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On Fatou's and Beurling's Theorems

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The purpose of the present paper is to give simple proofs for well known Fatou and Beurling's theorems for harmonic functions and to ameliorate the Beurling's theorem for analytic functions given in a previous paper[1]. Let R be a Riemann surface $\notin 0_g$ and $\{R_n\}: n=0, 1, \cdots$ be its exhaustion. We suppose α-Martin's topology is defined on $\bar{R}=R+\Delta^{\alpha}$, where $\alpha=K$ or N.

Let $U(z): z\in R$ be a harmonic function, i. e. $U(z)$ is a mapping from R into a real axis. Let Δ_1^{α} be the set of α-minimal points[2] of Δ^{α}. The fine cluster set $\overset{\alpha}{A}(U(p))$ at $p\in\Delta_1^{\alpha}$ is defined as

$$\overset{\alpha}{A}\big(U(p)\big)=\bigcap_{\tau}\overline{U(G_{\tau})}:\ G_{\tau}\overset{\alpha}{\ni}p\,,^{2)}$$

where G_{τ} is a fine neighbourhood of p with respect to α-Martin's topology. If $\overset{\alpha}{A}(U(p))$ is a single point, we say $U(z)$ has a fine limit denoted by $\overset{\alpha}{U}(p)$. Then the following Lemma is well known.

LEMMA 1.[1] *Let G be an open set in R and $v(p)$ be a neighbourhood of p relative to α-Martin's topology. Then*

1) *Let $p\in\Delta_1^{\alpha}$. Then* a) $v(p)\overset{\alpha}{\ni}p$. b) *There exists only one component G' of G such that $G'\overset{\alpha}{\ni}p$ and $G\overset{\alpha}{\ni}p$ implies $(CG)^0\overset{\alpha}{\not\ni}p$. If $G_i\overset{\alpha}{\ni}p$ $(i=1, 2, \cdots, i_0)$, $\left(\bigcap_{i=1}^{i_0}G_i\right)\overset{\alpha}{\ni}p$. Hence $\overset{\alpha}{A}(U(p))$ is a point or continuum.*

2) a) *Let $G'\subset G\subset R$ G' and G be open sets and let F be a closed set in Δ_1^K. If the H. M. (harmonic measure) of $F\cap\bar{G}'$ relative to $G>0$:*

$$w(F\cap G', z, G)>0\,,^{(2)}$$

then there exists at least a point $p\in F\cap\Delta_1^K$ such that $G\overset{K}{\ni}p$. b) *Let $G'\subset G$ and let F be a closed set in Δ_1^N. If the C. P. (capacity) $F\cap G'$ relative to $G>0$: i. e. $\omega(F\cap G', z, G)>0$, then there exists at least a point $p\in F\cap\Delta_1^N$ such that $G\overset{N}{\ni}p$.*

If an open set (not necessarily connected) G has ∂G consisting of at most enumerably infinite number of analytic curves clustering nowhere in R, we call G a subdomain of R. Let G be a subdomain. Then $K(z, p)$

$(N(z, p))$ is lower semicontinuous with respect to $p \in \bar{R}$ and $\{p \in \overset{\alpha}{\Delta}{}^{\alpha}_1 : G \overset{\alpha}{\ni} p\}$ is a G_δ set. We suppose E (on the real axis) is compact and $U(z) \in E$. Let I_i^n $(i=1, 2, \cdots, i(n) < \infty)$ be a system of intervals on the real axis such that any closed interval with length $< \frac{1}{3n}$ is contained in some I_i^n and that any interval of length $> \frac{3}{n}$ contains at least one I_i^n. Let $T_i^n = \{p \in \Delta_1^\alpha : U^{-1}(I_i^n) \overset{\alpha}{\not\ni} p\}$, $S = \{p \in \Delta_1^\alpha : \text{dia } \overset{\alpha}{A}(U(p)) > 0\}$.
Then

$$S = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{i(n)} T_i^n$$

and S is a $G_{\delta\sigma}$ set, where dia $\overset{\alpha}{A}(U(p))$ is the diameter of $\overset{\alpha}{A}(U(p))$.

3) *If* dia $A(U(p)) > \delta_0$, *any component of $U^{-1}(I_\delta) \overset{\alpha}{\not\ni} p$ for $\delta < \delta_0$ where $I_\delta = (\delta, \infty)$.*

4) *If $\overset{\alpha}{U}(p)$ exists, there exists a path L α-tending to p such that $U(z) \to \overset{\alpha}{U}(p)$ as $z \to p$ along L.*

1. Fatou's theorem for bounded harmonic functions.

THEOREM 1. *Let $U(z)$ be a bounded harmonic function. Then $U(z)$ has fine limit $U^K(p) : p \in \Delta^K$ except at most a set of harmonic measure zero.*

PROOF. Since H. M. of $\Delta - \Delta_1^K$ is zero, it is sufficient to show that $U^K(p)$ exists a. e. on Δ_1^K. Without loss of generality, we can suppose $\inf U(z) = 0$, $\sup U(z) = 1$. Let $I_i^n = \{u : (i-1)\,\delta < u < (i+1)\delta\}$, $\tilde{I}_i^n = \{u : (i-2)\,\delta < u < (i+2)\delta\}$, $i = 1, 2, \cdots, n$, $\delta = \frac{1}{n}$, and $G_i^n = \{z \in R;\ U(z) \in I_i^n\}$, $\tilde{G}_i^n = \{z \in R,\ U(z) \in \tilde{I}_i^n\}$. Then G_i^n and $\tilde{G}_i^n$ are subdomains. Put $T_i^n = \{p \in \Delta_1^K, p \overset{K}{\not\in} G_i^n\}$ and $S = \bigcap_{n=1}^{\infty} \bigcap_{i=1}^{i(n)} T_i^n$. Then the set of points p where $U^K(p)$ does not exist is contained in S. We show S is a set of H. M. zero. Assume S is of positive H. M. Then we can find a number n and a closed set F of positive H. M. in $\bigcap_{i=1}^{n-1} T_i^n$. Then dia $A(U(p)) \geqq \frac{1}{n}$ for any point $p \in F$. Let $m \geqq 5n$. Since $\bigcup_{i=1}^{m-1} G_i^m = R$,

$$\sum_{i=1}^{m-1} w(F \cap G_i^m, z, R) \geqq w(F \cap \sum G_i^m, z, R) = w(F, z, R) > 0\,.$$

Hence the exists at least one i such that $w(F \cap G_i^m, z, R) > 0$, where $1 \leqq i \leqq m^{-1}$. Let $s(z) = U(z)$ for $U(z) \leqq (i-1)\,\delta' : \delta' = \frac{1}{m}$, $s(z) = (i-1)\,\delta'$ for $(i-1)\,\delta'$

$<U(z)\leqq(i+1)\delta'$ and $s(z)=(i-1)\delta'-\{U(z)-(i+1)\delta'\}$ for $U(z)>(i+1)\delta'$. Then $s(z)$ is an S. P. H. (superharmonic function). Let $\alpha=\min(0, 2i\delta'-1)$ and

$$t(z)=\frac{s(z)-\alpha}{(i-1)\delta'-\alpha}.$$

Then $t(z)$ is a positive S. P. H. such that $t(z)=1$ on $\bar{G}_i^m$ and $t(z)=\dfrac{(i-2)\delta'-\alpha}{(i-1)\delta'-\alpha}$ $=\varepsilon_0<1$ on $\partial\tilde{G}_1^m$. Since $w(G_i^m, z, R)$ is the least positive S. P. H. not smaller than 1 on $\bar{G}_i^m$, $w(F\cap G_i^m, z, R)\leqq\varepsilon_0$ on $\partial\tilde{G}_i^m$. This implies

$$w(F\cap G_i^m, z, R)-w_{C\tilde{G}_i^m}(F\cap G_i^m, z, R)=w(F\cap G_i^m, z, \tilde{G}_i^m)>0.$$

By Lemma 1 there exists at least one point p in F such that $p\overset{K}{\in}\tilde{G}_i^m$. Next by $\sup_{z\in\tilde{G}_i^m}U(z)-\inf_{z\in\tilde{G}_i^m}U(z)\leqq\dfrac{4}{m}$ dia $A(U(p))<\dfrac{4}{m}<\dfrac{1}{n}$. This contradicts $p\in F$. Thus we have the theorem.

LEMMA 2. *Let $U(z)$ be a bounded harmonic function. If $U^K(z)=C$ (const) a. e. on Δ^K, then $U(z)=C$.*

PROOF. Suppose $U(z)\neq$const. Then we can suppose $\inf U(z)=0$ and $\sup U(z)=1$. Let $G_\delta=\{z: U(z)>\delta\}: 0<\delta<1$. By the maximum principle

$$U(z)\leqq w\Big(G_\delta\cap(R-R_n), z, R\Big)+\delta.$$

Let $n\to\infty$, then

$$U(z)\leqq w(G_\delta\cap\Delta, z, R)+\delta.$$

If $w(G_\delta\cap\Delta, z, R)=0$, $U(z)\leqq\delta<1$. This is a contradiction. Hence

$$w(G_\delta\cap\Delta, z, R)>0 \qquad \text{for } 0<\delta<1.$$

Put $s(z)=\min\left(1, \dfrac{U(z)}{\delta}\right)$. Then $s(z)$ is an S. P. H. and $=1$ on $\bar{G}_\delta$, $s(z)=\dfrac{5}{6}$ on $\partial G_{\frac{5}{6}\delta}$. By $w(G_\delta\cap\Delta, z, R)\leqq w(G_\delta, z, R)\leqq s(z)\leqq\dfrac{5}{6}$ on $\partial G_{\frac{5}{6}\delta}$,

$$w(G_\delta\cap\Delta, z, R)-W_{C\tilde{G}}(G_\delta\cap\Delta, z, R)=w(G_\delta\cap\Delta, z, \tilde{G})>0,$$

where $G_{\frac{5}{6}\delta}=\tilde{G}$.
Now $w(G_\delta\cap\Delta, z, R)$ is represented by a positive canonical mass μ on $\bar{G}_\delta\cap\Delta_1$ such that $w(G_\delta\cap\Delta, z, R)=\int K(z, p)\,d\mu(p)\leqq1$.
By $w_{C\tilde{G}}(G_\delta\cap\Delta, z, R)=\int K_{C\tilde{G}}(z, p)\,d\mu(p)$, we have

$$1\geqq\int\Big(K(z, p)-K_{C\tilde{G}}(z, p)\Big)\,d\mu(p)>0.$$

This implies that there exists a positive restriction μ' of μ on $\Delta_1(\tilde{G})=$

$\{p \in \Delta_1^K : p \overset{K}{\in} \tilde{G}\}$ and H. M. of $\Delta_1(\tilde{G})$ is positive. Suppose $U^K(p) : p \in \Delta_1(\tilde{G})$ exists, then evidently $U^K(p) \geqq \frac{5\delta}{6}$ by $p \overset{K}{\in} \tilde{G}$. Hence by Theorem 1, $U^K(p) \geqq \frac{5\delta}{6}$ a. e. on $\Delta_1(\tilde{G})$ and H. M. of $\left\{p ;\ U^K(p) \geqq \frac{5\delta}{6}\right\}$ is positive. Put $\delta = \frac{2}{3}$. Then H. M. of $\left\{p ;\ U^K(p) \geqq \frac{5}{9}\right\}$ is positive. Consider $1 - U(z)$. Then H. M. of $\left\{p :\ U^K(p) \leqq \frac{4}{9}\right\}$ is positive. This contradicts $U^K(p) = C$ a. e. on Δ and we have the Lemma.

LEMMA 3. *Let G be a subdomian. Let $U(z)$ be an S. P. H. such that $U(z) = w(\partial G, z, R - \bar{G})$ (H. M. of ∂G relative to $R - \bar{G}$) in $R - \bar{G}$ and $U(z) = 1$ on G. Then $U^K(p) = 0$ a. e. on $\Delta_1(R - \bar{G})$.*

PROOF. Assume $U(z)$ has not $U^K(p)$ at a set of positive H. M. in $\Delta_1(R - \bar{G})$. Then we can find a closed set F in $\Delta_1(R - \bar{G})$ such that dia $A(U(p)) > \delta > 0$ for any point p in F and $w(F, z, R) > 0$. Let μ be the canonical distribution of $w(F, z, R)$. Then

$$w(F, z, R - \bar{G}) = w(F, z, R) - w_{\bar{G}}(F, z, R)$$
$$= \int \big(K(z, p) - K_{\bar{G}}(z, p)\big)\, d\mu(p) > 0 \qquad (1)$$

by $K_{\bar{G}}(z, p) < K(z, p) : p \in F \subset \Delta_1(R - \bar{G})$.

Let $\varepsilon < \frac{\delta}{6}$ and $G_i = \{z \in R - \bar{G} :\ (i-1)\,\varepsilon < U(z) < (i+1)\,\varepsilon\}$ and

$$\tilde{G}_i = \left\{z \in R - \bar{G} :\ (i-2)\,\varepsilon < U(z) < (i+2)\,\varepsilon\right\}, \quad (i = 1, 2, \cdots, n)\,.$$

Consider $w(F \cap G_i, z, R - \bar{G})$. Then similarly as the proof of Theorem 1 we can find a domain $\tilde{G}_i$ such that $w(F \cap G_i, z, \tilde{G}_i) > 0$ and a point p in F such that $p \overset{K}{\in} \tilde{G}_i$. This implies dia $A(U(p)) < 4\varepsilon < \frac{2}{3}\delta$. This contradicts $p \in F$. Hence $U^K(p)$ exists a. e. on $\Delta(R - \bar{G})$.

Next we show $U^K(p) = 0$ a. e. on $\Delta_1(R - \bar{G})$. Assume there exists a closed set F in $\Delta_1(R - \bar{G})$ such that $U^K(p) > \delta > 0$ for any $p \in F$ and $w(F, z, R) > 0$. Then $G_\delta = \left\{z :\ U(z) > \frac{\delta}{2}\right\} \overset{K}{\ni} p$ for any $p \in F$. By $G_\delta \overset{K}{\ni} p$ (and $(V_n(p) \cap G_\delta) \overset{K}{\ni} p$), $K(z, p) = K_{\bar{G}_\delta \cap F}(z, p)$, whence

$$0 < w(F, z, R) = \int K(z, p)\, d\mu(p) = \int K_{\bar{G}_\delta \cap F}(z, p)\, d\mu(p)$$
$$= w_{\bar{G}_\delta \cap F}(F, z, R) \leqq w(G_\delta \cap F, z, R)$$
$$\leqq w(F, z, R)\,. \quad \text{Hence } w(G_\delta \cap F, z, R) > 0\,.$$

Consider $G_\delta \cap F$ instead of F in (1). Then we have similarly as above by $(F\cap G_\delta)\subset \Delta_1(R-\bar{G})$

$$w(F\cap G_\delta,\, z,\, R-\bar{G})>0\,. \tag{2}$$

By $U(z)\geqq\frac{\delta}{2}$ on G_δ and by the definition of $w(F\cap G_\delta, z, R-\bar{G})$

$$\left(\tilde{w}(z)=\right)U(z)-\frac{\delta}{2}w(F\cap G_\delta,\, z,\, R-\bar{G})>0$$

and $\tilde{w}(z)=1$ on ∂G. Hence by the definition of $U(z)$, $\tilde{w}(z)\geqq U(z)$ and $w(F\cap G_\delta, z, R-\bar{G})=0$. This contradicts (2). Thus we have the lemma.

LEMMA 4. *Let G be a subdomain and $\Omega=R-\bar{G}$. Then*

$$H.\,M.\ of\ (\bar{\Omega}\cap\Delta)-\Delta_1(\bar{\Omega})\leqq w(G,\, z,\, R)\,.$$

PROOF. Since G is a subdomain $w(G, z, R)=w(\bar{G}, z, R)$. Let F bea closed set in $(\bar{\Omega}\cap\Delta)-\Delta_1(\Omega)$ such that $w(F, z, R)>0$ and μ be its canonical distribution. Clearly $\mu=0$ on CF. $p\in F$ implies $p\overset{K}{\notin}\Omega$ and $K(z,p)=K_{C\Omega}(z,p)=K_{\bar{G}}(z,p)=K(z,p)$. Hence by $w(F, z, R)=\int K(z,p)\,d\mu(p)$, $w(F, z, R)=\int K_{\bar{G}}(z,p)\,d\mu(p)=w_{\bar{G}}(F, z, R)\leqq 1_{\bar{G}}=w(\bar{G}, z, R)=w(G, z, R)$. Now F is arbitrarily. Hence we have the lemma.

THEOREM 2. *Let $U(z)$ be a positive harmonic function. Then $U(z)$ is divided into two parts: quasibounded part $V(z)$ and a singular part $S(z)$ and $U(z)$ has fine limits a.e. on Δ such that $U^K(p)=V^K(p)$ a.e. on Δ.*

PROOF. Let $G_M=\{z\in R:\ U(z)>M\}$. Then by $U(z)<\infty$, we have at once $w(G_M, z, R)\downarrow 0$ as $M\to\infty$. Let $U_{M,n,n+i}(z)$ be a harmonic function in $R_{n+i}-(G_M\cap(R-R_n))$ such that $U_{M,n,n+i}(z)=M$ on $(\partial R_n\cap G_M)+\partial G_M\cap(R_{n+i}-R_n),=U(z)$ on $\partial R_{n+i}-G_M$. Then $U_{M,n,n+i+1}(z)\leqq U(z)=U_{M,n,n+i}(z)$ on $\partial R_{n+i}-G_M$ implies $U_{M,n,n+i}(z)\downarrow U_{M,n}(z)$ as $i\to\infty$. Similarly we have $U_{M,n}(z)\downarrow U_M(z)\leqq U(z)$ as $n\to\infty$ and

$$U_M(z)\nearrow V(z)\leqq U(z)\ \text{ as }\ M\to\infty\,,$$

where $V(z)$ is quasibounded.

Since $U_M(z)$ is a bounded harmonic function, $U_M^K(p)$ exists a.e. on Δ. We denote by E_M the set in $\Delta_1(R-\bar{G}_M)$ where both $w^K(G_M, z, R)\,(=0)$ and $U^K(z)$ exists. Then by Lemma 3, H.M. of $\Delta_1(R-\bar{G}_M)-E_M=0$. Since the intersection of two fine neighbourhoods is also a fine neighbourhood and since $0\leqq U(z)-U_M(z)\leqq Mw(G_M, z, R)$ in $R-G_M$, $U^K(p)$ exists on E_M and $U^K(p)=U_M^K(p)$ on E_M. $U(p)\geqq V(p)\geqq U_M(p)$ on E_M. Thus $V^K(p)=U_M^K(p)=U^K(p)$ on $\bigcup_M^{\infty} E_M$.

Let $\Omega_M=CG_M$. Then by Lemma 4

$$\text{H. M. of } \Delta\cap\Omega_M-E_M\leqq w(G_M, z, R)\downarrow 0 \text{ as } M\to\infty\,. \qquad (3)$$

On the other hand,

$$w(\Omega_M\cap\Delta, z, R)\geqq 1-w(G_M\cap\Delta, z, R)\geqq 1-w(G_M, z, R) \qquad (4)$$

$w(\Omega_M\cap\Delta, z, R)\to 1$ as $M\to\infty$ and by (3) $w(E_M, z, R)\to 1$ as $M\to\infty$ and $V^K(p)=U^K(p)$ a. e. on Δ and $U^K(p)$ exists a. e. on Δ.

Put $S_M(z)=U(z)-U_M(z)$. Then $S_M(z)\downarrow S(z)$ and $S(z)$ is harmonic. Let $t(z)$ be a bounded positive harmonic function $\leqq S(z)$. Then $t^K(p)=0$ a. e. on Δ. By Lemma 2 we have $t(z)=0$. Hence $S(z)$ is singular. The uniqueness $V(z)$ and $S(z)$ is well known.

Remark. If R is a unit circle $|z|<1$, $e^{i\theta}$ is a minimal point. Let $V(z)$ be the conjugate harmonic function of $U(z)$. Put $g(z)=\frac{1}{U(z)+iV(z)}$. Then $\operatorname{Re} g(z)$ and $\operatorname{Im} g(z)$ are bounded. There exists a set E on $|z|=1$ such that mes $E=2\pi$ and both of them have fine limits and there exists a curve $L(e^{i\theta})$ terminating at $e^{i\theta}$ along which they converge to the fine limits. Hence by Lindelöf's theorem $g(z)$ has angular limits at $e^{i\theta}$ in E. This implies $U(z)$ has angular limits a. e. on $|z|=1$.

2. Beurling's theorem for harmonic functions.

Let $U(z)$ be a Dirichlet bounded harmonic function. Then $V(z)=\frac{1}{2}\left(\frac{U(z)}{1+|U(z)|}\right)+\frac{1}{2}$ is Dirichlet bounded and $0<V(z)<1$. In fact, $\left|\frac{\partial}{\partial x}V(z)\right|\leqq\frac{1}{2}\left(\frac{\left|\frac{\partial}{\partial x}U\right|}{1+|U|}+\frac{|U|\frac{\partial}{\partial x}U}{(1+|U|)^2}\right)\leqq\left|\frac{\partial}{\partial x}U\right|$ and similarly $\left|\frac{\partial}{\partial y}V\right|\leqq\left|\frac{\partial}{\partial y}U\right|$ and $D(V(z))\leqq D(U(z))$.

Theorem 3. *Let $U(z)$ be a Dirichlet bounded harmonic function in R. Then $U(z)$ has N-fine limt $U^N(p)$* on Δ^N *except at most a set of capacity zero.*

Proof. It is sufficient to show that the assertion holds for $V(z)$ instead of $U(z)$. Let $G_i^n=\{z:(i-1)\,\delta<V(z)<(i+1)\,\delta\}:\delta=\frac{1}{n}$. Put $T_i^n=\{p\in\Delta_1^N: p\notin \overset{N}{G_i^n}\}$ and $S=\bigcup_{i=1}^{\infty}\bigcap_{i}^{n-1}T_i^n$. Then S is the set of point p such that dia $A(V(p))>0$. Since $\Delta^N-\Delta_1^N$ is of capacity zero, we shall show S is of capacity zero. If it were not so, we can find a number δ and a closed set F in Δ_1^N such that $\omega(F, z, R-R_0)>0$ dia $A(V(p))>\delta>0$ for any $p\in F$, where $\{R_n\}$ is an exhaustion of R and R_0 is a compact disk. Let $m>\frac{\delta}{8}$. Then by $\omega(F, z, R-R_0) \leqq \sum_{i=1}^{m}\omega(F\cap G_i^m, z, R-R_0)$ there exists a number i such that

$$\omega(F\cap G_i^m, z, R-R_0)>0\,.$$

Let $s(z)=V(z)$ for $V(z)\leqq(i-1)\,\delta'$: $\delta'=\frac{1}{m}$, $s(z)=(i-1)\,\delta'$ for $(i-1)\,\delta'\leqq V(z)<(i+1)\,\delta'$ and $s(z)=(i-1)\,\delta'-\{V(z)-((i+1)\,\delta'\}$ for $V(z)\geqq(i+1)\,\delta'$. Then $t(z)=\frac{s(z)-(i-2)\,\delta'}{\delta'}$ satisfies $t(z)=1$ on G_i^m, $=0$ on $\partial\tilde{G}_i^m$ and $D(t(z))<\infty$, where $\tilde{G}_i^m=\{z: (i-2)\,\delta'<V(z)<(i+2)\,\delta'\}$. Since R_0 and R_1 are compact and G_i^m and $\tilde{G}_i^m$ are subdomains, we can easily contruct a Dirichlet function $\alpha(z)$ in R from $t(z)$ such that $\alpha(z)=1$ on G'^m_i, $=0$ $\partial\tilde{G}'^m_i$ and

$$D\big(\alpha(z)\big)<\infty\,, \tag{5}$$

where $G'^m_i=(R-R_1)\cap G_i^m$ and $\tilde{G}'^m_i=(R-R_0)\cap\tilde{G}_i^m$.

Since $F\subset\Delta$, $\omega(F\cap G_i^m, z, R-R_0)=\omega(F\cap G'^m_i, z, R-R_0)$. By (5) $\omega(G'^m_i, z, \tilde{G}'^m_i)$ can be defined and by the Dirichlet principle

$$D\big(\omega(G'^m_i, z, \tilde{G}'^m_i)\big)\leqq D\big(\alpha(z)\big)<\infty\,.$$

$$0<D\big(\omega(G'^m_i\cap F, z, R-R_0)\big)\leqq D\big(\omega(G'^m_i\cap F, z, \tilde{G}'^m_i)\big)\leqq D\big(\omega(G'^m_i, z, \tilde{G}'^m_i)\big)\,.$$

By the maximum principle

$$\omega(G'^m_i\cap F, z, R-R_0)\geqq\omega(G'^m_i\cap F, z, \tilde{G}'^m_i)>0\,.$$

Hence by Lemma 1 there exists at least a point $p\in F$ such that $p\overset{N}{\in}\tilde{G}'^m_i$. Now dia $V(p)\leqq$ dia $V(\tilde{G}'^m_i)<\frac{4}{m}<\delta$. This contradict $p\in F$. Hence we hve the theorem.

Remark. If R is a unit circle: $|z|<1$, $e^{i\theta}$ is N-minimal. Let $V(z)$ be a conjugage harmonic function of $U(z)$. Then there exists a set E on $|z|=1$ such that both $U(z)$ and $V(z)$ have fine limits on E, CE is a set of capacity zero. We can find a path $L(e^{i\theta})$ tending to $e^{i\theta}$ for any $e^{i\theta}\in E$ along which both $U(z)$ and $V(z)$ converge. The area of $|z|<1$ by $f(z)=U(z)+iV(z)$ is finite and there exist 3 points not taken by $f(z)$ near $|z|=1$. Hence by Lindelöf's theorem we have the following.

Theorem. *If $D(U(z))<\infty$ in $|z|<1$. Then $U(z)$ has angular limits on $|z|=1$ except a set of capacity zero.*

Lemma 5. 1) *Let G be a subdomain in $R-R_0$ and $\Omega_1=\Big\{z\in R-R_0$:*

$\omega(G, z, R-R_0)>\frac{1}{3}\Big\}$ and $\Omega_2=\Big\{z\in R-R_0:\ \omega(G, z, R-R_0)<\frac{2}{3}\Big\}$. Then $(\Delta_1-\Delta_1(\Omega_1))\cap(\Delta_1-\Delta_1(\Omega_2))$ is a set of capacity zero, where $\Delta_1(\Omega_i)=\{p\overset{N}{\in}\Delta_1^N:\ \Omega_i\ni p\}$: $i=1, 2$.

2) Let G_M be a subdomain in $R-R_0$ such that G_M decreases as $M\to\infty$ and $\omega(G_M, z, R-R_0)\downarrow 0$ as $M\to\infty$. Then the capacity of $(\Delta_1-\Delta_1(\Omega^M))\downarrow 0$ as $M\to\infty$, where $\Omega^M=\Big\{z\in R-R_0:\ \omega(G^M, z, R-R_0)<\frac{2}{3}\Big\}$.

PROOF. Assume there exists a closed set F in $\{(\Delta_1-\Delta_1(\Omega_1))\}\cap\{\Delta_1-\Delta_1(\Omega_2)\}$ such that $\omega(F, z, R-R_0)>0$. Put $\omega(G, z)=\omega(G, z, R-R_0)$, $\Omega_2^\varepsilon=\Big\{z\in R-R_0:\ \omega(G, z)<\frac{2}{3}-\varepsilon\Big\}$ and $\Omega_1^\varepsilon=\Big\{z:\ \omega(G, z)>\frac{1}{3}+\varepsilon\Big\}$: $0\leqq\varepsilon<\frac{1}{6}$ such that $C_1=\partial\Omega_1^\varepsilon$ and $C_2=\partial\Omega_2^\varepsilon$ are regular level curves[2)] of $\omega(G, z)$, i.e.

$$\int_{C_1}\frac{\partial}{\partial n}\omega(G, z)\,ds=\int_{C_2}\frac{\partial}{\partial n}\omega(G, z)\,ds=D\big(\omega(G, z)\big).$$

By $\Omega_1^\varepsilon\subset\Omega_1$, $F\cap\Delta_1(\Omega_1^\varepsilon)=0$ and $p\notin\Delta_1(\Omega_1^\varepsilon)$ for $p\in F$. Let $\omega(F, z, R-R_0)=U(z)$ and μ be its canonical distribution. Then $\mu=0$ on CF and $N(z, p)=N_{C\Omega_1^\varepsilon}(z, p)$ by $p\overset{N}{\notin}\Omega_1^\varepsilon$, whence

$$U_{C\Omega_1^\varepsilon}(z)=U(z).$$

Hence $U(z)$ has minimal Dirichlet integral (M. D. I) over Ω_1^ε among all functions with the same value as $U(z)$ on $\partial\Omega_1^\varepsilon$. Hence $U_n(z)\Rightarrow U(z)$, where $U_n(z)$ is a harmonic function in $\Omega_1^\varepsilon\cap R_n$ such that $U_n(z)=U(z)$ on $\partial\Omega_1^\varepsilon\cap R_n$ and $\frac{\partial}{\partial n}U_n(z)=0$ on $\Omega_1^\varepsilon\cap\partial R_n$ and $\Rightarrow$ means mean convergence and convergencce. On the other hand $\omega(G, z)$ has M. D. I. over $\Omega_2^\varepsilon\cap\Omega_1^\varepsilon$ and $\omega_n(z)\Rightarrow\omega(G, z)$, where $\omega_n(z)$ is a harmonic function in $(\Omega_2^\varepsilon\cap\Omega_1^\varepsilon)\cap R_n$ such that $\omega_n(z)=\frac{1}{3}+\varepsilon$ on $C_1\cap R_n$, $=\frac{2}{3}-\varepsilon$ on $C_2\cap R_n$ and $\frac{\partial}{\partial n}\omega_n(z)=0$ on $\partial R_n\cap(\Omega_2^\varepsilon\cap\Omega_1^\varepsilon)$. Since $\int_{C_1\cap R_n}\frac{\partial}{\partial n}U_n(z)\,ds=\int_{\Omega_1^\varepsilon\cap\partial R_n}\frac{\partial}{\partial n}U_n(z)\,ds=0$ also $\int_{C_2\cap R_n}\frac{\partial}{\partial n}U_n(z)\,ds=0$, we have by Green's formula

$$\int_{C_1\cap R_n}U_n(z)\frac{\partial}{\partial n}\omega_n(z)\,ds+\int_{C_2\cap R_n}U_n(z)\frac{\partial}{\partial n}\omega_n(z)\,ds=0.$$

By the regularity of C_1 and C_2 and $U_n(z)\to U(z)$ on C_1+C_2, we have by letting $n\to\infty$[2)]

$$\int_{C_1} U(z)\frac{\partial}{\partial n}\,\omega(G, z)\,ds+\int_{C_2} U(z)\frac{\partial}{\partial n}\,\omega(G, z)\,ds=0\,. \qquad (6)$$

By $F\cap\Delta_1(\Omega_2^e)=0$, we have $U_{C\Omega_2^e}(z)=U(z)$ and $U(z)$ has M. D. I. over Ω_2^e with the same value on $C_2+\partial R_0$. Let $U_n(z)$ be a harmonic function in $(R_n-R_0)\cap\Omega_2^e$ such that $U_n(z)=U(z)$ on $C_2\cap R_n+\partial R_0$ and $\frac{\partial}{\partial n}U_n(z)=0$ on $\Omega_2^e\cap\partial R_n$. Then $U_n(z)\Rightarrow U(z)$. Let $\omega_n(z)$ be the function as before. Apply Green's formula in $R_n\cap(\Omega_2^e\cap\Omega_1^e)$. Then

$$\int_{(C_1+C_2)\cap R_n+\partial R_n\cap(\Omega_2^e\cap\Omega_1^e)} U_n(z)\frac{\partial}{\partial n}\,\omega_n(z)\,ds=\int_{(C_1+C_2)\cap R_n+\partial R_n\cap(\Omega_2^e\cap\Omega_1^e)} \omega_n(z)\frac{\partial}{\partial n}\,U_n(z)\,ds\,.$$

$$\int_{C_1\cap R_n} U_n(z)\frac{\partial}{\partial n}\,\omega_n(z)ds+\int_{C_2\cap R_n} U_n(z)\frac{\partial}{\partial n}\,\omega_n(z)\,ds$$
$$=\left(\frac{2}{3}-\varepsilon\right)\int_{C_2\cap R_n}\frac{\partial}{\partial n}\,U_n(z)\,ds+\left(\frac{1}{3}+\varepsilon\right)\int_{C_1\cap R_n}\frac{\partial}{\partial n}\,U_n(z)\,ds\,.$$

Now $\int_{C_1\cap R_n}\frac{\partial}{\partial n}U_n(z)\,ds=-\int_{R_0}\frac{\partial}{\partial n}U_n(z)\,ds$, $\int_{C_2\cap R_n}\frac{\partial}{\partial n}U_n(z)\,ds=\int_{\partial R_0}\frac{\partial}{\partial n}U_n(z)\,ds$ and both of them$\rightarrow\int_{\partial R_0}\frac{\partial}{\partial n}\,\omega(F, z)\,ds=D(\omega(F, z))>0$ as $n\rightarrow\infty$. Hence by letting $n\rightarrow\infty$,

$$\int_{C_1} U(z)\frac{\partial}{\partial n}\,\omega(G, z)\,ds+\int_{C_2} U(z)\frac{\partial}{\partial n}\,\omega(G, z)\,ds$$
$$=\left\{-\left(\frac{2}{3}-\varepsilon\right)+\left(\frac{1}{3}+\varepsilon\right)\right\}D\big(\omega(F, z)\big)<0\,. \qquad (7)$$

(7) contradicts (6). Hence we have (1).

Proof of (2) Let $G_M=G$, Ω_1 and Ω_2 be the domains defined in (1) with respect to G_M. Assume C. P. of $(\Delta_1-\Delta_1(\Omega_2))\downarrow\delta>0$ as $M\rightarrow\infty$. By (1) there exists a closed set F in $\Delta_1(\Omega_1)$ such that

$$D\big(\omega(F, z, R-R_0)\big)\geqq\delta>0 \qquad \text{for every } M. \qquad (8)$$

Let $\omega(z)=\omega(F, z, R-R_0)$. Then $\omega(z)=\int N(z, p)\,d\mu(p)$ and by $p\in\Delta_1(\Omega_1)$, $N(z, p)=N_{\bar{\Omega}_1}(z, p)$ and $\omega_{\bar{\Omega}_1}(z)=\omega(z)$.

$$\omega(\Omega_1, z, R-R_0)=1_{\bar{\Omega}_1}\geqq\omega_{\bar{\Omega}_1}(z)=\omega(z) \text{ and}$$
$$\text{C. P. of } \Omega_1=D\big(\omega(\Omega_1, z, R-R_0)\big)\geqq D\big(\omega(z)\big)\geqq\delta>0\,. \qquad (9)$$

Since $\omega(\Omega_1, z, R-R_0)$ and $\omega(G, z, R-R_0)$ have M. D. I.s over $R-R_0-\Omega_1$ and $\omega(\Omega_1, z, R-R_0)=1$ and $\omega(G, z, R-R_0)=\frac{1}{3}+\varepsilon$ on $\partial\Omega_1$, $\omega(\Omega_1, z, R-R_0)=\frac{3}{1+3\varepsilon}$ $\omega(G, z, R-R_0)$ in $R-R_0-\Omega_1$. Hence

$$\text{C. P. of } \Omega_1=\int_{\partial R_0}\frac{\partial}{\partial n}\omega(\Omega_1, z, R-R_0)\,ds$$

$$=\frac{3}{1+3\varepsilon}\times\text{C. P. of } G_M\downarrow 0 \text{ as } M\to\infty\,. \tag{10}$$

(9) contradicts (10). Thus we have (2).

LEMMA 6. *Let G be a subdomain in $R-R_0$ and let F be a closed set of positive capacity in $\Delta_1(R-\bar{G})$. Then there exists a subdomain Ω in $R-G$ and a Dirichlet function $\alpha(z)$ in $R-\bar{G}$ such that $\alpha(z)=1$ on Ω and $\alpha(z)=0$ on ∂G and $\omega(F\cap\Omega, z, R-G_0)>0$.*

PROOF. $N(z, p)-N_{\bar{G}}(z, p)>0$ for $p\in\Delta_1(R-\bar{G})$. Let $\omega(z)=\omega(F, z, R-R_0)$ >0 and μ be its canonical mass. Then $\mu>0$ on F and

$$\omega(z)-\omega_{\bar{G}}(z)=U(z)>0\,.$$

Let $\Omega=\left\{z\in R-R_0:\ U(z)>\frac{\delta}{2}\right\}$: $\delta=\sup U(z)\,(>0)$. Then

$$\omega_{F\cap C\Omega}(z)+\omega_{F\cap\bar{\Omega}}(z)\geqq\omega_F(z)=\omega(z)\,.^{2)}$$

Assume $\omega_{F\cap\bar{\Omega}}(z)=0$. Then

$$\omega(z)\geqq\omega_{F\cap C\Omega+\bar{G}}(z)\geqq\omega_{F\cap C\Omega}(z)=\omega(z)$$

Now $\omega(z)\leqq\omega_{\bar{G}}(z)+\frac{\delta}{2}$ on $C\Omega\supset G$ and

$$\omega(z)=\omega_{F\cap C\Omega+\bar{G}}(z)\leqq\left(\frac{\delta}{2}\right)_{F\cap C\bar{\Omega}+\bar{G}}+{}_{F\cap C\Omega+\bar{G}}\Big(\omega_{\bar{G}}(z)\Big)\leqq\frac{\delta}{2}+\omega_{\bar{G}}(z)\,.$$

Hence $U(z)\leqq\frac{\delta}{2}$. This contradicts $\sup U(z)=\delta$. Hence

$$0<\omega_{F\cap\bar{\Omega}}(z)\leqq\omega(F\cap\Omega, z, R-R_0)\,.$$

Since $D(U(z))<\infty$, $\omega(\Omega, z, R-\bar{G})$ can be defined and by the Dirichlet principle

$$\begin{aligned}0<D\Big(\omega(\Omega\cap F, z, R-R_0)\Big)&\leqq D\Big(\omega(\Omega\cap F, z, R-\bar{G})\Big)\\&\leqq D\Big(\omega(\Omega, z, R-\bar{G})\Big)\\&\leqq\frac{4}{\delta^2}D\Big(U(z)\Big)<\infty\,.\end{aligned}$$

Hence Ω is a domain required. Let $\alpha(z)=1$ on $\bar{\Omega}$ and $=\left(\frac{1}{\delta/2}\right)U(z)$ in $R-\Omega$. Then $\alpha(z)$ is a required Dirichlet function.

Let $U(z)$ be a harmonic function in R. If there exists a closed set E in $[0, 1]$ such that $\omega(G_n, z, R-R_0)\downarrow 0$ as $n\to\infty$ and $D_{R-G_n}(U(z))<\infty$ for any n, then we call $U(z)$ an almost Dirichlet bounded harmonic function where $G_n=\left\{z\in R:\ \text{dist}\left\{E, \frac{1}{2}\left(\frac{U(z)}{1+|U(z)|}+\frac{1}{2}\right)<\frac{1}{n}\right\}\right.$. Then

THEOREM 4. *Let $U(z)$ be an almost Dirichlet bounded harmonic function. Then $U(z)$ has N-fine limit $U^N(p)$ on Δ except a set of capacity zero.*

PROOF. It is sufficient to prove the assertion for $V(z)=\frac{1}{2}\left(\frac{U(z)}{1+|U(z)|}+\frac{1}{2}\right)$. Suppose there exists a closed set E in $[0, 1]$ satisfying the condition of the theorem. We show $V(z)$ has fine limit on $\Delta_1(R-\overline{G_n})$ except a set of capacity zero. Assume $V(p)$ does not exist on a set of positive capacity in $\Delta_1(R-\overline{G_n})$, then we can find a number $\delta>0$ and a closed set F of positive capacity in $\Delta_1(R-\overline{G_n})$ such that dia $A(V(p))>\delta$ for any p in F and $\omega(F, z, R-R_0)>0$. Since we consider $V(z)$ near Δ, we can suppose without loss of generality that $\bar{G}_n\cap R_0=0$. Then by Lemma 6 there exists a domain Ω in $R-R_0-G_n$ and a Dirichlet function $\alpha(z)$ such that

$$\omega(F\cap\Omega, z, R-R_0-G_n)>0 \text{ and}$$

$\alpha(z)=1$ on Ω, $=0$ on $\partial R_0+\partial G_n$ (or $=0$ on $\bar{R}_0+G_n$).

Let $m>\frac{6}{\delta}$ and $G_i^m=\left\{z\in R:\ \frac{i-1}{m}<V(z)<\frac{i+1}{m}\right\}$, $\tilde{G}_i^m=\left\{z\in R:\ \frac{i-2}{m}<V(z)<\frac{i+2}{m}\right\}$. Then there exists at least one i such that $\omega(F\cap\Omega\cap G_i^m, z, R-R_0-G_n)>0$.

Let $t(z)$ be the function in the proof of Theorem 3. Then $t(z)=1$ on $\bar{G}_i^m$, $=0$ on $R-\tilde{G}_i^m$ and $\underset{R-R_0-G_n}{D}(t(z))<\infty$ by $\underset{R-R_0-G_n}{D}(V(z))<\infty$.

Put $\beta(z)=\min(\alpha(z), t(z))$. Then $\beta(z)=1$ on $\bar{\Omega}\cap\bar{G}_i^m$, $=0$ on $\partial R_0+\partial G_n+\{\partial\tilde{G}_i^m\cap(R-R_0)\}$ and $D_R(\beta(z))<\infty$. Hence $\omega(F\cap\Omega\cap G_i^m, z, (R-R_0-G_n)\cap\tilde{G}_i^m)$ can be defined. By the Dirichlet principle

$$0<D\Big(\omega\Big((F\cap\Omega\cap G_i^m, z, (R-R_0-G_n)\Big)\Big)$$
$$\leqq D\Big(\omega\Big(F\cap\Omega\cap G_i^m, z, (R-R_0-G_n)\cap\tilde{G}_i^m\Big)\Big).$$

Hence by Lemma 1, there exists a point p in F with $\{(R-R_0-G_n)\cap\tilde{G}_i^m\}\overset{N}{\ni}p$ i. e. $A(V(p))\leqq\frac{4}{m}<\delta$. This contradicts $p\in F$. Whence $V(p)$ exists on

$\Delta_1(R-G_n)$ except a set of capacity zero. Let $\Omega_n^1=\{z\in R-R_0:\ \omega(G_n, z, R-R_0) <\frac{1}{3}\}$. *Then* $\Omega_n^1\subset(R-R_0-G_n)$ and $V(p)$ exists on $\Delta_1(\Omega_n^1)$ except capacity zero. $\Omega_n^1\nearrow$ as $n\to\infty$. By Lemma 5, capacity of $(\Delta_1-\Delta_1(\Omega_n^1))\downarrow 0$ as $n\to\infty$. Thus $U^N(p)$ exists on Δ_1 except a set of capacity zero.

3. Beurling's Theorem for analytic functions.

Suppose a metric d is given on $R+\Delta$ such that d is compactible in R to the one defined by local parameters and that $\bar{R}=R+\Delta$ and Δ are compact with respect to d. If d satisfies, for any $p\in\bar{R}$ and $r_1<r_2$ the condition 1) and 1′), it is called H. B. separative and H. D. separative respectively.

Let $C(r_2, p)\supset C(r_1, p):\ r_2>r_1$ be two circles: $C(r, p)=\{z\in\bar{R}:\ d(z, p)<r\}$

1) *Let* $\Omega_{1-\varepsilon}=\{z:\ w_{CG}(C(r_1, p)\cap\Delta, z)>1-\varepsilon\}:\ G=C(r_2, p)$, then

$$\lim_{\varepsilon\to 0} w(\Omega_{1-\varepsilon}\cap C(r, p)\cap\Delta, z)=0\,.$$

1′) Let $\Omega_{1-\varepsilon}=\{z\in R-R_0:\ \omega_{CG}(C(r_1, p)\cap\Delta, z, R-R_0)>1-\varepsilon\}$. Then

$$\lim_{\varepsilon\to 0} \omega(\Omega_{1-\varepsilon}\cap C(r_1, p)\cap\Delta, z, R-R_0)=0\,.$$

We proved if d is H. D. separative, then it is H. B. separative. K-Martin's topology is H. B. separative and N-Martin's is H. D. separative.

2) Let d be a metric. If for any two compact set F_1 and F_2: $F_1\cap F_2=0$, there exists a continuous (in $\bar{R}$) Dirichlet function on R, $U(z)$ exists such that $U(z)=1$ on F_1 and $U(z)=0$ on F_2. We call a metric satisfying the condition (2) a D-disjoint metric. Then we have.

LEMMA 7. *If d is D-disjoint, it is H. D. separative. N-Martin's is D-disjoint.*

PROOF. Let d be D-disjoint. Since H. D. separability depends on Δ, we can suppose $C(r_2, p)\cap\bar{R}_0=0$. Let $C(r, p)=C(r_1, p)$ and $G=C(r_2, p)$. Then by $\{\Omega_{1-\varepsilon}\cap C(r, p)\}\subset C(r, p)$ and by the Dirichlet principle

$$\begin{aligned} D\Big(\omega(\Omega_{1-\varepsilon}\cap C(r, p)\cap\Delta, z, R-R_0)&\leqq D\Big(\omega(\Omega_{1-\varepsilon}\cap C(r, p)\,\Delta, z, G)\Big)\\ &\leqq D\Big(\omega(C(r, p), z, G)\Big)<\infty\,.\end{aligned}$$

Assume $\hat{\omega}(z)=\lim_{\varepsilon\to 0}\omega(\Omega_{1-\varepsilon}\cap C(r, p)\cap\Delta, z, G)>0$. Let $\omega(z)=\omega(C(r, p)\cap\Delta, z, R-R_0)$. Now since $\omega_{CG}(z)\geqq 1-\varepsilon$ on $\Omega_{1-\varepsilon}$,

$$\begin{aligned}\omega_{CG}(z)&\geqq(1-\varepsilon)\ \omega(\Omega_{1-\varepsilon}, z, R-R_0)\\ &\geqq(1-\varepsilon)\ \omega(\Omega_{1-\varepsilon}\cap C(r, p)\cap\Delta, z, G)\,.\end{aligned}$$

Let $\varepsilon\to 0$. Then

$$\omega_{CG}(z) \geqq \hat{\omega}(z)\,.$$

$\hat{\omega}(z)$ is a C. P. of $\{\Omega_{1-\varepsilon} \cap C(r, p) \cap \Delta\}$: $\varepsilon \to 0$ in G. Let $C_{1-\varepsilon}$ and C_δ be regular level curves of $\hat{\omega}(z)$. Now $\omega_{CG}(z)$ has M. D. I. over G. By Green's formula

$$\int_{C_\delta} \omega_{CG}(z) \frac{\partial}{\partial n} \hat{\omega}(z)\, ds = \int_{C_{1-\varepsilon}} \omega_{CG}(z) \frac{\partial}{\partial n} \hat{\omega}(z)\, ds\,.$$

Since $\omega_{CG}(z) < 1$ in R,

$$D\big(\hat{\omega}(z)\big) = \int_{C_\delta} \frac{\partial}{\partial n} \hat{\omega}(z)\, ds \gneqq \int_{C_\delta} \omega_{CG}(z) \frac{\partial}{\partial n} \hat{\omega}(z)\, ds = \int_{C_{1-\varepsilon}} \omega_{CG}(z) \frac{\partial}{\partial n} \hat{\omega}(z)\, ds$$

$$\geqq (1-\varepsilon) \int_{C_{1-\varepsilon}} \frac{\partial}{\partial n} \hat{\omega}(z)\, ds = (1-\varepsilon)\, D\big(\hat{\omega}(z)\big)\,.$$

Let $\varepsilon \to 0$, then we have a contradiction. Hence $\hat{\omega}(z) = 0$ and d is H. D. separative.

We shall show N-Martin's topology is D-disjoint. At first we suppose F_1 and F_2 are contained in $\overline{R-R_3}$. $F_1 \cap F_2 = 0$ implies dist $(F_1, F_2) > 0$ and $N(z, p) \neq N(z, q)$ for $p \in F_1$ and $q \in F_2$. Assume $N(z, q) \geqq N(z, p)$ on ∂R_1 and there exists a point on ∂R_1 with $N(z, q) > N(z, p)$, then

$$2\pi = \int_{\partial R_0} \frac{\partial}{\partial n} N(z, q)\, ds > \int_{\partial R_0} \frac{\partial}{\partial n} N(z, p)\, ds = 2\pi\,.$$

Also assume $N(z, p) = N(z, q)$ on ∂R_1, then by the harmonicity $N(z, p) = N(z, q)$ these contradict dist $(p, q) > 0$. Then there exists at least one point z' on ∂R_1 such that

$$N(q, z') = N(z', q) < N(z', p) = N(p, z')\,.$$

Let $A_{p,q}(z) = \max\left(0,\ \min\left(1, \dfrac{N(z, z') - N(q, z')}{N(p, z') - N(q, z')}\right)\right)$. Then since $z' \in \partial R_1$, $A_{p,q}(z)$ is continuous in $\bar{R} - R_0$, $D(A_{p,q}(z)) < \infty$ and $A_{p,q}(z)$ is a Dirichlet function in $R - R_0$ with $A_{p,q}(z) = 0$ on ∂R_0. Now $A_{p,q}(p) = 1$, $A_{p,q}(q) = 0$. For any given point q in F_2 and $\frac{1}{3} > \varepsilon > 0$, there exists a neighbourhood $v(q)$ such that $A_{p,q}(z) < \varepsilon$ in $v(q)$. We cover F_2 by $\sum_{i=1}^{i_0} v(q_i)$: $i_0 < \infty$ and put $A_p(z) = \min_i (A_{p,q_i}(z))$. Then $A_p(p) = 1$ and $A_p(z) < \varepsilon$ on F_2. Also we can cover F_2 by $\sum_{j=1}^{j_0} v(p_j)$ such that $A_{p_j}(z) > 1 - \varepsilon$ in $v(p_j)$. Let $A(z) = \max_j^{j_0} (A_{p_j}(z))$. Then

$$A(z)=\min\left(1,\ \max\left(0,\ \frac{A(z)-\varepsilon}{1-2\varepsilon}\right)\right)$$

is continuous in $\bar{R}-R_0$, $A(z)=1$ on $F_1,=0$ on F_2 and $D(A(z))<\infty$. Next let F_1 and F_2 be compact in $\bar{R}$ such that $F_1\cap F_2=0$. Let $F'_i=(\overline{R-R_3})\cap F_i$ and $F''_i=F_i\cap\bar{R}_3$: $i=1, 2$. Then since F''_i is compact in R, evidently there exists a Dirichlet function $V_1(z)$ such that $V_1(z)=0$ on $F''_2,=1$ on F_1. Similarly there exists a Dirichlet function $V_2(z)$ such that $V_2=1$ on $F''_1,=0$ on F_2. Let $A(z)$ be a Dirichlet function such that $A(z)=1$ on $F'_1=0$ on F'_2. Then

$$U(z)=\max\left(V_2(z),\ \min\left(A(z),\ V_1(z)\right)\right)$$

is a Dirichlet function with value 1 on F_1 and 0 on F_2. Hence N-Martin's topology is D-disjoint.

Let $\tilde{R}$ be a Riemann surface and let $w=f(z)$ be an analytic function from $\tilde{R}(\notin 0_g)$ into R: $w\in R$, $z\in\tilde{R}$. For any point p of R there exists a local parameter disk $C(p)$ such that the area of $\tilde{R}$ over $C(p)$ is fininite and there exists a number m such that $R-R_m$ is covered by $\tilde{R}$ only a finite number of times, then we say $\tilde{R}$ is an almost finitely sheeted covering surface, where $\{R_m\}$ is an exhaustion of R. Then we proved.

THEOREM 5. *Suppose on R N-Martin's topology is defined and an H. D. separative metric is given on R. If $\tilde{R}$ is an almost finitely sheeted covering surface over R, then $w=f(z)$ has N-fine limits at Δ except a set of capacity zero.*

In the following we shall extend this theorem but we suppose a D-disjoint metric rather than H. D. separative metrics.

Non thick-property at a point $p\in\bar{R}$.

Suppose a D-disjoint metric is given on $\bar{R}$. Then

$$C_{n,n+i}(p)=\left\{w\in\bar{R}:\ \frac{1}{2^{n+i}}<\operatorname{dist}(w, p)<\frac{1}{2^n}\right\}$$

is a ring. We can find a ring G which is a subdomain such that ∂G consists of $(\partial G)_1$ and $(\partial G)_2$ and

1) $(\partial G)_1$ separates ∂C_n: $C_n(p)=\left\{w\in\bar{R}:\ \operatorname{dist}(w, p)<\frac{1}{2^n}\right\}$ and $(\partial G)_2$.
2) $(\partial G)_2$ separates (∂G_1) and $\partial C_{n+i}(p)$ in every components of $C_{n,n+i}(p)$
3) $\operatorname{dist}((\partial G)_1, (\partial G)_2)>0$.

Then there exists a Dirichlet function $H(w)$ in G such that $H(w)=0$ on $(\partial G)_1$, $H(w)=\alpha$ on $(\partial G)_2$, $H(w)$ has M. D. I. over G and $D(H(w))=2\pi\alpha$, i. e.

$$H(w)=\frac{2\pi\omega((\partial G)_2, w, G)}{D(\omega(\partial G)_2, z, G))}\ \text{and}\ \alpha=\frac{2\pi}{D(\omega(\partial G)_2, z, G)}\,.$$

Let $J(w)$ be the conjugate harmonic function of $H(w)$. Put $\zeta=\zeta(w)=\exp(-(H(w)+iJ(w)))$: $\zeta=\xi+i\eta$. Then $\zeta(w)$ maps G onto $1>|\zeta|>\exp(-\alpha)$ conformally with radial slits whose areal measure $=0$, because $H(w)$ has M. D. I. and $\int_{C_\delta}\frac{\partial}{\partial n}H(w)\,ds=2\pi$ for almost δ: $0<\delta<\alpha$. Let Ω be a subdomain in $\tilde{R}$ and let $n(w)$ be the number of times when w is covered by $f^{-1}(G)-\Omega$. Then the area of $f^{-1}(G)-\Omega$ over $1>|\zeta|>\exp(-\alpha)$ is given by

$$A\left(f^{-1}(G)-\Omega\right)=\iint n(\zeta)\,d\xi d\eta:\ n(\zeta)=n(w).$$

$(\partial G)_1$ divides R into two parts: E_1 and E_2 such that any component of E_1 contains at least one comonent of $C_n(p)$ and any component of E_2 contains at least one component of $C_{n+i}(p)$. Let $H(w)=0$ on E_1. Similarly $(\partial G)_2$ divides R into two parts E_3 and E_4 such that any component of E_4 contains at least one component of $C_{n+i}(p)$. Let $H(w)=\alpha$ on E_4. Then $H(w)$ is a Dirichlet function in R. Let $U(z)=H(f(z))$. Then $U(z)$ is harmonic in $f^{-1}(G)$, $U(z)=0$ on $f^{-1}((\partial G)_1)$, $U(z)=\alpha$ on $f^{-1}((\partial G)_2)$ and continuous in R. Now in $f^{-1}(G)$

$$U(z)=-\log|\zeta|:\ 1>|\zeta|>\exp(-\alpha),\ \xi=\zeta\left(f(z)\right)$$

$D_{f^{-1}(G)-\Omega}(U(z))$ is given by

$$\iint n(\zeta)\left(\frac{\partial}{\partial r}U(\zeta)\right)^2 r\,dr\,d\theta\leqq\sup_{1>|\zeta|>\exp(-\alpha)}\left(\frac{\partial}{\partial r}U(\zeta)\right)^2\iint n(\zeta)\,r\,dr\,d\theta$$
$$\leqq\exp 2\alpha\,A\left(f^{-1}(G)-\Omega\right);\ \zeta=r\,e^{i\theta}.$$

If we can find a ring G in $C_{n,n+i}(p)$ satisfying the conditions 1), 2) and 3) and the quantity $A(f^{-1}(G)-\Omega)$ (defined with respect to G) is finite, we say $C_{n,n+i}(p)$ is non thickly covered by $\tilde{R}-\Omega$. Further if there exists a sequence $n_1, n_2, \cdots$; $\lim_i n_i=\infty$ such that $C_{n_i,n_{i+1}}(p)$ is non thickly covered, we say p is non thickly covered by $\tilde{R}-\Omega$.

REMARK. Let $\tilde{R}$ be a covering surface of almost finitely sheeted over R, then it is easy to see every point of $\bar{R}$ is non thickly covered by $\tilde{R}$, in this case Ω is empty.

THEOREM 6. *Let $\tilde{R}(\notin 0_g)$ and R the Riemann surface and $w=f(z)$ be an analytic function from $\tilde{R}$ into R. Suppose on $\bar{R}$ a D-disjoint metric is defined. Let $\{\Omega_n\}$ be a decreasing sequence of subdomains such that*

$$\omega\left(\Omega_n\cap(\tilde{R}-\tilde{R}_0),\tilde{R}-\tilde{R}_0\right)\downarrow 0\ \ \textit{as}\ \ n\to\infty$$

and every point p of $\bar{R}$ is non thickly covered by $\tilde{R}-\Omega_n$ for every n. Then $f(z)$ has N-fine limits at Δ of $\tilde{R}$ except a set of capacity zero.

PROOF. We consider the behaviour of $f(z)$ near Δ of $\tilde{R}$. We can suppose without loss of generality that Ω_n is contained in $\tilde{R}-\tilde{R}_3$. We show $f(z)$ has finite limit $f^N(p)$ in $\Delta_1(\tilde{R}-R_0-\Omega_n)$ except a set of capacity zero.

Assume dia $\overset{N}{A}(f(p))>0$ at a set of positive capacity in $\Delta_1(\tilde{R}-\tilde{R}_0-\Omega_n)$, then we can find a closed set F in $\Delta_1(\bar{R}-\bar{R}_0-\Omega_n)$ of positive capacity: $\omega(F, z, \tilde{R}-\tilde{R}_0)>0$ and dia $\overset{N}{A}(f(p))>\delta$: $p\in F$. Fix m_0: $2^{m_0}>\frac{2}{\delta}$. Since for any point $p\in\bar{R}$, there exists a ring $C_{m(p),m(p)+j(p)}(p)$: $m(p)\geqq m_0$ such that $C_{m(p),m(p)+j(p)}(p)$ is non thickly covered by $\tilde{R}-\Omega_n$. Then we can find a finite number of $C_{m(p_i),m(p_i)+j(p_i)}(p_i)$ such that $\sum^{i_0} C_{m(p_i)+j(p_i)}(p_i)\supset\bar{R}$. Now $F\subset\Delta_1(\tilde{R}-\tilde{R}_0-\Omega_n)$ and $\omega(F, z, \tilde{R}-\tilde{\bar{R}}_0)>0$. By Lemma 6 there exists a domain Ω in $\tilde{R}-\tilde{R}_0-\Omega_n$ such that $\omega(F\cap\Omega, z, \tilde{R}-\tilde{R}_0-\bar{\Omega}_n)>0$ and a Dirichlet function $V_1(z)$ in $\tilde{R}$ with $V_1(z)=1$ on $\partial\Omega$, $V_1(z)=0$ on $\partial\tilde{R}_0+\partial\Omega_n$. Now $\sum^{i_0}\omega(F\cap\Omega\cap f^{-1}(C_{m(p_i)+j(p_i)}(p_i), z, \tilde{R}-\tilde{R}_0-\bar{\Omega}_n)\geqq\omega(F\cap\Omega, z, \tilde{R}-\tilde{\bar{R}}_0-\Omega_n)>0$. There exists at least one point p such that

$$\omega\Big(F\cap\Omega\cap f^{-1}\Big(C_{m(p)+j(p)}(p), z, \tilde{R}-\tilde{\bar{R}}_0-\bar{\Omega}_n\Big)\Big)>0\,. \tag{11}$$

Let $m(p)=m$. Since $C_{m,m+j(p)}(p)$ is non thickly covered by $\tilde{R}-\tilde{R}_0-\Omega_n$, there exists a continuous function $U(z)$ in $\tilde{R}-\Omega_n$ such that $U(z)=0$ on $f^{-1}(\partial C_m(p)-\Omega_n)$, $=1$ on $f^{-1}(\partial C_{m+j(p)}(p))-\Omega_n$ and $D(U(z))<\infty$. Let $V(z)=\min(V_1(z), U(z))$. Then $V(z)=1$ on $\{\Omega\cap f^{-1}(C_{m+j(p)}(p))\}$, $=0$ on $\Omega_n+R-f^{-1}(C_{m(p)})+R_0$ and $D(V(z))<\infty$. Hence $\omega(f^{-1}(C_{m+j(p)}(p)\cap\Omega, z, f^{-1}(C_m(p)\cap(\tilde{R}-\tilde{R}_0-\Omega_n))$ can be considered. By the Dirichlet principle and by (11)

$$D\Big(\omega\Big(F\cap f^{-1}C_{m+j(p)}(p)\Big)\cap\Omega, z, (\tilde{R}-\tilde{R}_0-\bar{\Omega}_n)\cap f^{-1}\Big(C_m(p)\Big)\Big)$$
$$\geqq D\Big(\omega(F\cap f^{-1}\Big(C_{m+j(p)}(p)\cap\Omega, z, \tilde{R}-\tilde{R}_0-\Omega_n)\Big)\Big)>0\,.$$

Hence there exists at least one point p such that

$$\Big\{(\tilde{R}-\tilde{R}_0-\bar{\Omega}_n)\cap f^{-1}\Big(C_m(p)\Big)\Big\}\overset{N}{\ni}p : p\in F\,.$$

Hence there exists at least one component of $f^{-1}(C_m(p))$ which is a fine neighbourhood of p and dia $A(f(p))\leqq\frac{1}{2^{m_0}}<\frac{\delta}{2}$. This contradicts $p\in F$. Hence $f(p)$ exists on $\Delta_1(\tilde{R}-\tilde{R}_0-\bar{\Omega}_n)$ except a set of capacity zero. Next by Lemma 4, similarly as Theorem 4, $f^N(p)$ exists on Δ of $\tilde{R}$ except a set of capacity zero and we have Theorem 6.

REMARK. In case we consider fine limits of $f(z)$ relative to a D-disjoint metric (or N-Martin's topology), Beurling's theorem holds under weaker condition than almost finitely sheeted. Especially in case $\tilde{R}$ is a unit circle: $|z|<1$ and R is a Riemann sphere, every point $e^{i\theta}$ is N-minimal and N-Martin's topology is compactible to Euclidean metric on $|z|=1$. Then by Lindelöf's theorem and by Lemma 1, 4) we have the following.

THEOREM 7. *Let $w=f(z)$ be analytic function from $|z|<1$ into a w-Riemann sphere R. If $w=f(z)$ does not take 3 points near $|z|=1$ and every point of R is non thickly covered by $\tilde{R}$. Then $w=f(z)$ has angular limits on $|z|=1$ except a set of capacity zero.*

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