*Hokkaido Mathematical Journal Vol. 42 (2013) p. 113–120*

# Takeshita's examples for Leray's Inequality

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(Received April 1, 2011; Revised August 18, 2011)

**Abstract.** It is well known that Leray's Inequality holds under stringent outflow condition ($SOC$). But Leray's Inequality does not hold under general outflow condition ($GOC$). This fact has been proved by Takeshita [8]. But, Takeshita's argument is very complicated. The author succeeds in giving an alternative proof which is simpler than Takeshita's. Moreover, the result is an improvement of Takeshita's result.

*Key words*: Stationary Navier-Stokes equations, Leray's Inequality.

## 1. Problem and Main Theorem

We consider a stationary flow of an incompressible viscous fluid with the Dirichlet boundary conditions. Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$ which has multiply connected components $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$. $\Gamma_1, \ldots, \Gamma_J$ lie inside of $\Gamma_0$. $\Omega$ is filled with an incompressible viscous fluid. $\boldsymbol{u} = (u_1(x), u_2(x))$ is the unknown velocity of the fluid motion and $p = p(x)$ is the unknown pressure of the fluid in $\Omega$, while $\nu > 0$ is the given kinematic viscosity. Then the fluid motion governed by the Navier-Stokes equations is

$$-\nu\Delta\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega, \tag{1.1}$$

$$\operatorname{div}\boldsymbol{u} = 0 \quad \text{in} \quad \Omega \tag{1.2}$$

with the Dirichlet boundary conditions

$$\boldsymbol{u} = \boldsymbol{\beta} \quad \text{on} \quad \partial\Omega, \tag{1.3}$$

where $\boldsymbol{f}$ is the prescribed external force and $\boldsymbol{\beta}$ is the given function defined on $\partial\Omega$. The boundary condition $\boldsymbol{\beta}$ satisfies the compatibility condition

$$\int_{\partial\Omega} \boldsymbol{\beta}\cdot\boldsymbol{n}dS = 0, \tag{1.4}$$

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*2000 Mathematics Subject Classification* : 35Q30, 76D05.

where $\boldsymbol{n}$ is the unit outward normal to $\partial\Omega$. We call the condition (1.4) "*General Outflow Condition*", (*GOC*) in short. Moreover if the boundary condition $\boldsymbol{\beta}$ satisfies

$$\int_{\Gamma_j} \boldsymbol{\beta} \cdot \boldsymbol{n} dS = 0 \quad (j = 0, \ldots, J), \tag{1.5}$$

the condition (1.5) is called "*Stringent Outflow Condition*", (*SOC*) in short.

Let $\boldsymbol{b}$ be an extension of $\boldsymbol{\beta}$ with divergence free. The proof of existense of the solution of the Navier-Stokes equations (1.1)–(1.3) depends on whether there is an extension $\boldsymbol{b}$ of $\boldsymbol{\beta}$ such that the term $((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b})$ is small. For example, the following Proposition is well known.

**Proposition 1.1** *Suppose that $\Omega$ is a two or three dimensional smooth and bounded domain and $\boldsymbol{\beta} \in \mathbb{H}^{1/2}(\partial\Omega)$ satisfies (SOC).*

*Then for any $\varepsilon > 0$, there exists an extension $\boldsymbol{b}_\varepsilon \in \mathbb{H}^1(\Omega)$ of $\boldsymbol{\beta}$ with divergence free such that $\boldsymbol{b}_\varepsilon$ satisfies*

$$|((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}_\varepsilon)| < \varepsilon \|\nabla \boldsymbol{v}\|_2^2 \quad (\boldsymbol{v} \in \mathbb{H}^1_{0,\sigma}(\Omega)). \tag{1.6}$$

We introduce some function spaces.

$$\mathbb{C}^\infty_{0,\sigma}(\Omega) = \big\{\boldsymbol{\varphi} \in \mathbb{C}^\infty_0(\Omega); \operatorname{div} \boldsymbol{\varphi} = 0 \text{ in } \Omega\big\},$$
$$\mathbb{H}^1_{0,\sigma}(\Omega) = \overline{\mathbb{C}^\infty_{0,\sigma}(\Omega)}^{\|\nabla\cdot\|}.$$

$\|\cdot\|_2$ and $(\cdot,\cdot)$ denotes the usual $\mathbb{L}^2(\Omega)$ norm and inner product on $\Omega$ respectively.

We call the estimate (1.6) "Leray's Inequality". We refer the proof to R. Finn [2], H. Fujita [4] or R. Temam [9, chap. II Section 1. Lemma 1.8]. But in this paper for the convenience of the readers, we define strictly "Leray's Inequality".

**Definition 1.1** Suppose that the boundary condition $\boldsymbol{\beta}$ satisfies (*GOC*). We say that $\boldsymbol{\beta}$ satisfies "Leray's Inequality" if for any $\varepsilon > 0$, there exists an extension $\boldsymbol{b}_\varepsilon \in \mathbb{H}^1(\Omega)$ of $\boldsymbol{\beta}$ with divergence free such that the estimate (1.6) holds true for any $\boldsymbol{v} \in \mathbb{H}^1_{0,\sigma}(\Omega)$.

If $\boldsymbol{\beta}$ satisfies (*GOC*) but not (*SOC*), A. Takeshita [8] proves that for the two dimensional annular domain $\{x \in \mathbb{R}^2; R_1 < |x| < R_2\}$, there exists a $\boldsymbol{v} \in \mathbb{H}^1_{0,\sigma}(\Omega)$ such that the value of $((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}, \boldsymbol{b})$ does not depend on extensions $\boldsymbol{b}$ of $\boldsymbol{\beta}$. Furthermore he obtained the following Proposition.

**Proposition 1.2** (A. Takeshita [8, Theorem 2]) *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary $\partial\Omega = \cup_{i=0}^{J}\Gamma_i$, $\Gamma_i$ being the connected component of $\partial\Omega$. Suppose that for each $\Gamma_i$ $(i = 0, \ldots, J)$ there exists a diffeomorphism $\varphi_i$ of $S^{N-1} \times [0,1]$ into $\overline{\Omega}$ such that $\varphi_i(S^{N-1} \times \{0\}) = \Gamma_i$ and $\varphi_i(S^{N-1} \times \{1\})$ is a sphere contained in $\Omega$ and $\boldsymbol{\beta} \in \mathbb{C}^\infty(\partial\Omega)$ satisfies (GOC).*

*Then the necessary and sufficient condition that $\boldsymbol{\beta}$ satisfies "Leray's Inequality" is $\boldsymbol{\beta}$ satisfies (SOC).*

In this paper we use the following condition.

**Definition 1.2** (Domain Condition) We call a domain $\Omega$ satisfies "*Domain Condition*" if there exists at least one inner boundary $\Gamma_j$ of $\Omega$ such that for a certain $P \in \mathbb{R}^2$ and $l > 0$, $\Gamma_j$ is contained in the ball $B(P,l)(:= \{x \in \mathbb{R}^2; |x-P| < l\})$ and the sphere $\partial B(P,l)(:= \{x \in \mathbb{R}^2; |x-P| = l\}) \subset \Omega$.

Hereafter we represent this condition as $(DC)$.

**Theorem 1.1** *Let $\Omega$ be a smooth and bounded domain with two boundaries $\Gamma_0$ and $\Gamma_1$ in $\mathbb{R}^2$. Suppose that $\Omega$ satisfies $(DC)$ and $\boldsymbol{\beta} \in \mathbb{H}^{1/2}(\partial\Omega)$ satisfies (GOC).*

*Then $\boldsymbol{\beta}$ satisfies "Leray's Inequality" if and only if $\boldsymbol{\beta}$ fulfills (SOC).*

**Corollary 1.1** *Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^2$ with several boundaries $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$. Suppose that $\Omega$ satisfies $(DC)$, that is to say, $\Gamma_1, \ldots, \Gamma_N$ are contained in $B(P,l)$ for a certain $P \in \mathbb{R}^2$, $l > 0$ in such a way that $\partial B(P,l) \subset \Omega$ and rests $\Gamma_0, \Gamma_{N+1}, \ldots, \Gamma_J$ lie outside of $B(P,l)$, and that $\boldsymbol{\beta} \in \mathbb{H}^{1/2}(\partial\Omega)$ satisfies (GOC).*

*If $\boldsymbol{\beta}$ satisfies "Leray's Inequality", then*

$$\sum_{i=1}^{N} \int_{\Gamma_i} \boldsymbol{\beta}\cdot\boldsymbol{n}dS = 0 \tag{1.7}$$

*holds true.*

**Corollary 1.2** *Let $\Omega$ be a smooth and bounded domain in $\mathbb{R}^2$ with several boundaries $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$. Suppose that there are $J$ balls $B(P_1, l_1)$,*

*..., $B(P_J, l_J)$ such that each $B(P_i, l_i)$ contains only one $\Gamma_i$ and such that $\partial B(P_i, l_i) \subset \Omega$ for all $i = 1, \ldots, J$, $\boldsymbol{\beta} \in \mathbb{H}^{1/2}(\partial\Omega)$ satisfies $(GOC)$.*

*Then $\boldsymbol{\beta}$ satisfies "Leray's Inequality" if and only if $\boldsymbol{\beta}$ fulfills $(SOC)$.*

**Remark 1.1** From the condition of Corollary 1.2, each of the inner boundaries $\Gamma_j (j = 1, \ldots, J)$ satisfies $(DC)$.

**Remark 1.2** From Definition 1.2, for a two dimensional bounded domain we slightly loosen the condition of the boundaries $\Gamma_0, \Gamma_1, \ldots, \Gamma_J$ for Proposition 1.2. A domain $\Omega$ may not be necessarily smooth. For example, the boundaries are $C^1$ class or locally lipschitzian, because Gauss-Green Theorem holds true. Definition 1.2 is useful if $\Gamma_1, \ldots, \Gamma_J$ are smooth but not diffeomorphically formed to the sphere.

**Remark 1.3** Corollary 1.1 and Corollary 1.2 hold true for various unbounded domains. For example, extrior domains, perturbed half spaces, infinite channels, etc. But "Domain Condition" are slightly different from respective domains.

**Remark 1.4** Although Theorem 1.1 does not directly imply non-existence of solutions of the Navier-Stokes equations under $(GOC)$, it suggests us to find an essentially new approach which is different from such a usual technique as subtracting solenoidal extensions of $\boldsymbol{\beta}$.

**Remark 1.5** In a certain two dimensional symmetric bounded domain, C. J. Amick [1] proved that there exist the symmetric solutions of the Navier-Stokes equations with the symmetric Dirichlet boundary conditions which satisfy $(GOC)$. Under the same conditions, H. Fujita [3] proved that any symmetric function defined on the boundary with $(GOC)$ has the extensions which satisfy symmetric types of "*Leray's Inequality*". For two or three dimensional bounded domains, H. Fujita and H. Morimoto [5] proved that there exist a weak solution of the stationary Navier-Stokes equations with the special boundary conditions. H. Kozono and T. Yanagisawa [7] proved that in three dimensional bounded domains there exist stationary solutions of the Navier-Stokes equations with the Dirichlet boundary conditions satisfying $(GOC)$, using Helmholtz-Weyl decomposition and the harmonic part of the solenoidal extension of the given boundary data which is sufficiently small in the $\mathbb{L}^3$ space compared with the viscosity constant $\nu$.

## 2. Proof

### 2.1. Proof of Theorem 1.1

First of all, let us prove Theorem 1.1 for the following domain. For certain $l$ and $l'$ with $0 < l < l'$

$$\Omega = B(O, l') \backslash \overline{B(O, l)}$$

holds true, where $O$ is the origin. $\Gamma_0$ equals $\partial B(O, l')$ and $\Gamma_1$ equals $\partial B(O, l)$. We use the basis in the polar coordinate. Since $\boldsymbol{\beta} \in \mathbb{H}^{1/2}(\partial\Omega)$ satisfies $(GOC)$, we set

$$\mu_1 := \int_{\Gamma_1} \boldsymbol{\beta} \cdot \boldsymbol{n} dS (\neq 0).$$

Let $l < r < l'$ and $\omega_r = B(0, r) \backslash B(0, l)$. Since $\boldsymbol{b} \in \mathbb{H}^1(\Omega)$ is an arbitrary extension of $\boldsymbol{\beta}$ with divergence free, we see

$$0 = \int_{\omega_r} \operatorname{div} \boldsymbol{b} dx = \int_{\Gamma_0} \boldsymbol{\beta} \cdot \boldsymbol{n} dS + \int_{|x|=r} \boldsymbol{b} \cdot \boldsymbol{n} dS.$$

Since $\boldsymbol{n} = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ for $\boldsymbol{x} \in \Gamma_1$, it should be

$$\mu_1 = \int_0^{2\pi} \boldsymbol{b}(r, \theta) \cdot \boldsymbol{e}_r r d\theta.$$

We take a vector $\boldsymbol{v}$ which has the form

$$\boldsymbol{v} = v_\theta(r) \boldsymbol{e}_\theta.$$

If $v_\theta \in C_0^\infty(l, l')$, then $\boldsymbol{v} \in \mathbb{H}^1_{0,\sigma}(\Omega)$. For such a $\boldsymbol{v}$, we obtain

$$\boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{1}{r} v_\theta^2 \boldsymbol{e}_r$$

by easy computations. Then for such a $\boldsymbol{v}$, we obtain

$$((\boldsymbol{v} \cdot \nabla) \boldsymbol{v}, \boldsymbol{b}) = -\mu_1 \int_l^{l'} \frac{1}{r} v_\theta^2 dr. \tag{2.1}$$

Lastly, let $\Omega$ be a bounded domain satisfying $(DC)$.

We suppose that the inner boundary $\Gamma_1$ is contained in $B(P,l)$ and $\partial B(P,l)$ is included in $\Omega$ for a certain $P \in \mathbb{R}^2$ and $l > 0$. Then we can find a certain $l' > 0$ such that $B(P,l) \subset B(P,l')$ and $\partial B(P,l')$ is contained in $\Omega$. Here we set

$$\mu_1 := \int_{\Gamma_1} \boldsymbol{\beta} \cdot \boldsymbol{n} dS.$$

Using the polar coordinate of the origin $P$, then we can check easily that

$$((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}) = -\mu_1 \int_l^{l'} \frac{v_\theta^2(r)}{r} dr, \tag{2.2}$$

where $\boldsymbol{v} = v_\theta(r)\boldsymbol{e}_\theta$, $v_\theta \in C_0^\infty(l,l')$.

From (2.1) and (2.2) if the boundary condition $\boldsymbol{\beta}$ satisfies "Leray's Inequality", then for $\boldsymbol{v} = v_\theta(r)\boldsymbol{e}_\theta$, $v_\theta \in C_0^\infty(l,l')$ we have

$$\left| \mu_1 \int_l^{l'} \frac{v_\theta^2(r)}{r} dr \right| < \varepsilon \|\nabla \boldsymbol{v}\|_2^2.$$

Therefore $\boldsymbol{\beta}$ fulfills $(SOC)$. $\square$

### 2.2. Proof of Corollary 1.1 and 1.2

Let us prove Corollary 1.1.

Let $\Omega$ be a bounded domain satisfying $(DC)$.

We suppose that the several inner boundaries $\Gamma_1, \ldots, \Gamma_N$ $(N \leq J)$ are contained in $B(P,l)$ and $\partial B(P,l)$ is included in $\Omega$ for a certain $P \in \mathbb{R}^2$ and $l > 0$. Then we can find a certain $l' > 0$ such that $B(P,l) \subset B(P,l')$ and $\partial B(P,l')$ is contained in $\Omega$. Here we set

$$\mu_j := \int_{\Gamma_j} \boldsymbol{\beta} \cdot \boldsymbol{n} dS \quad (j = 1, \ldots, N).$$

Let use the polar coordinate of the origin $P$. We suppose that $\boldsymbol{v} = v_\theta(r)\boldsymbol{e}_\theta$ with $v_\theta \in C_0^\infty(l,l')$. Then $\boldsymbol{v} \in \mathbb{H}_{0,\sigma}^1(\Omega)$ and we obtain

$$((\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \boldsymbol{b}) = -\sum_{j=1}^N \mu_j \int_l^{l'} \frac{v_\theta^2(r)}{r} dr. \tag{2.3}$$

Therefore if the boundary condition $\boldsymbol{\beta}$ satisfies "Leray's Inequality",

$$\sum_{i=1}^{N} \mu_i = 0$$

holds true.

Let us prove Corollary 1.2.

Suppose that there are $J$ balls $B(P_1, l_1), \ldots, B(P_J, l_J)$ such that each $B(P_i, l_i)$ contains only one $\Gamma_i$ and such that $\partial B(P_i, l_i) \subset \Omega$ for all $i = 1, \ldots, J$. Then we can find a certain $l'_i > 0$ such that $B(P_i, l_i) \subset B(P_i, l'_i)$ and $\partial B(P_i, l'_i)$ is contained in $\Omega$. Let us use the polar coordinate of the origin $P_i$. We suppose that $\boldsymbol{v}_i = v^i_\theta(r)\boldsymbol{e}^i_\theta$ with $v^i_\theta \in C^\infty_0(l_i, l'_i)$. Then $\boldsymbol{v}^i \in \mathbb{H}^1_{0,\sigma}(\Omega)$ and we obtain

$$((\boldsymbol{v}^i \cdot \nabla)\boldsymbol{v}^i, \boldsymbol{b}) = -\mu_i \int_{l_i}^{l'_i} \frac{(v^i_\theta(r))^2}{r} dr. \tag{2.4}$$

Therefore if the boundary condition $\boldsymbol{\beta}$ satisfies "Leray's Inequality", then $\boldsymbol{\beta}$ fulfills $(SOC)$. $\square$

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