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# A special type of finite groups with an automorphism of prime order

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## § 0. Introduction and notation

In this paper, $G$ denotes a finite group with an automorphism $\sigma$ of prime order $r$, where $(r, |G|)=1$. Let $H=C_G(\sigma)$. We shall use the following notation:

Let $K$ be a $\sigma$-invariant subgroup of $G$.

$$\tilde{T}_K=\left\{x^{-1}x^{\sigma}\middle| x\in K\right\}.$$

For an element $h\in K\cap H^{\#}$, where $H^{\#}=H-\{1\}$,

$$T_K(h)=\left\{y\in\tilde{T}_K\middle|[y, h, \cdots, h]=1\right\},$$

$$M_K(h)=hT_K(h),$$

$$L_K(h)=\bigcup_{g\in K}M_K(h)^{g}.$$

Especially, we delete the suffix $G$ in the case $K=G$. That is:

$$\tilde{T}=\tilde{T}_G.$$

For $h\in H^{\#}$,

$$T(h)=T_G(h),$$

$$M(h)=M_G(h),$$

$$L(h)=L_G(h).$$

Furthermore, $\sigma^g$ denotes $g^{-1}\sigma g$ in the semidirect product $G\langle\sigma\rangle$.

If $\tilde{T}$ is a subgroup of $G$, then we say $G$ is *of splitting type with respect to* $\sigma$. If $T(h)$ is a subgroup of $G$ for every element $h$ of $H^{\#}$, then we say $G$ is *of locally splitting type with respect to* $\sigma$. Suppose $G$ is of splitting type (with respect to $\sigma$, then $\tilde{T}$ is a nilpotent normal complement of $H$ in $G$ by [3]. Hence by Lemma 1. 5, $G$ is of locally splitting type. The converse holds under the additional condition. More precisely, we shall show the following:

THEOREM A. *Suppose $G$ is a solvable group of locally splitting type with respect to $\sigma$. Then $G$ is of splitting type with respect to $\sigma$.*

THEOREM B. *Suppose $G$ is of locally splitting type with respect to $\sigma$ and suppose $G$ satisfies the following condition:*

*Whenever $h\in H^{\#}$ and $g\in G$, $M(h)\cap M(h)^{g}=\phi$ or $M(h)$. Then $G$ is of splitting type with respect to $\sigma$.*

All groups considered are assumed to be finite. Other notation is standard and taken from [1].

## § 1. Preliminary results and proof of Theorem A

LEMMA 1.1. *Let $P$ be a $p$-group and let $Q$ be a subgroup of $P$, where $p$ is a prime. If $Q$ is not normal in $P$, then there exists $x\in P-N_{P}(Q)$ such that both $Q$ and $Q^{x}$ are contained in $N_{P}(Q)\cap N_{P}(Q^{x})$.*

PROOF. Let $Q^{*}=N_{P}(Q)$ and let $x\in N_{P}(Q^{*})-Q^{*}$. Then $Q$ and $Q^{x}$ satisfy the assertion of Lemma 1.1.

Now we restate the Definition in [2].

DEFINITION 1.2. *Let $Y$ be a subgroup of a group $X$ which controls fusion in $Y$ with respect to $X$. Let $\Psi$, $\Gamma$ and $\Delta$ be mappings from $Y^{\#}$ to the family of subsets of $X$. If $\Psi$, $\Gamma$ and $\Delta$ satisfy the following conditions, then we say that $(\Psi, \Gamma, \Delta)$ is a complementary triple of $Y$ in $X$.*

(1.2.1) *For every $y\in Y^{\#}$;*

(i) *$\Psi(y)$ is a subgroup of $X$ and $\Psi(y)^{w}=\Psi(y^{w})$ for $w\in Y$,*

(ii) *$\Gamma(y)=y\Psi(y)$,*

(iii) *$\Delta(y)=\bigcup_{x\in X}\Gamma(y)^{x}$,*

(iv) *$N_{X}(\Gamma(y))=\Psi(y)C_{Y}(y)$.*

(1.2.2) *Whenever $y\in Y^{\#}$ and $x\in X$, $\Gamma(y)\cap\Gamma(y)^{x}=\phi$ or $\Gamma(y)$.*

(1.2.3) *$(X-\bigcup_{z\in Y^{\#}}\Delta(z))\cap N_{X}(\Gamma(y))=\Psi(y)$ for every $y\in Y^{\#}$.*

(1.2.4) *Whenever $y_1$ and $y_2$ are elements of $Y^{\#}$ which are not conjugate in $X$, then $\Delta(y_1)\cap\Delta(y_2)=\phi$.*

LEMMA 1.3. *Let $Y$ be a subgroup of a group $X$ which controls fusion in $Y$ with respect to $X$. Suppose there exists a complementary triple $(\Psi, \Gamma, \Delta)$ of $Y$ in $X$. Then $X-\bigcup_{z\in Y^{\#}}\Delta(z)$ is a normal complement of $Y$ in $X$.*

PROOF. See the proof of Theorem in [2].

In the succeeding part of this section, we show some properties of $G$.

LEMMA 1.4. *Let $z=g^{-1}g^{\sigma}$. Then $\tilde{T}z^{-1}=\tilde{T}^{g}$.*

PROOF. Since $x^{-1}x^{\sigma}(g^{-1}g^{\sigma})^{-1}=g^{-1}gh^{-1}(hg^{-1})^{\sigma}g$, Lemma 1.4 is immediate.

LEMMA 1.5. *Suppose $G$ is of splitting type. Then $(T, M, L)$ is a complementary triple of $H$ in $G$.*

PROOF. By [3], the result follows from Theorem (3.3) in [2].

LEMMA 1.6. *Let $h \in H^{\#}$. Suppose $T(h)$ is a subgroup of $G$. Then the following hold:*

(i) $N_G(M(h)) = T(h)\, C_H(h)$.

(ii) *If $u$ is an element of $H^{\#}$ which is not conjugate to $h$ in $H$, then $M(h) \neq M(u)^g$ for every element $g$ of $G$.*

PROOF. For (i), let $K = N_G(M(h))$. Then $K$ is a $\sigma$-invariant subgroup of $G$. Let $x \in \tilde{T}_K$. As $[x, h] \in T(h)$, we get $x \in T(h)$. Hence, $\tilde{T}_K \subseteqq T(h)$, which implies that $K$ is of splitting type. If $y \in H \cap K$, then $[y, h] \in T(h) \cap H = \{1\}$, resulting $H \cap K \subseteqq C_H(h)$. Therefore $K$ is contained in $T(h)\, C_H(h)$. Since the converse inclusion is obvious, (i) is verified. Next, suppose (ii) is false. Then there exists an element $g$ of $G$ with $M(h) = M(u)^g$. Because $M(h)$ is both $\sigma$-invariant and $\sigma^g$-invariant, there exists an element $y$ of $K$ with $\sigma^y = \sigma^g$. It follows that $H^g \cap M(h)$ contains $h^y$. Thus we have $h^y = u^g$, a contradiction. This proved Lemma 1.6.

LEMMA 1.7. *Let $W = \bigcap_{g \in G} \tilde{T}^g$. Then the following hold:*

(i) *$W$ is a normal subgroup of $G$.*

(ii) *$\tilde{T}W = W\tilde{T} = \tilde{T}$.*

PROOF. To prove (i), since $W$ is a $\sigma$-invariant normal subset of $G$, it suffices to show that $W$ is a subgroup. Let $w \in W$. Then $w$ is contained in $\tilde{T}^g$ for every element $g$ of $G$. By Lemma 1.4, we get $\tilde{T}^g w^{-1} = \tilde{T}^z$ for some element $z$ of $G$. Hence, $Ww^{-1} = \bigcap_{g \in G} \tilde{T}^g w^{-1}$ is cotained in $W$. It follows that $Ww^{-1} = W$ from $|Ww^{-1}| = |W|$. Thus (i) is proved. For (ii), let $x^{-1}x^{\sigma} \in \tilde{T}$ and $y \in W$. Since $W$ is a normal subgroup of $G$, there exists an element $u$ of $W$ with $(u^{-1}u^{\sigma})^x = y$. Then, $yx^{-1}x^{\sigma} = (ux)^{-1}(ux)^{\sigma}$, which concludes $\tilde{T}W = W\tilde{T} = \tilde{T}$. Thus Lemma 1.7 is proved.

LEMMA 1.8. *Let $K$ be a $\sigma$-invariant nilpotent subgroup of $G$. Suppose $G$ is of locally splitting type. Then $K$ is of splitting type.*

PROOF. Since $\tilde{T}_K$ is contained in $T_K(h)$ for every $h \in K \cap H^{\#}$, the conclusion is obvious.

LEMMA 1.9. *Suppose $G$ is of locally splitting type and $F(G) \cap H \neq 1$, where $F(G)$ is the Fitting subgroup of $G$. Then $G$ is of splitting type.*

PROOF. Let $h \in H^{\#} \cap F(G)$. Then $[g, h, \cdots, h] = 1$ for every $g \in G$. Thus we have $\tilde{T} \subseteqq T(h)$, which implies Lemma 1.9.

In the following Lemma 1. 10, let $W=\bigcap_{g\in G}\tilde{T}^g$ and $\bar{G}=G/W$.

LEMMA 1. 10. *Suppose $G$ is of locally splitting type and not of splitting type. Assume furthermore, $K$ is of splitting type for every $\sigma$-invariant proper subgroup $K$ of $G$. Then $T_{\bar{G}}(\bar{h})=\overline{T(h)}$ for every $h\in H^{\sharp}$. Especially, $\bar{G}$ is of locally splitting type.*

PROOF. First we show $W$ is contained in $Z(G)$, the center of $G$. Suppose false. Then we can choose a Sylow $p$-subgroup $V$ of $W$ which is not contained in $Z(G)$. Let $K=C_G(V)$. Since $K$ is a $\sigma$-invariant proper normal subgroup of $G$, $K$ is of splitting type and $F(K)$ is normal in $G$, where $F(K)$ is the Fitting subgroup of $K$. Suppose $F(K)\cap H\neq 1$. Since $F(K)\subseteqq F(G)$, we get $F(G)\cap H\neq 1$. Applying Lemma 1. 9, we conclude that $G$ is of splitting type, a contradiction. So we may assume $F(K)\cap H=1$. On the other hand, as $\tilde{T}_K$ is contained in $F(K)$, it follows that $F(K)=\tilde{T}_K$. Now let $q$ be a prime divisor of $|G|$ distinct from $p$ and Let $Q$ be a $\sigma$-invariant Sylow $q$-subgroup of $G$. Set $Q^*=\tilde{T}_Q$. Since $Q^*V$ is nilpotent, we get $Q^*\subseteqq K$. Let $P$ be a $\sigma$-invariant Sylow $p$-subgroup of $G$ and let $P^*=\tilde{T}_P$. Then $\tilde{T}_K P^*$ is a $\sigma$-invariant subgroup of $G$. However, as $\tilde{T}_K$ contains $Q^*$, we have $\tilde{T}_K P^*=\tilde{T}$. A contradiction. Thus we assert $W\subseteqq Z(G)$. Next, let $\bar{x}\in T_{\bar{G}}(\bar{h})$ for $h\in H^{\sharp}$. Since $\tilde{T}_{\bar{G}}=\tilde{T}/W$ by Lemma 1. 7, we get $x\in\tilde{T}$. Then $[x, h, \cdots, h]$ is contained in $Z(G)$, follows $x\in T(h)$, which implies $T_{\bar{G}}(\bar{h})\subseteqq\overline{T(h)}$. Since the converse inclusion is immediate, Lemma 1. 10 is proved.

PROPOSITION 1. 11. *Suppose $H$ is a Hall subgroup of $G$. Then $G$ is of splitting type.*

PROOF. See Corollary (3. 5) in [2].

PROOF OF THEOREM A. Let $G$ be a minimal counterexample. Since $G$ satisfies the assumption of Lemma 1. 10, we conclude $\bigcap_{g\in G}\tilde{T}^g=1$. Let $F=F(G)$. Then $F\cap H=1$ by Lemma 1. 9, which implies $F\subseteqq\bigcap_{g\in G}\tilde{T}^g$, a contradiction. Thus we proved Theorem A.

## § 2. Proof of Theorem B

Let $G$ be a minimal counterexample to Theorem B. We show that $(T, M, L)$ is a complementary triple of $H$ in $G$, which yields a contradiction.

(2. 1) *Let $K$ be a $\sigma$-invariant proper subgroup of $G$. Then $K$ is of splitting type.*

PROOF. Let $h\in K\cap H^{\sharp}$. Then obviously, $T_K(h)\subseteqq T(h)$. On the other hand, since $\tilde{T}\cap K=\tilde{T}_K$, we have $T(h)\cap K\subseteqq T_K(h)$, follows $T_K(h)=T(h)\cap K$. Thus $K$ is of locally splitting type. Now, suppose $M_K(h)\cap M_K(h)^y\neq\phi$ for $h\in K\cap H^\sharp$ and $y\in K$. Then $M(h)\cap M(h)^y\neq\phi$ it follows that $M(h)=M(h)^y$ by the assumption. As $M_K(h)=M(h)\cap K$, we conclude $M_K(h)=M_K(h)^y$. But then $K$ satisfies the assumption of Theorem B. Hence by the minimality of $G$, $K$ is of splitting type. This proves (2. 1).

(2. 2) $\bigcap_{g\in G}\tilde{T}^g=1$.

PROOF. Let $W=\bigcap_{g\in G}\tilde{T}^g$ and $\bar{G}=G/W$. Suppose $W\neq 1$. Then by Lemma 1. 10 and (2. 1), $G$ is of locally splitting type. Assume $M_{\bar{G}}(\bar{h})\cap M_{\bar{G}}(\bar{h})^{\bar{g}}\neq\phi$ for $h\in H^\sharp$ and $g\in G$. Since $T_{\bar{G}}(\bar{h})=\overline{T(h)}$, $M_{\bar{G}}(\bar{h})=\overline{M(h)}$. Hence, $\overline{M(h)}\cap\overline{M(h)^g}\neq\phi$. Let $K=\langle h\rangle T(h)\,W$. As $K$ is of splitting type, $M(h)\,W$ is contained in $L_K(h)$ by Lemma 1. 5. Therefore, we conclude $M(h)^{z_1}\cap M(h)^{gz_2}\neq\phi$ for some elements $z_1$ and $z_2$ of $W$. By the assumption of Theorem B, it follows that $M(h)^{z_1}=M(h)^{gz_2}$. But then $\overline{M(h)}=\overline{M(h)^{z_1}}=\overline{M(h)^{gz_2}}=\overline{M(h)^g}$, which implies that $\bar{G}$ satisfies the assumption of Theorem B. Then by the minimality of $G$, we conclude $\bar{G}$ is of splitting type, a contradiction. Thus (2. 2) is verified.

(2. 3) *Let $P$ be a $\sigma$-invariant Sylow $p$-subgroup of $G$ and let $P^*=\tilde{T}_P$, where $p$ is a prime. Then $P^*$ is weakly closed in $P$ with respect to $G$.*

PROOF. Suppose false. The by Lemma 1. 1, there exists some element $g$ of $G$ with $P^*\neq(P^*)^g$, $(P^*)^g\subseteqq P$ and $P^*\subseteqq N_G((P^*)^g)$. Set $K=P^*(P^*)^g$. Replacing $g$ by another element if necessary, we may assume $g\in N_G(K)$. Since $P^*\subsetneqq K$, we get $K\cap H\neq 1$. Let $h\in K\cap H^\sharp$. Then there exists an element $x$ of $(P^*)^g$ with $x\in hP^*$. On the other hand, as $h^g\in K\cap(H^g)^\sharp$, there exists an element $y$ of $P^*$ with $y\in h^g(P^*)^g$. Hence we conclude $xy\in hP^*\cap(hP^*)^g\subseteqq M(h)\cap M(h)^g$. By the assumption, it follows that $M(h)=M(h)^g$. But then $T(h)=T(h)^g$. However, as $P^*$ is the Sylow $p$-subgroup of $T(h)$ and $(P^*)^g$ is the Sylow $p$-subgroup of $T(h)^g$, we obtain $P^*=(P^*)^g$. A contradiction. Thus we proved (2. 3).

(2. 4) *Let $P^*$ be as defined in* (2. 3) *and let $K$ be a $\sigma$-invariant proper subgroup of $G$ which contains $(P^*)^g$ for some $g\in G$. Then $(P^*)^g$ is the Sylow $p$-subgroup of $\tilde{T}_K$. Especially, $(P^*)^g$ is normal in $K$.*

PROOF. Let $P_1$ be a $\sigma$-invariant Sylow $p$-subgroup of $K$ and let $P_2$ be a $\sigma$-invariant Sylow $p$-subgroup of $G$ which contains $P_1$. Then there exists some $x\in K$ with $(P^*)^{gx}\subseteqq P_1$. Hence by (2. 3), we have $(P^*)^{gx}=P_2^*=\tilde{T}_{P_2}$. But then, $P_2^*$ is contained in $\tilde{T}_K$ and hence normal in $K$. It follows that $(P^*)^g=P_2^*$. This implies (2. 4).

(2. 5) *Let $K$ be a maximal $\sigma$-invariant subgroup of $G$ with $\tilde{T}_K\neq 1$.*

*Let* $\tilde{T}_K = V_1 \times V_2 \times \cdots \times V_s$, *where* $V_i$ *is a Sylow* $p_i$*-subgroup of* $\tilde{T}_K$ *for a prime* $p_i$, $i=1, 2, \cdots, s$. *Let* $P_i$ *be a Sylow* $p_i$*-subgroup of* $G$ *which contains* $V_i$. *Then* $V_i = \tilde{T}_{P_i}$, $i=1, 2, \cdots, s$.

PROOF. *Let* $P_i^* = \tilde{T}_{P_i}$, $i=1, 2, \cdots, s$. Since $V_i \subseteqq P_i^*$ and $N_G(V_i)=K$, we have $N_{P_i^*}(V_i) \subseteqq \tilde{T}_K$. It follows that $V_i = N_{P_i^*}(V_i)$, resulting $V_i = P_i^*$. This concludes (2. 5).

(2. 6) $(G - \bigcup_{z \in H^\sharp} L(z)) \cap N_G(M(h)) = T(h)$ *for every element* $h$ *of* $H^\sharp$.

PROOF. Let $S = \bigcup_{z \in H^\sharp} L(z)$. Since $N_G(M(h)) - T(h)$ is contained in $S$, it suffices to show $S \cap T(h) = \phi$. Suppose to the contrary that $S \cap T(h) \neq \phi$ for some element $h$ of $H^\sharp$. Consequently, $T(h) \neq 1$. Let $K$ be a maximal $\sigma$-invariant subgroup of $G$ which contains $T(h)$. Set $V = \tilde{T}_K$. Then by (2. 5), $V = P_1^* \times \cdots \times P_s^*$ with $P_i^* = \tilde{T}_{P_i}$, where $P_i$ is a $\sigma$-invariant Sylow $p_i$-subgroup of $G$ for a prime $p_i$, $1 \leqq i \leqq s$. Let $1 = V_0 \subset V_1 \subset \cdots \subset V_t = V$ be the upper central series of $V$. Then there exists some $m$, $0 \leqq m < t$, with $S \cap V_m = \phi$ and $S \cap V_{m+1} \neq \phi$. Choose elements $u \in H^\sharp$ and $g \in G$ satisfying $M(u)^g \cap V_{m+1} \neq \phi$. Fix an element $w$ of $M(u)^g \cap V_{m+1}$. Let $X = N_G(M(u))$ and $Y = \tilde{T}_X$. Now we show that $V_j$ is contained in $Y^g$, $0 \leqq j \leqq m$, by induction on $j$. For $j=0$, the assertion is obvious. Suppose $V_{j-1}$ is contained in $Y^g$. Then $[V_j, w] \subseteqq V_{j-1} \subseteqq Y^g$. It follows that $V_j$ is contained in $X^g$, hence we get $V_j \subseteqq Y^g$ by the choice of $m$. Thus the assertion is proved. Therefore, $V_m \subseteqq Y^g$. But then, as $[V, w] \subseteqq [V, V_{m+1}] \subseteqq V_m \subseteqq Y^g$, follows $V \subseteqq X^g$. However by (2. 4), we obtain $P_i^* \subseteqq Y^g$, $1 \leqq i \leqq s$. This implies $w \in V \subseteqq Y^g$, a contradiction. Thus (2. 6) is proved.

(2. 7) *Let* $h_1$ *and* $h_2$ *be elements of* $H^\sharp$ *which are not conjugate in* $H$. *Then* $L(h_1) \cap L(h_2) = \phi$.

PROOF. Suppose false. Then there exists an element $g$ of $G$ with $M(h_1) \cap M(h_2)^g \neq \phi$. Fix an element $w$ of $M(h_1) \cap M(h_2)^g$. Let $X_i = N_G(M(h_i))$, $i=1, 2$. Since $\langle h_1 \rangle T(h_1)$ is nilpotent, there exists a central series $1 = Z_0 \subseteqq Z_1 \subseteqq \cdots \subseteqq Z_t \subseteqq Z_{t+1} = \langle h_1 \rangle T(h_1)$, with $Z_t = T(h_1)$. We show $Z_j \subseteqq T(h_2)^g$, $0 \leqq j \leqq t$, by induction. For $j=0$, the result follows immediately. Assume $Z_{j-1} \subseteqq T(h_2)^g$. Then $[Z_j, w] \subseteqq Z_{j-1} \subseteqq T(h_2)^g$, follows $Z_j \subseteqq X_2^g$. By (2. 6), $Z_j \subseteqq T(h_2)^g$. Therefore, we conclude the assertion. Whence we get $Z_t = T(h_1) \subseteqq T(h_2)^g$. Conversely, exchanging $T(h_1)$ for $T(h_2)^g$ and applying a similar argument, we have $T(h_2)^g \subseteqq T(h_1)$. Thus $T(h_1) = T(h_2)^g$. Consequently, $M(h_1) = M(h_2)^g$. However, this contradicts (ii) of Lemma 1. 6. Hence we proved (2. 7).

(2. 8) $(T, M, L)$ *is a complementary triple of* $H$ *in* $G$.

PROOF. By the definition of $T$, $M$ and $L$, (i), (ii) and (iii) of (1. 2. 1) is satisfied. From (i) of Lemma 1.6, follows (iv) of (1.2.1). On the other hand, (2.6) and (2.7) imply (1.2.3) and (1.2.4), respectively. Since (1.2.2) is assumed, we conclude (2.8).

(2.9) *A contradiction.*

Let $N=G-\bigcup_{z\in H^{\#}} L(z)$. By Lemma 1.3, $N$ is a normal complement of $H$ in $G$. Since $L(h)$ is $\sigma$-invariant for $h\in H^{\#}$, $N$ is a $\sigma$-invariant subgroup of $G$. Considering $N\cap H=1$, we have $N=\tilde{T}$, contradictory to our assumption. This completes the proof of Theorem B.

APPENDIX. The following question has been left unsolved.

QUESTION: *Suppose $G$ is of locally splitting type with respect to an automorphism $\sigma$ of prime order. Is $G$ a group of splitting type?*

## References

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