{hj} - g_{ij} g_{hk}) \, .$$

We then consider an m-dimensional Riemannian space V^m covered by a system of coordinate neighborhoods $\{V; u^\alpha\}$ and denote by $g_{\alpha\beta}$, $\Gamma'^\gamma_{\alpha\beta}$ and $R'_{\delta\alpha\beta\gamma}$ the metric tensor, the Christoffel symbols formed with $g_{\alpha\beta}$ and the curvature tensor respectively.

We assume that V^m is isometrically immersed in $M^{m+p}(c)$ by the immersion: $V^m \rightarrow M^{m+p}(c)$ and represent the immersion by

2) $N^i_P (P = m+1, \cdots, m+p)$ denote the contravariant components of p mutually orthogonal unit vectors normal to V^m.

$$x^i = x^i(u^\alpha) \qquad (i=1, 2, \cdots, m+p\,;\ \alpha=1, 2, \cdots, m)\,.^{3)}$$

Since the immersion is isometric, we have

$$g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta\,. \qquad (B^i_\alpha = \partial x^i/\partial u^\alpha)$$

We choose p mutually orthogonal unit vectors N^i_P $(P=m+1, \cdots, m+p)$[4] normal to V^m. Denoting by the symbol ; the covariant differentiation along V^m due to van der Waerden-Bortolotti, we have the following formulas of Gauss and Weingarten for V^m:

(1. 2) $$B^i_{\alpha;\beta} = \sum_P b_{P\alpha\beta} N^i_P\,,$$

(1. 3) $$N^i_{P;\alpha} = -b_{P\alpha}^{\ \ \beta} B^i_\beta + \Gamma''^{\,Q}_{P\alpha} N^i_Q\,,$$

where $b_{P\alpha\beta}$ denotes the second fundamental tensor with respect to the normal vector N^i_P, $b_{P\alpha}^{\ \ \beta} = g^{\beta\gamma} b_{P\alpha\gamma}$ and $\Gamma''^{\,Q}_{P\alpha}$ indicate components of a connection induced on the normal bundle of V^m, that is

$$\Gamma''^{\,Q}_{P\alpha} = (N^i_{P;\alpha} + \Gamma^i_{kj} N^k_P B^j_\alpha)\, N_{Qi}\,.$$

By means of (1. 1)-(1. 3) we have the following Gauss and Codazzi equations:

(1. 4) $$R'_{\delta\alpha\beta\gamma} = c(g_{\alpha\beta} g_{\delta\gamma} - g_{\alpha\gamma} g_{\delta\beta}) + \sum_P (b_{P\alpha\beta} b_{P\delta\gamma} - b_{P\alpha\gamma} b_{P\delta\beta})\,,$$

(1. 5) $$b_{P\alpha\beta;\gamma} - b_{P\alpha\gamma;\beta} - b_{Q\alpha\beta} \Gamma''^{\,Q}_{P\gamma} + b_{Q\alpha\gamma} \Gamma''^{\,Q}_{P\beta} = 0\,.$$

When there exist p mutually orthogonal unit normal vector fields N^i_P such that $\Gamma''^{\,Q}_{P\alpha}=0$ $(P, Q=m+1, \cdots, m+p\,;\ \alpha=1, \cdots, m)$, we asy that the connection induced on the normal bundle of V^m is trivial. It has been shown that the connection induced on the normal bundle of V^m is trivial if and only if

(1. 6) $$b_{P\alpha}^{\ \ \gamma} b_{Q\gamma}^{\ \ \beta} = b_{Q\alpha}^{\ \ \gamma} b_{P\gamma}^{\ \ \beta} \qquad (P,\ Q = m+1, \cdots, m+p)$$

be satisfied.

Let N^i be an arbitrary vector field in the normal bundle of V^m. When $(N^i_{;\alpha})^\perp = 0$, i. e., the vector $N^i_{;\alpha}$ is tangent to V^m everywhere, the vector field N^i is said to be parallel with respct to the connection induced on the normal bundle of V^m, where $(\ \ )^\perp$ denotes the normal part of a vector in the round bracket.

The invariant normal vector field H^i defined by

(1. 7) $$H^i = \frac{1}{m} \sum_P b_{P\alpha}^{\ \ \alpha} N^i_P$$

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3) The Latin indices $i, j, k, \cdots$ and the Greek indices $\alpha, \beta, \gamma, \cdots$ run over the range $1, 2, \cdots, m+p$ and $1, 2, \cdots, m$, respectively.

4) The capital Latin indices $P, Q, R, \cdots$ run over the range $m+1, \cdots, m+p$.

is called the mean curvature vector field of V^m and its magnitude H is called the mean curvature.

Denote by $\kappa_{P1}, \kappa_{P2}, \cdots, \kappa_{Pm}$ the eigen values of $b_{P\alpha\beta}$ relative to $g_{\alpha\beta}$ and put

$$H_P=\frac{1}{m}\sum_{\alpha}\kappa_{P\alpha}\left(=\frac{1}{m}b_{P\alpha}^{\alpha}\right). \tag{1.8}$$

H_P is called the 1-st mean curvature of V^m with respect to N_P. At a point of V^m, if we have $\kappa_{P1}=\kappa_{P2}=\cdots=\kappa_{Pm}$ for a fixed integer P then the point is said to be umbilic with respect to N_P. A point of V^m is umbilic with respect to N_P if and only if

$$b_{P\alpha\beta}=H_P g_{\alpha\beta} \tag{1.9}$$

be satisfied at the point. When (1. 9) holds good for $P=m+1, \cdots, m+p$ at every point of V^m, the submanifold V^m is said to be totally umbilic. From the identity

$$b_{P\alpha\beta}b_P{}^{\alpha\beta}-\frac{1}{m}(b_{P\gamma}^{\gamma})^2=\left(b_{P\alpha\beta}-\frac{1}{m}b_{P\gamma}^{\gamma}g_{\alpha\beta}\right)\left(b_P{}^{\alpha\beta}-\frac{1}{m}b_{P\gamma}^{\gamma}g^{\alpha\beta}\right),$$

and the positive definiteness of the Riemannian metric $g_{\alpha\beta}$ we have

LEMMA 1. 1. *Let V^m be a submanifold in M^{m+p}. Then V^m is totally umbilic if and only if*

$$b_{P\alpha\beta}b_P{}^{\alpha\beta}=\frac{1}{m}(b_{P\gamma}^{\gamma})^2 \qquad (P=m+1, \cdots, m+p)$$

be satisfied at every point of V^m.

§ 2. **Submanifolds with parallel mean curvature vector field.** By virtue of (1. 3) and (1. 7) we get

LEMMA 2. 1. *Let V^m be a submanifold in M^{m+p}. Then the mean curvature vector field H^i is parallel with respect to the connection induced on the normal bundle of V^m if and only if $b_{P\alpha;\beta}^{\alpha}=b_{Q\alpha}^{\alpha}\Gamma'{}^{Q}_{P\beta}$.*

Furthermore, by means of (1. 5) and Lemma 2. 1 it follows that

LEMMA 2. 2. Let *V^m be a submanifold in $M^{m+p}(c)$ and the mean curvature vector field H^i is parallel with respect to the connection induced on the normal bundle of V^m, then $b_{P\alpha;\beta}^{\beta}=b_{Q\alpha}^{\beta}\Gamma'{}^{Q}_{P\beta}$.*

If the mean curvature H of V^m does not vanish everywhere on V^m, we have a uniquely determined unit normal vector at every point of V^m which has the same direction with the mean curvature vector H^i at the point. This is called the Euler vector field and we denote it by N_E^i. Thus, for a submanifold V^m with $H\neq 0$ we can choose p vectors $\{N^i_E, N^i_{m+2}, \cdots, N^i_{m+p}\}$ as a set of p mutually orthogonal unit vectors normal to V^m. In this case, from (1.7) and (1.8) we have

$$H_P=\frac{1}{m}b_{P\alpha}^{\alpha}=0 . \qquad (P=m+2, \cdots, m+p) \tag{2.1}$$

Then we get

$$H=H_E=\frac{1}{m}b_{E\alpha}^{\alpha}, \tag{2.2}$$

where we have used the index E in place of $m+1$. Then we have

LEMMA 2.3. (K. Yano [10]) *Let V^m be a submanifold of M^{m+p} and $H\neq 0$ everywhere on V^m. Then the following statements* (i) *and* (ii) *are equivalent*:

(i) *$H=const.$ and $\Gamma''^{P}_{E\alpha}=0$.* $(P=m+2, \cdots, m+p\,;\ \alpha=1, \cdots, m)$

(ii) *The mean curvature vector field H^i is parallel with respect to the connection induced on the normal bundle of V^m.*

Furthermore, we have

LEMMA 2.4. *Let V^m be a totally umbilical submanifold in $M^{m+p}(c)$ and $H\neq 0$ everywhere on V^m. Then the connection induced on the normal bundle of V_m is trivial and the mean curvature vector field H^i is parallel with respect to the connection induced on the normal bundle of V^m.*

§ 3. Characterizations of totally umbilical submanifolds.

THEOREM 3.1. *Let V^m be a closed orientable submanifold in $M^{m+p}(c)$ $(m\geqq 2,\ p\geqq 3)$ and the mean curvature H does not vanish everywhere on V^m. Then V^m is totally umbilic if and only if*

(i) *there exists a skew symmetric tensor field $T_{i_1\cdots i_p}$ along V^m such that*

$$T_{i_1\cdots i_a\cdots i_p;\alpha}N^{i_1}_{m+1}\cdots B^{i_a}_{\beta}\cdots N^{i_p}_{m+p} + T_{i_1\cdots i_a\cdots i_p;\beta}N^{i_1}_{m+1}\cdots B^{i_a}_{\alpha}\cdots N^{i_p}_{m+p}=\Phi_{m+a}g_{\alpha\beta}$$

for some functions Φ_{m+a} $(a=1, \cdots, p)$ and

$$T_{i_1\cdots i_p}N^{i_1}_{m+1}\cdots N^{i_p}_{m+p}$$

has fixed sign on V^m,

(ii) *the mean curvature vector field H^i is parallel with respect to the connection induced on the normal bundle of V^m,*

(iii) *the connection induced on the normal bundle of V^m is trivial.*

PROOF. The condition (i) for a skew symmetric tensor field $T_{i_1\cdots i_p}$ is independent on the choice of a set of p mututally orthogonal unit normal vector fields $\{N^i_{m+1}, \cdots, N^i_{m+p}\}$ (Cf. [6]). Then we prove Theorem 3.1 with respect to a suitable set of p mutually orthogonal unit normal vector fields.

Let V^m be a totally umbilical submanifold in $M^{m+p}(c)$ and $H\neq 0$ everywhere on V^m. Then we can choose $\{N^i_E, N^i_{m+2}, \cdots, N^i_{m+p}\}$ as a set of p mutually orthogonal unit normal vector fields along V_m. From Lemma 2.4, V^m satisfies (ii) and (iii). Now, we put

$$(3.1) \qquad T_{i_1 i_2\cdots i_p} = \sum_{\sigma\in S(i_1,\cdots i_p)} \mathrm{sgn}\,(\sigma)\, N_{E\sigma(i_1)}\, N_{m+2\sigma(i_2)}\cdots N_{m+p\sigma(i_p)}\,,$$

where $S(i_1, \cdots, i_p)$ denotes the symmetric group of all permutations of p integers $i_1, \cdots, i_p$. Then $T_{i_1\cdots i_p}$ is a skew symmetric tensor field along V^m. In this case $T_{i_1\cdots i_p}N^{i_1}_E N^{i_2}_{m+2}\cdots N^{i_p}_{m+p}=1$ on V^m and we can easily verify that $T_{i_1\cdots i_p}$ satisfies the first relation of (i) for $\Phi_{m+1}=-2H$ and $\Phi_{m+a}=0$ ($a=2, \cdots, p$). Therefore the skew symmetric tensor field $T_{i_1\cdots i_p}$ defined by (3.1) satisfies (i).

Next, we show that if we assume (i)-(iii), then V^m is a totally umbilical submanifold.

Let $\{N^i_{m+1}, \cdots, N^i_{m+p}\}$ be a set of p mutually orthogonal unit normal vector fields and with respect to this set $\Gamma''^{Q}_{P\alpha}=0$ ($P, Q=m+1, \cdots, m+p$; $\alpha=1, \cdots, m$). Then, by means of Lemma 2.1 and Lemma 2.2 it follows that

$$(3.2) \qquad b_{P\alpha;\beta}^{\alpha}=0,\ b_{P\alpha;\beta}^{\beta}=0\,. \qquad (P=m+1, \cdots, m+p)$$

Now we put

$$\xi_\alpha=\sum_{a=1}^{p} b_{m+a\alpha}^{\ \ \gamma}\, T_{i_1\cdots i_a\cdots i_p} N^{i_1}_{m+1}\cdots B^{i_a}_{\gamma}\cdots N^{i_p}_{m+p}\,,$$

$$\eta_\alpha=\sum_{a=1}^{p} b_{m+a\gamma}^{\ \ \gamma}\, T_{i_1\cdots i_a\cdots i_p} N^{i_1}_{m+1}\cdots B^{i_a}_{\alpha}\cdots N^{i_p}_{m+p}\,.$$

By means of (1.2), (1.3), (3.2) and skew symmetric property of $T_{i_1\cdots i_p}$ we obtain

$$\xi^{\alpha}{}_{;\alpha}=f\sum_{P} b^{\beta\gamma}_{P} b_{P\beta\gamma}+\frac{1}{2}\sum_{a=1}^{p} b_{m+a}^{\ \ \beta\gamma}(T_{i_1\cdots i_a\cdots i_p;\beta}N^{i_1}_{m+1}\cdots B^{i_a}_{\gamma}\cdots N^{i_p}_{m+p}$$
$$+T_{i_1\cdots i_a\cdots i_p;\gamma}N^{i_1}_{m+1}\cdots B^{i_a}_{\beta}\cdots N^{i_p}_{m+p})\,,$$

$$\eta^{\alpha}{}_{;\alpha}=f\sum_{P}(b_{P\gamma}^{\ \gamma})^2+\frac{1}{2}\sum_{a=1}^{p} b_{m+a\gamma}^{\ \ \gamma}(T_{i_1\cdots i_a\cdots i_p;\alpha}N^{i_1}_{m+1}\cdots B^{i_a}_{\beta}\cdots N^{i_p}_{m+p}$$
$$+T_{i_1\cdots i_a\cdots i_p;\beta}N^{i_1}_{m+1}\cdots B^{i_a}_{\alpha}\cdots N^{i_p}_{m+p})\, g^{\alpha\beta}\,,$$

where we put $f=T_{i_1\cdots i_p}N^{i_1}_{m+1}\cdots N^{i_p}_{m+p}$. Making use of our assumption (i), we get from the avove equations

$$\xi^{\alpha}{}_{;\alpha}=f\sum_{P} b_{P}^{\beta\gamma} b_{P\beta\gamma}+\frac{1}{2} \sum_{a=1}^{p} \Phi_{m+a} b_{m+a\gamma}^{\gamma},$$

$$\eta^{\alpha}{}_{;\alpha}=f\sum_{P}(b_{P\gamma}^{\gamma})^{2}+\frac{m}{2} \sum_{a=1}^{p} \Phi_{m+a} b_{m+a\gamma}^{\gamma}.$$

Integrating both sides of these equations over V^m and applying the Green's theorem we get

$$\int_{V^m}\left\{f\sum_{P} b_{P}^{\beta\gamma} b_{P\beta\gamma}+\frac{1}{2} \sum_{a=1}^{p} \Phi_{m+a} b_{m+a\gamma}^{\gamma}\right\} dV=0, \tag{3.3}$$

$$\int_{V^m}\left\{f\sum_{P}(b_{P\gamma}^{\gamma})^{2}+\frac{m}{2} \sum_{a=1}^{p} \Phi_{m+a} b_{m+a\gamma}^{\gamma}\right\} dV=0, \tag{3.4}$$

where dV denotes the volume element of V^m. Then, from (3.3) and (3.4) we obtain

$$\int_{V^m} f\left\{m\sum_{P} b_{P}^{\beta\gamma} b_{P\beta\gamma}-\sum_{P}(b_{P\gamma}^{\gamma})^{2}\right\} dV=0. \tag{3.5}$$

Therefore, from our assumption (i) and Lemma 1.1, we can see that V^m is a totally umbilical submanifold.

When $p=2$, that is, V^m is a closed orientable submanifold of codimension 2 with $H\neq 0$, by virtue of Lemma 2.3 the connection induced on the normal bundle is trivial if the mean curvature vector field H^i is parallel with respect to the connection induced on the normal bundle. Then we get

COROLLARY 3. *Let V^m be a closed orientable submanifold in $M^{m+2}(c)$ and $H\neq 0$ everywhere on V^m. Then V^m is totally umbilic if and only if*

(i) *there exists a skew symmetric tensor field T_{ij} along V^m such that*

$$T_{ij;\alpha} B_{\beta}^{i} N_{m+a}^{j}+T_{ij;\beta} B_{\alpha}^{i} N_{m+a}^{j}=\Phi_{m+a} g_{\alpha\beta}$$

for some functions Φ_{m+a} $(a=1, 2)$ and $T_{ij} N_{m+1}^{i} N_{m+2}^{j}$ has fixed sign on V^m,

(ii) *the mean curvature vector field H^i is parallel with respect to the connection induced on the normal bundle of V^m.*

Finally we cansider the case of $p=1$. When V^m is a hypersurface in M^{m+1}, the Gauss and Weingarten formulas are

$$B_{\alpha;\beta}^{i}=b_{\alpha\beta} N^{i}, \qquad N^{i}{}_{;\alpha}=-b_{\alpha}^{\beta} B_{\beta}^{i}, \tag{3.6}$$

and the mean curvature vector field H^i is

$$H^{i}=HN^{i}=\frac{1}{m} b_{\gamma}^{\gamma} N^{i}, \tag{3.7}$$

where $b_{\alpha\beta}$, N^i and H denotes the second fundamental tensor, contravariant component of a unit normal vector and mean curvature of V^m respectively. By means of (3. 6) and (3. 7) it follows that $H=$ const. if and only if $(H^i{}_{;\alpha})^{\perp}=0$.

Making use of the method of Y. Katsurada [3] for the study of hypersurface in an Einstein space, we get

COROLLARY 3. 3. *Let V^m be a closed orientable hypersurface in an Einstein space R^{m+1} and $H\neq 0$ everywhere on V^m. Then V^m is an umbilical hypersurface if and only if*

(i) *there exists a vector field T_i along V^m such that*

$$T_{i;\alpha}B^i_\beta+T_{i;\beta}B^i_\alpha=\Phi g_{\alpha\beta}$$

for some function Φ, and T_iN^i has fixed sign on V^m,

(ii) $H=const.$

PROOF. Let V^m be an umbilical hypersurface in an Einstein space R^{m+1}. In this case the Codazzi equation is

$$R_{ihjk}N^iB^h_\alpha B^j_\beta B^k_\gamma=b_{\alpha\beta;\gamma}-b_{\alpha\gamma;\beta}\,. \tag{3. 8}$$

Since $b_{\alpha\beta}=Hg_{\alpha\beta}$, from (3. 8) we get (ii). Furthermore, if we put $T_i=N_i$, we have (i).

Next, we show that if we assume (i) and (ii), a closed orientable hypersurface V^m in R^{m+1} is an umbilical hypersurface. We put

$$\xi_\alpha=T_iB^i_\alpha\,,\qquad \eta_\alpha=b^\gamma_\alpha T_iB^i_\gamma\,.$$

By means of (3. 6) it follows that

$$\xi^\alpha{}_{;\alpha}=\frac{1}{2}g^{\alpha\beta}(T_{i;\beta}B^i_\alpha+T_{i;\alpha}B^i_\beta)+b^\gamma_\gamma T_iN^i\,,$$

$$\eta^\alpha{}_{;\alpha}=b^{\gamma\alpha}{}_{;\alpha}T_iB^i_\gamma+b^\gamma_\alpha b^\alpha_\gamma T_iN^i+\frac{1}{2}b^{\alpha\beta}(T_{i;\beta}B^i_\alpha+T_{i;\alpha}B^i_\beta)\,.$$

On the other hand, from (3. 8) we get $b^{\gamma\alpha}{}_{;\alpha}=g^{\gamma\beta}b^\alpha_{\alpha;\beta}$. Then from our assumptions (i) and (ii) we have

$$\xi^\alpha{}_{;\alpha}=\frac{m}{2}\Phi+b^\gamma_\gamma T_iN^i\,,\qquad \eta^\alpha{}_{;\alpha}=\frac{1}{2}b^\gamma_\gamma\Phi+b^\gamma_\alpha b^\alpha_\gamma T_iN^i\,.$$

Integrating both sides of above equations over V^m and applying the Green's theorem we have

$$\int_{V^m}\left\{\frac{m}{2}\Phi+b^\gamma_\gamma T_iN^i\right\}dV=0\,,\qquad \int_{V^m}\left\{\frac{1}{2}b^\gamma_\gamma\Phi+b^\gamma_\alpha b^\alpha_\gamma T_iN^i\right\}dV=0\,.$$

Then, from our assumption (ii) we obtain

$$\int_{V^m}\left\{mb^\gamma_\alpha b^\alpha_\gamma-(b^\gamma_\gamma)^2\right\}T_i N^i dV=0\,.$$

Thus, from our assumption (i) and Lemma 1.1, we can see that V^m is an umbilical hypersurface.

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