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On H-separable extensions of two sided simple rings II

Dedicated to Professor Hisao Tominaga on his 60th birthday

Kozo SUGANO

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1. The aim of this paper is to give an improvement of Theorem 2 in the author's previous paper [8]. Let A be a two sided simple ring with its center C, B a two sided simple subring of A such that $B=V_{A}(V_{A}(B))$ and $V_{A}(B)$ is a finite dimensional simple C-algebra. This condition is equivalent to the condition that B is a two sided simple ring and A is an H-separable extension of B and left B-finitely generated projective (See Theorem 1 [8]). Denote the class of simple C-subalgebras of $V_{A}(B)$ by $\mathfrak{T}$, and the class of two sided simple subrings of A which are left relatively separable extensions of B in A by $\mathfrak{S}_{l}$ (See [8] for the definition of relatively separable extension). Theorem 2 [8] shows that we can obtain mutually inverse one to one correspondences between $\mathfrak{T}$ and $\mathfrak{S}_{l}$ by letting each member of $\mathfrak{T}$ or $\mathfrak{S}_{l}$ correspond with its centralizer in A. In this paper we will show that a two sided simple subring S of A which contains B belongs to $\mathfrak{S}_{l}$ if and only if A is left S-finitely generated projective.

All notations and symbols in this paper are same as those in [8] and [10].

2. Throughout this paper A will be a ring with 1, B a subring of A containing 1. We will denote the center of A by C, and $V_{A}(B)$, the centralizer of B in A, by D.

THEOREM 1. *Let B be a two sided simple ring, and A an H-separable extension of B such that A is left B-finitely generated projective. Denote the class of simple C-subalgebras of D by $\mathfrak{T}$, and the class of two sided simple subrings S of A containing B such that A is left S-finitely generated projective by* S_{l}*, respectively. Then for each S in $\mathfrak{S}_{l}$, we have $V_{A}(S)\in\mathfrak{T}$ and $V_{A}(V_{A}(S))=S$, while for each T in $\mathfrak{T}$, we have $V_{A}(T)\in\mathfrak{S}_{l}$ and $V_{A}(V_{A}(T))=T$.*

PROOF Since A is an H-separable extension of B, there exists a ring isomorphism η_{l} of $D\otimes_{C}A^{\circ}$ to $\mathrm{Hom}({}_{B}A, {}_{B}A)$ such that $\eta_{l}(d\otimes a^{\circ})(x)=dxa$, for each $a, x\in A$, $d\in D$. Here A° is the opposite ring of A. Now let $S\in\mathfrak{S}_{l}$. Since A° is central two sided simple over C and $\eta_{l}^{-1}(\mathrm{Hom}({}_{S}A, {}_{S}A))\supseteq A^{\circ}$, we have $\eta_{l}^{-1}(\mathrm{Hom}({}_{S}A, {}_{S}A))=M\otimes_{C}A^{\circ}$, for some C-subspace M of D by

Noether-Krosch Theorem (See e. g., Lemma 4. 1 [10]). Then since A is left S-finitely generated projective, we have the following isomorphisms as A-A-module;

$$\begin{aligned} A\otimes_S A &\cong \mathrm{Hom}(A_A,\ A_A)\otimes_S A\cong \mathrm{Hom}(\mathrm{Hom}({}_SA,\ {}_SA)_A,\ A_A) \\ &\cong \mathrm{Hom}(M\otimes_C A_A,\ A_A) \\ &\cong \mathrm{Hom}((A\oplus A\oplus \cdots \oplus A)_A,\ A_A)\cong A\oplus A\oplus \cdots \oplus A \end{aligned}$$

Thus A is an H-separable extension of S. Then by Theorem 1 [8] we have that $V_A(S)$ is simple and $S=V_A(V_A(S))$. The rest of the proof has been shown in Proposition 2 [8]. That $T=V_A(V_A(T))$ for each T in $\mathfrak{T}$ is due to Theorem 3. 7 [2].

Of course $\mathfrak{S}_l$ coincides with the class of two sided simple subrings S of A containing B such that A is right S-finitely generated projective. Furthermore combining Theorem 2 [8] and Theorem 1, we have

PROPOSITION 1 (Proposition 2 (3) [9]) *Under the same assumption as Theorem* 1, *if we assume furthermore that A contains a minimal A-B-submodule, then a two sided simple subring S of A which contains B belongs to $\mathfrak{S}_l$ if and only if A is A-S-completely reducible..*

PROOF. See Proposition 2 [9].

3. A subring S of A is said to be a left relatively separable extension of B in A if $B\subset S\subset A$ and the map π of $S\otimes_B A$ to A such that $\pi(s\otimes a)=sa$, for $s\in S$, $a\in A$, splits as S-A-map. This is the case if and only if there exists $\sum s_i\otimes a_i$ in $S\otimes_B A$ such that $\sum ss_i\otimes a_i=\sum s_i\otimes a_is$ for each $s\in S$ and $\sum s_ia_i=1$. We will denote $\{\sum s_i\otimes a_i\in S\otimes_B A\mid \sum ss_i\otimes a_i=\sum s_i\otimes a_is$ for each $s\in S\}=(S\otimes_B A)^S$. Right realtively separable extensions are similarly defined.

On the other hand, A is said to be a Frobenius extension of B if A is right B-finitely generated projective and A is B-A- isomorphic to $\mathrm{Hom}(A_B,\ B_B)$. This is the case if and only if there exist $h\in\mathrm{Hom}({}_BA_B,\ {}_BB_B)$ and finite x_i, $y_i\in A$ such that $x=\sum h(xx_i)y_i=\sum x_ih(y_ix)$ holds for each $x\in A$. We will call $\{h, x_i, y_i\}$ a Frobenius system of A over B. In this case the map θ of A to $\mathrm{Hom}(A_B, B_B)$ such that $\theta(x)=h\circ x$, for each $x\in A$, gives the B-A-isomorphism (See [6]). Concerning with the relation between relatively separable extensions and Frobenius extensions we have an extension of Proposition 2. 18 [3] as follows.

PROPOSITION 2. *Assume that a subring S of A is a Frobenius extension of B, and let $\{h, s_i, t_i\}$ be a Frobenius system of S over B. Then the following conditions are equivalent;*

(i) *S is a left relatively separable extension of B in A*
(ii) *S is a right relatively separable extension of B in A*
(iii) $\sum s_i D t_i = V_A(S)$ (*See Proposition* 2.4 [4])

PROOF. Since S is right B-finitely generated projective, there exists an S-A-isomorphism ψ of $S \otimes_B A$ to $\mathrm{Hom}({}_B\mathrm{Hom}(S_B, B_B), {}_BA)$ such that $\psi(s \otimes a)(f) = f(s)a$, for $a \in A$, $s \in S$ and $f \in \mathrm{Hom}(S_B, B_B)$. On the other hand the isomorphism θ of S to $\mathrm{Hom}(S_B, B_B)$ such that $\theta(t) = h \cdot t$, for each $t \in S$, yields an S-A-isomorphism $\theta^* = \mathrm{Hom}(\theta, A)$ of $\mathrm{Hom}({}_B\mathrm{Hom}(S_B, B_B), {}_BA)$ to $\mathrm{Hom}({}_BS, {}_BA)$. Then the inverse map of $\theta^* \circ \psi$ is given by $(\theta^* \circ \psi)^{-1}(g) = \sum s_i \otimes g(t_i)$, for $g \in \mathrm{Hom}({}_BS, {}_BA)$. $\theta^* \circ \psi$ induces an isomorphism of $(S \otimes {}_BA)^S$ to $\mathrm{Hom}({}_BS_S, {}_BA_S) \cong V_A(B)$. Therefore, there exists $\sum r_i \otimes s_i \in (S \otimes_B A)^S$ such that $\sum r_i s_i = 1$ if and only if there exists $d \in D$ such that $\sum s_i d t_i = 1$. It is already known that $\sum s_i \otimes t_i \in (S \otimes_B S)^S$. In fact, in $S \otimes_B S$ we have $\sum s s_i \otimes t_i = \sum_i \sum_j s_j h(t_j s s_i) \otimes t_i = \sum_j s_j \otimes \sum_i h(t_j s s_i) t_i = \sum s_j \otimes t_j s$ for each $s \in S$ (See [6]). Then we see that $\sum s_i D t_i$ is an ideal of $V_A(S)$. Thus we have shown (i) $\Longleftrightarrow$(iii). Similarly we can show (ii)$\Longleftrightarrow$(iii).

THEOREM 2. *Let A be an H-separable extension of B such that A is flat as left B-module. If a subring S of A is a Frobenius extension of B and $V_A(S)$ is a two sided simple ring, then S is a left and right relatively separable extension of B.*

PROOF. Let $\{h, s_i, t_i\}$ be a Frobenius system of S over B. Since A is an H-separable extension of B, there exists an A-A-isomorphism η of $A \otimes_B A$ to $\mathrm{Hom}({}_CD, {}_CA)$ such that $\eta(x \otimes y)(d) = xdy$, for $x, y \in A, d \in D$. Now suppose that $\sum s_i D t_i = 0$. Then $\sum s_i \otimes t_i = 0$ in $A \otimes_B A$. But we have $S \otimes_B S \subset A \otimes_B A$ by assumption. Hence we have $\sum s_i \otimes t_i = 0$ in $S \otimes_B S$. Then we have $\sum s_i \otimes h(t_i) = 0$, and $1 = \sum s_i h(t_i) = 0$, a contradiction. Thus we see $\sum s_i D t_i$ is a non zore ideal of a two sided simple ring $V_A(S)$, and have $\sum s_i D t_i = V_A(S)$.

Now we can apply the above results to H-separable extensions of two sided simple rings, and obtain

THEOREM 3. *Let A, B, $\mathfrak{S}_l$ and $\mathfrak{T}$ be as in Theorem 1. Then all subrings of A which belong to $\mathfrak{S}_l$ are Frobenius extensions of B. Conversely, if S is a two sided simple subring of A such that S is a Frobenius extension of B and $V_A(S)$ is simple, then S belongs to $\mathfrak{S}_l$.*

PROOF. Let $S \in \mathfrak{S}_l$ and $T = V_A(S)$. $D \otimes_Z T^\circ$ is simple artinian, where Z is the center of D, and both $\mathrm{Hom}({}_TD, {}_TT)$ and D are direct sum of copies of a simple left $D\otimes_C T^\circ$-module I. But they are free right T-modules of the same finite rank, since $[D:C]<\infty$ and T is a simple artinian subring of D. Hence they are direct sum of the same number of copies of I. This means that $\mathrm{Hom}({}_TD, {}_TT)$ and D are left $D\otimes_Z T^\circ$-isomorphic. Thus D is a Frobenius extension of T. Let $\{\tilde{h}, d_i, e_i\}$ be a Frobenius system of D over T. Now there exists an A-A-isomorphism η of $A\otimes_B A$ to $\mathrm{Hom}({}_CD, {}_CA)$ as is defined in Theorem 2. η induces an isomorphism $\tilde{\eta}$ of $S\otimes_B S$ to $\mathrm{Hom}({}_TD_T, {}_TA_T)$, since both S_B and ${}_BA$ are finitely generated projective (See Proposition 2.1 [7]). Now we can use the same method as in [5] to show that S is a Frobenius extension of B. Put $\tilde{\eta}^{-1}(\tilde{h})=\sum s_j\otimes t_j$, and $h(s)=\sum d_i s e_i$, for each $s\in S$. Then h is a B-B-homomorphism of S to B, since $\sum d_i\otimes e_i\in(D\otimes_T D)^D$, and we have $\sum s_j\otimes t_j\in(S\otimes_B S)^S$ by virtue of $\tilde{h}(D)\subset V_A(S)$. Now we have $\sum h(ss_j)t_j=\sum d_i s s_j e_i t_j=\sum d_i s_j e_i t_j s=\sum d_i \tilde{h}(e_i)s=s$, and similarly, $\sum s_j h(t_j s)=s$. Thus we see that $\{h, s_j, t_j\}$ is a Frobenius system of S over B. The converse is Theorem 2 [8] and Theorem 2.

After submitting this paper, the author was informed by Professor H. Tominaga that Proposition 2 is essentially included in Proposition 2.4 [4]. The author gives him hearty thanks for the kind information.

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Hokkaido University