Hokkaido Mathematical Journal Vol. 16 (1987), p. 127~134

# On representations of the maximal unramified Galois extension of a field of positive characteristic

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(Received February 3, 1986, Revised February 19, 1987)

## 1. Introduction

The aim of this paper is to give an expression of representations of the Galois group for the maximal unramified extension of a field of positive characteristic. In fact, we express the set of representations in a group theoretic means.

To be more precise, let $K$ be a field containing an algebraically closed field $k$ of positive characteristic $p$. Let $T$ be the set of all equivalence classes of discrete valuations of $K$ trivial on $k$ whose residue fields are isomorphic to $k$. For each $v$ of $T$, let $K_v$ be the completion of $K$ at $v$ and $O_v$ the integer ring in $K_v$. Put

$$A=\prod_{v\in T} K_v,\ G_n(A)=\prod_{v\in T} GL_n(K_v),$$

and

$$O=\prod_{v\in T} O_v,\quad C_n=\prod_{v\in T} GL_n(O_v),$$

where $GL_n$ denotes the general linear group of degree $n$. Let $K_T$ be the maximal Galois extension of $K$ which is unramified at every element of $T$. Then there is an imbedding $i_v$ of $K_T$ into $K_v$ because the residue field of $O_v$ is algebraically closed. Thus $K_T$ and $K$ can be imbedded into $A$ diagonally; hence, $GL_n(K_T)$ and $GL_n(K)$ can be regarded as subgroups of $G_n(A)$. We use the same symbol $i_v$ to denote the imbedding of $GL_n(K_T)$ into $GL_n(K_v)$ induced by $i_v: K_T\rightarrow K_v$. Put

$$\mathcal{G}_n=C_n\backslash G_n(A)/GL_n(K).$$

For each element $u$ of $G_n(A)$, we denote by $[[u]]$ the double coset $C_nuGL_n(K)$. For each commutative ring $R$ of characteristic $p$, and for each $p$-power $q$, we can define an endomorphism $f_q$ of $M_n(R)$ by $(a_{ij})\longmapsto(a_{ij}^q)$. This endomorphism induces a map from $\mathcal{G}_n$ into itself, which will be again denoted by $f_q$. We often write $f=f_q$ if no confusion arises. We denote by

$G_n^f$ the set of all $f$-fixed points of $G_n$. For each group $G$, let $Rep(G, GL_n(F_q))$ be the set of all $GL_n(k)$-equivalence classes of representations of $G$ into $GL_n(k)$ whose images are isomorphic to some subgroups of $GL_n(F_q)$. Then our main result in this paper is

THEOREM 1. *Assume that the intersection $O \cap K$ in $K$ coincides with $k$. Then there exists a bijection $\phi$ from $Rep(Gal(K_T/K), GL_n(F_q))$ onto $G_n^f$.*

We shall prove this theorem in § 2. When $K$ is a function field, we can express $Rep(Gal(K_T/K), GL_n(F_q))$ in terms of adeles using Theorem 1. We shall explain this in more details. Let $X$ be an algebraic variety over $k$. Assume that $X$ is complete and normal. We take as $K$ the function field of $X$ over $k$. Then the set $T$ can be regarded as the set of all equivalence classes of valuations of prime divisors on $X$. Let $GL_n(K)_A$ be the adelization of $GL_n(K)$, and put

$$G_n' = C_n \backslash GL_n(K)_A / GL_n(K).$$

Then the assumptions of Theorem 1 are verified, and we have

THEOREM 2. *The bijection $\phi$ in Theorem 1 induces a bijection from $Rep(Gal(K_T/K), GL_n(F_q))$ onto the set $G_n'^f$ of all $f$-fixed points of $G_n'$.*

Now let $V_n$ be the set of all isomorphism classes of locally free sheaves on $X$ of rank $n$. Then the $q$-th power absolute Frobenius map on $X$ induces a map $f$ from $V_n$ into itself. Let $\pi_1(X)$ be the algebraic fundamental group of $X$. Then, as an equivalent statement to Theorem 2, we have

COROLLARY TO THEOREM 2.*) *There exists a bijection from $Rep(\pi_1(X), GL_n(F_q))$ onto the set $V_n^f$ of all $f$-fixed points of $V_n$.*

We shall prove these results in § 3. We remark that this corollary is a slight generalization of a result of H. Lange and U. Stuhler [5], § 1.

## § 2. Proof of Theorem 1

Let $k$ be a field containing a finite field $F_q$ and $K$ a field containing $k$. For a moment, we do not assume $k$ to be algebraically closed as we have done in § 1. We use the same notations as in the introduction if otherwise mentioned. We denote by $\bar{K}$ the algebraic closure of $K$, and let $K_s$ be the separable algebraic closure of $K$. For a subfield $L$ of $\bar{K}$, put

$$GL_n(L)' = \{a \in GL_n(L)\ ;\ a^{-1}f(a) \in GL_n(K)\}.$$

*) This was also announced in Katsurada [3] for the case where $K$ is an algebraic function field of one variable over $k$.

For each element $a=(a_{ij})_{1\leqq i,j\leqq n}$ of $GL_n(\bar{K})$, let $K(a)$ be the field generated over $K$ by all components of $a$. Then for each $a\in GL_n(\bar{K})'$, we have $K(a)^q K=K(a)$, where $K(a)^q$ is the image of $K(a)$ under the $q$-th power homomorphism. Thus $K(a)$ is a separable extension of $K$ (cf. Proposition 1 of Lang [4]). Since $a^{-1}f(a)$ belongs to $GL_n(K)$, we have

$$\sigma(a)^{-1}f(\sigma(a))=a^{-1}f(a),$$

where $\sigma(a)$ denotes the matrix $(\sigma(a_{ij}))_{1\leqq i,j\leqq n}$ for each element $\sigma$ of the Galois group $Gal(K_s/K)$. Hence we have

$$a\sigma(a)^{-1}=f(a\sigma(a)^{-1}).$$

This shows that $a\ \sigma(a)^{-1}$ belongs to $GL_n(F_q)$ for each $\sigma\in Gal(K_s/K)$. Hence we can define a map $\rho_a$ from $Gal(K_s/K)$ into $GL_n(F_q)$ by

$$\sigma\longmapsto a\sigma(a)^{-1}.$$

We remark that $K(a)$ is a Galois extension of $K$, and for each $\sigma\in Gal\ (K_s/K)$,

$$\rho_a(\sigma)=a\bar{\sigma}(a)^{-1},$$

where $\bar{\sigma}$ denotes the image of $\sigma$ under the natural surjection $t: Gal(K_s/K)\longrightarrow Gal(K(a)/K)$. Now for all $\sigma$, $\tau$ of $Gal(K_s/K)$, we have

$$\begin{aligned}\rho_a(\sigma\tau)&=a(\sigma\tau(a))^{-1}=a(\sigma(\tau(a)))^{-1}=a(\sigma(\rho_a(\tau)^{-1}a))^{-1}\\&=a(\rho_a(\tau)^{-1}\sigma(a))^{-1}=a\sigma(a)^{-1}\rho_a(\tau)=\rho_a(\sigma)\rho_a(\tau).\end{aligned}$$

This implies that $\rho_a$ is a homomorphism from $Gal(K_s/K)$ into $GL_n(F_q)$.

Now for a subfield $L$ of $\bar{K}$ containing $K$, put

$$\overline{GL_n(L)}=GL_n(k)\backslash GL_n(L)/GL_n(K),$$

and

$$\overline{GL_n(L)'}=\{GL_n(k)aGL_n(K)\ ;\ a\in GL_n(L)'\}.$$

For each element $a$ of $GL_n(L)$, we denote by $[a]$ the double coset $GL_n(k)aGL_n(K)$. Moreover for each representation $\rho$ of a group in $GL_n(k)$, we denote by $[\rho]$ the $GL_n(k)$-equivalence class of $\rho$. We note that two elements $a$, $b$ of $GL_n(\bar{K})'$ belong to the same double coset if and only if $\rho_a$ and $\rho_b$ are $GL_n(k)$-equivalent. In fact, if $a$ and $b$ belong to the same double coset, we have

$$b=uav$$

with $u\in GL_n(k)$, $v\in GL_n(K)$. Then, for each element $\sigma$ of $Gal(K_s/K)$, we have

$$\rho_b(\sigma)=b\sigma(b)^{-1}=uav\sigma(uav)^{-1}=ua\sigma(a)^{-1}u^{-1}$$
$$=u\rho_a(\sigma)u^{-1}.$$

Thus $\rho_a$ and $\rho_b$ are $GL_n(k)$-equivalent. Conversely, if $\rho_a$ and $\rho_b$ are $GL_n(k)$-equivalent, there is an element $u$ of $GL_n(k)$ such that

$$\rho_b(\sigma)=u\rho_a(\sigma)u^{-1}$$

for all $\sigma\in Gal(K_s/K)$. Put $v=a^{-1}u^{-1}b$. Then for each $\sigma$ of $Gal(K_s/K)$, we have

$$\sigma(v)=\sigma(a^{-1}u^{-1}b)=\sigma(a)^{-1}u^{-1}\sigma(b)=a^{-1}\rho_a(\sigma)u^{-1}\rho_b(\sigma)^{-1}b$$
$$=a^{-1}u^{-1}b=v.$$

Hence, $v$ belongs to $GL_n(K)$. This implies that $a$ and $b$ belong to the same double coset. Thus we can define an injective map $\psi$ from $\overline{GL_n(\bar{K})'}$ into $Rep(Gal(K_s/K), GL_n(F_q))$ by

$$[a]\longmapsto[\rho_a].$$

The following proposition is essentially obtained by Inaba [2], but the formulation and the proof are slightly different.

PROPOSITION 3. *$\psi$ is bijective.*

PROOF. It suffices to show that $\psi$ is surjective. Let $[\rho]$ be an element of $Rep(Gal(K_s/K), GL_n(F_q))$ and $L_\rho$ the field corresponding to the kernel of $\rho$. Then $\rho$ is factored through $Gal(L_\rho/K)$ like

$$Gal(K_s/K)\xrightarrow{t}Gal(L_\rho/K)\xrightarrow{\bar{\rho}}GL_n(F_q).$$

Since we have $F_q\subset K$, $\bar{\rho}$ gives an element of the set of 1-cocycles $Z^1(Gal(L_\rho/K), GL_n(L_\rho))$. Since we have $H^1(Gal(L_\rho/K), GL_n(L_\rho))=\{1\}$ (for example, see Serre [6], Chp. X, § 1), there is an element $a$ of $GL_n(L_\rho)$ such that $\bar{\rho}(\sigma)=a\sigma(a)^{-1}$ for all $\sigma\in Gal(L_\rho/K)$. Put $\alpha=a^{-1}f(a)$. Then we have $\sigma(\alpha)=\alpha$ for all $\sigma\in Gal(L_\rho/K)$, that is, $\alpha$ belongs to $GL_n(K)$. Moreover we have $\psi([a])=[\rho]$. This proves the surjectivity.

Now let $k$, $K$ be as in § 1. Let $\phi'$ be the inverse map of $\psi$. Let $\delta: Gal(K_s/K)\longrightarrow Gal(K_T/K)$ be the natural surjection. Then for each element $[\rho]$ of $Rep(Gal(K_T/K), GL_n(F_q))$, we can take an element $a$ of $GL_n(K_s)'$ such that $[a]=\phi'([\rho\circ\delta])$. (Recall $GL_n(\bar{K})'=GL_n(K_s)'$). Let $\rho_a$ be the homomorphism from $Gal(K_s/K)$ into $GL_n(F_q)$ defined above. Then by our construction, we have $\psi([a])=[\rho_a]$. Thus two representations $\rho\circ\delta$ and $\rho_a$ are $GL_n(k)$-equivalent each other. Consequently, $a$ belongs to $GL_n(K_T)$. Thus $a$ belongs to $G_n(A)$, and we have $[[f(a)]]=[[a]]$ in $G_n$. Since $k$ is contained in $O_v$ for each $v\in T$, we can define a map $\phi$ from $Rep(Gal(K_T/K), GL_n(F_q))$ into $G_n^f$ by

$$[\rho]\longmapsto[[a]].$$

Then to prove Theorem 1, it suffices to prove

THEOREM 1′. *$\phi$ is bijective.*

To prove this, we need the following fact.

LEMMA 4. *The map $\xi$ from $C_n$ into itself defined by $u\longmapsto f(u)u^{-1}$ is surjective.*

PROOF. Since we have $C_n=\prod_{v\in T} GL_n(O_v)$, it suffices to show that for each $v\in T$, the map

$$GL_n(O_v)\ni u\longmapsto f(u)u^{-1}\in GL_n(O_v)$$

is surjective. Let $M$ be the maximal ideal of $O_v$. Let $b$ be an element of $GL_n(O_v)$. We claim that we can take a sequence $\{a_i\}(i=0,1,\ldots)$ of $M_n(O_v)$ such that

$$a_0\in GL_n(O_v),$$
$$f(a_i)\equiv ba_i \bmod M^{i+1},$$

and

$$a_{i+1}\equiv a_i \bmod M^{i+1}.$$

In fact, since $O_v/M$ is algebraically closed, there is an element $a_0$ of $GL_n(O_v)$ such that

$$b\equiv f(a_0)a_0^{-1} \bmod M$$

(cf. Lang [4]). Assume that $a_0,\ldots,a_i$ are elements of $M_n(O_v)$ such that

$$f(a_j)\equiv ba_j \bmod M^{j+1},$$

and

$$a_j\equiv a_{j-1} \bmod M^j$$

for $1\leqq j\leqq i$. Put $a_{i+1}=b^{-1}f(a_i)$. Then we have

$$f(a_{i+1})-ba_{i+1}=f(b^{-1})f(f(a_i))-bb^{-1}f(a_i)$$
$$\equiv f(b^{-1})f(ba_i)-f(a_i)\equiv 0 \bmod M^{i+2},$$

and
$$a_{i+1}-a_i=b^{-1}(f(a_i)-ba_i)\equiv 0 \ mod \ M^{i+1}.$$
Thus by induction, we obtain a sequence $\{a_i\}$ with the required properties.

Since $O_v$ is a complete ring, this sequence $\{a_i\}$ has a limit a in $M_n(O_v)$. It follows from the construction that we have
$$a\equiv a_0 \ mod \ M,$$
and
$$f(a)\equiv ba \ mod \ M^i$$
for $i=1, 2, \ldots$. Thus a belongs to $GL_n(O_v)$, and we have $b=f(a)a^{-1}$. This proves the lemma.

**Proof of Theorem 1′.** First we prove the injectivity of $\phi$. Let $[\rho_i]$ $(i=1,2)$ be an element of $Rep(Gal(K_T/K), GL_n(F_q))$, and let $a_i$ be an element of $GL_n(K_T)'$ such that $[[a_i]]=\phi([\rho_i])$. Assume that we have
$$[[a_1]]=[[a_2]].$$
Then we have

(1) $a_i^{-1}f(a_i)\in K$,

for $i=1$, 2, and we have

(2) $a_2=aa_1b$

with $a\in C_n$, and $b\in GL_n(K)$. Then by (1) and (2), there is an element $\beta$ of $GL_n(K)$ such that

(3) $a^{-1}f(a)=a_1\beta a_1^{-1}$.

Now, as remarked at the beginning of this section, for each $\sigma\in Gal(K(a)/K)$, there is an element $\tau_\sigma$ of $GL_n(F_q)$ such that

(4) $\sigma(a_1)=\tau_\sigma a_1$.

Thus by (3) and (4), for each element $\sigma$ of $Gal(K(a_1)/K)$, we have

(5) $\sigma(a_1\beta a_1^{-1})\in C_n$.

Put $a_1\beta a_1^{-1}=(u_{ij})_{1\leq i,j\leq n}$. Let $f_{ij}(X)$ be the irreducible polynomial of $u_{ij}$ over $K$. Then by (5), $f_{ij}(X)$ belongs to $O_v[X]$ for all $v\in T$. Since we have $O\cap K=k$, $f_{ij}(X)$ belongs to $k[X]$. Since $k$ is algebraically closed, $u_{ij}$ belongs to $k$. This implies that $a^{-1}f(a)$ belongs to $GL_n(k)$. Thus a belongs to $GL_n(k)$ by Corollary to Theorem 1 of [4]. Hence by Proposition 3, we have $[\rho_1]=[\rho_2]$. This shows that $\phi$ is injective.

Now to prove the surjectivity of $\phi$, let $[[a]]$ be an f-fixed point of $G_n$. Then we have
$$f(a)=uad$$
with $u\in C_n$, $d\in GL_n(K)$. By Lemma 4, there is an element $w$ of $C_n$ such

that

$$f(w)w^{-1}=u.$$

Then $b=w^{-1}a$ satisfies

$$b^{-1}f(b)=d.$$

Now let $c$ be an element of $GL_n(\bar{K})$ such that

$$c^{-1}f(c)=d.$$

Let $T(L)$ be the set of all extensions of elements of $T$ to the field $L=K(c)$. For each element $v$ of $T(L)$, let $j_v$ be the imbedding of $GL_n(L)$ into $GL_n(K_v)$. Then $j_v(c)b^{-1}$ belongs to $GL_n(F_q)$. Thus $j_v(c)$ belongs to $GL_n(K_{v'})$, where $v'$ is the restriction of $v$ to $K$. This implies that $K(c)$ is contained in $K_T$, and we have

$$[[c]]=[[a]]$$

in $G_n$. Put $[\rho]=\psi([c])$. Then there is a homomorphism $\bar{\rho}$ from $Gal(K_T/K)$ into $GL_n(F_q)$ such that $\rho=\bar{\rho}\circ\delta$. Since we have $[c]=\phi'([\bar{\rho}\circ\delta])$, we have $[[a]]=[[c]]=\phi([\bar{\rho}])$ by our construction. This proves the surjectivity of $\phi$.

## § 3. Proof of Theorem 2 and its corollary

Now let $K$ be the function field of a complete normal irreducible algebraic variety $X$ over $k$. To prove Theorem 2, it suffices to show the following fact.

PROPOSITION 5. *Let $G_n^f$ and $G_n'^f$ be as in § 1. Then we have $G_n^f=G_n'^f$.*

PROOF. Clearly we have $G_n'^f\subset G_n^f$. We prove the converse inclusion. Let $[[a]]$ be an element of $G_n^f$. Then we have $f(a)=uad$ with $u\in C_n$, $d\in GL_n(K)$. Let $c$ an element of $GL_n(\bar{K})$ such that $c^{-1}f(c)=d$. Then $c$ belongs to $GL_n(K_T)$, and we have $[[c]]=[[a]]$ (see the proof of Theorem 1′). Since $K(c)$ is a finite extension of the function field $K$ of a normal variety, we have $i_v(c)\in GL_n(O_v)$ for all except finitely many elements $v$ of $T$. This shows that $c$ belongs to $GL_n(K)_A$. Hence $[[a]]$ belongs to $G_n'^f$. This proves the assertion.

**Proof of Corollary to Theorem 2.** First we show that there exists a bijection from $G_n'^f$ onto $V_n^f$. We use the same symbol $P$ to denote the equivalence class of the valuation of a prime divisor $P$ on $X$. Fix a vector space $V$ over $k$ together with a basis $e_1, \dots, e_n$. Let $M_O=\{M_{O,P}\}_P$ be the family of $O_P$-lattices in $V \underset{K}{\otimes} K_P$ such that

$$M_{O,P} = O_P e_1 \oplus \ldots \oplus O_P e_n$$

for all $P \in T$. We use the same symbol $i_P$ to denote the homomorphism from $V$ into $V \underset{K}{\otimes} K_P$ induced by the imbedding $i_P : K_T \longrightarrow K_P$. Then for each $a = (a_P)$ of $GL_n(K)_A$, we define a presheaf $E(a)$ on $X$ by $\Gamma(U, E(a)) = \{x \in V ; i_P(x) \in a_P^{-1} M_{O,P}$ for all $P \subset U\}$, where $U$ is an open subset of $X$. It is easily checked that $E(a)$ is a locally free sheaf of rank $n$. Moreover we have $E(a) \cong E(b)$ if we have $[[a]] = [[b]]$ in $G_n'$. Thus we can define a map $\theta$ from $G_n'$ into $V_n$ by

$$[[a]] \longmapsto \textit{the isomorphism class of } E(a).$$

It is easily checked that $\theta$ is a bijection (cf. § 1 of G. Harder and D. A. Kazhdan [1]). Now by our construction, we have $\theta(G_n'^{f}) = V_n^f$. On the other hand, we have $Gal(K_T/K) \cong \pi_1(X)$. Thus the assertion follows immediately from Theorem 2.

ACKNOWLEDGEMENTS. The author would like to thank the referee for his valuable advice.

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