Hokkaido Mathematical Journal Vol. 25 (1996) p. 521–536

Interpolating sequences and embedding theorems in weighted Bergman spaces

Masahiro YAMADA

(Received August 16, 1995; Revised October 30, 1995)

Abstract. For $0<p<\infty$, let $L^{p}{}_{a}(\mu)$ denote the weighted Bergman space on the unit disk D in the complex plane, where μ is a finite positive Borel measure on D. When μ is an absolutely continuous measure which satisfies an (A_{p})-condition, we study interpolating sequences on $L^{p}{}_{a}(\mu)$ and give several sufficient conditions in order that such a sequence exists in $L^{p}{}_{a}(\mu)$. Using them, we obtain embedding theorems for weighted Bergman spaces between $L^{p}{}_{a}(\mu)$ and $L^{q}{}_{a}(\nu)$, where ν is a finite positive Borel measure on D and $0<q<\infty$.

Key words: interpolating sequence, (A_{p})-condition, Carleson inequality, Bergman space, analytic function.

1. Introduction

Let D denote the open unit disk in the complex plane and H a set of all analytic functions on D. For $0<p<\infty$, let $L^{p}(\mu)$ denote an L^{p}-space on D with respect to a finite positive Borel measure μ on D and set $L^{p}{}_{a}(\mu)=L^{p}(\mu)\cap H$, which is called a weighted Bergman space on D.

For any a in D, let ϕ_{a} be the Möbius function on D, that is, $\phi_{a}(z)=(a-z)/(1-\bar{a}z)$ $(z\in D)$, and put $\beta(a,z)=1/2\{\log(1+|\phi_{a}(z)|)(1-|\phi_{a}(z)|)^{-1}\}$ $(a,z\in D)$. For $0<r<\infty$ and a in D, let $D_{r}(a)=\{z\in D;\beta(a,z)<r\}$ be the Bergman disk with "center" a and "radius" r, and m be the Lebesgue area measure on D. We define an average of a finite positive measure μ on $D_{r}(a)$ by

$$\widehat{\mu}_{r}(a)=\frac{1}{m(D_{r}(a))}\int_{D_{r}(a)}d\mu \quad (a\in D),$$

and if there exists a non-negative function w in $L^{1}(m)$ such that $d\mu=wdm$, then we may write it $\widehat{w}_{r}$ instead of $\widehat{\mu}_{r}$.

Let ν and μ be finite positive Borel measures on D, and for $0<p$, $q<\infty$, let $i: L^{p}{}_{a}(\mu)\rightarrow L^{q}{}_{a}(\nu)$ be an inclusion mapping. Our purpose of this paper is to study a necessary and sufficient condition on ν and μ so

---

1991 Mathematics Subject Classification : 30D99, 42A50.

that the inclusion mapping i is continuous. We say that ν and μ satisfy a (ν,μ)-Carleson inequality of (q,p), if there is a constant $C>0$ such that $(\int|f|^q d\nu)^{1/q} \leqq C(\int|f|^p d\mu)^{1/p}$ for all f in H. When $1 \leqq p,\ q<\infty$, clearly i is continuous if and only if ν and μ satisfy the (ν,μ)-Carleson inequality of (q,p). Hence, naturally we will study a necessary and sufficient condition on ν and μ so that ν and μ satisfy the (ν,μ)-Carleson inequality of (q,p) for $1 \leqq p,\ q<\infty$ and remaining cases, namely $0<p<1$ or $0<q<1$.

Particularly, put $d\mu=(1-|z|^2)^\alpha dm$ for $\alpha>-1$. When $0<p \leqq q<\infty$, Oleinik-Pavlov [11] showed that ν and μ satisfy the (ν,μ)-Carleson inequality of (q,p) if and only if there exists $0<r<\infty$ such that $(1-|a|^2)^{2(1-q/p)}\widehat{\nu}_r(a)/\widehat{\mu}_r(a)^{q/p}$ is bounded for $a \in D$. And when $0<q<p<\infty$, Luecking [8] showed that ν and μ satisfy the (ν,μ)-Carleson inequality of (q,p) if and only if there exists $0<r<\infty$ such that $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ is in $L^t(\mu)$, where $1/t+1/(p/q)=1$. In the result of [8], roughly speaking, if $p \to q$, then $t \to \infty$, hence we obtain that $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ is bounded for $a \in D$. Therefore, we can find the common property between two inequalities which are the (ν,μ)-Carleson inequalities of (q,p) when $0<p \leqq q<\infty$ and $0<q<p<\infty$. Conversely, when $0<p<\infty$, $\mu=m$ and $d\mu=\chi_G d\mathrm{m}$, where χ_G is a characteristic function of a measurable subset G of D, Luecking [5] showed the equivalence between the (ν,μ)-Carleson inequality of (p,p) and the condition that $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ is bounded for $a \in D$. A necessary and sufficient condition for the (ν,μ)-Carleson inequality of (q,p) is not known completely when ν, μ, p, and q are general. Therefore, it is interesting to study this condition. However, the result for this investigation is known only in Nakazi-Yamada [9]. When $p=q=2$, $d\mu=wdm$, and w satisfies the $(A_2(0))_\partial$-condition (See §3.), Nakazi-Yamada [9] showed the equivalence between the (ν,μ)-Carleson inequality of (2,2) and the condition that $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ is bounded for $a \in D$. Since $w(z)=(1-|z|^2)^\alpha$ does not satisfies the $(A_2(0))_\partial$-condition if $\alpha \geqq 1$, this result does not contain Oleinik-Pavlov's one.

In §2 of this paper, we give two sufficient conditions for the (ν,μ)-Carleson inequality of (q,p) when $0<p \leqq q<\infty$ and $0<q<p<\infty$. In §3, observing interpolating sequences in weighted Bergman spaces, we show that two sufficient conditions in §2 are also necessary when μ satisfies some conditions. This interpolation problem was studied by Amar [1], Amar gives a sufficient condition for a sequence in D in order that it is an interpolating sequence in ${L^p}_a(m)$. And Rochberg [12] extended it when D is a symmetric domain in C^n. The proofs in [12] are based on the results in [2]. Here, it is difficult that the results in [2] can be extend to the weighted case. Therefore, we use quantities ε_R and δ_R in order to avoid the difficulty (See §3.). Using the quantities, we give a sufficient condition for μ in order that an interpolating sequence exists in $L^p{}_a(\mu)$, and give necessary and sufficient conditions, which can be unifiable, on ν and μ in order that (ν, μ)-Carleson inequalities of (q, p) are satisfied when $0 < p \leqq q < \infty$ and $0 < q < p < \infty$.

The author wishes to thank Professor Takahiko Nakazi for his advice and indispensable help while this work was in progress.

2. (ν, μ)-Carleson inequality

Let $w \geqq 0$ be an integrable function on D. For $1 < p < \infty$, we say that w satisfies an (A_p)-condition if there exist $0 < r$, $C < \infty$ such that

$$\widehat{w}_r(a)(w^{-1/(p-1)})\widehat{\ }_r(a)^{p-1} \leqq C$$

for all a in D (This condition is often called condition C_p [7].). Moreover, for $\alpha > -1$, put $dm_\alpha = (1-|z|^2)^\alpha dm$, and throughout, C will denote a positive constant whose value is not necessarily the same at each occurrence; it may vary even within a line. We give sufficient conditions on ν and μ which satisfy the (ν, μ)-Carleson inequality of (q, p). The following lemma 1 is a consequence of corollary 3.6 and corollary 3.8 in [7].

Lemma 1 *Suppose that $d\mu = wdm_\alpha$ and w satisfies the (A_p)-condition for some $0 < p < \infty$. Then, for any $0 < r(1)$, $r(2)$, $r(3) < \infty$, there is a constant $0 < C < \infty$ such that $C^{-1}\widehat{\mu}_{r(1)}(a) \leqq \widehat{\mu}_{r(2)}(z) \leqq C\widehat{\mu}_{r(1)}(a)$ for all a, z in D such that $\beta(a, z) < r(3)$.*

Lemma 1 implies that the (A_p)-condition is independent of choice of r and $\widehat{w}_r(a)$ is equivalent to $\widehat{w}_r(z)$ for $z \in D_r(a)$ when w satisfies the (A_p)-condition. The following proposition 1 gives sufficient conditions on ν and μ which satisfy the (ν, μ)-Carleson inequality of (q, p). In order to prove them we use ideas in [8], [9] and [14; p109].

Proposition 1 *Suppose that $d\mu = wdm_\alpha$ and w satisfies the (A_s)-condition for some $1 < s < \infty$.*

(1) *Suppose that $0 < p \leqq q < \infty$. If there exists $0 < r < \infty$ such that $(1-|a|^2)^{2(1-q/p)}\widehat{\nu}_r(a)/\widehat{\mu}_r(a)^{q/p}$ is bounded for $a \in D$, then ν and μ satisfy the (ν, μ)-Carleson inequality of (q, p).*

(2) *Suppose that* $0 < q < p < \infty$. *If there exists* $0 < r < \infty$ *such that* $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ *is in* $L^t(\mu)$, *here* $1/t + 1/(p/q) = 1$, *then* ν *and* μ *satisfy the* (ν, μ)*-Carleson inequality of* (q, p).

Proof. (1) Suppose that there exists $0 < r < \infty$ such that $(1 - |a|^2)^{2(1-q/p)}\widehat{\nu}_r(a)/\widehat{\mu}_r(a)^{q/p}$ is bounded for $a \in D$. Proposition 4.3.8 in [14; p62] and Hölder's inequality imply that there exists a constant $C > 0$ such that

$$\begin{aligned} |f(a)|^{p/s} &\leqq \frac{C}{m(D_r(a))} \int_{D_r(a)} |f|^{p/s} dm \\ &\leqq \frac{C}{m(D_r(a))} \left(\int_{D_r(a)} |f|^p w dm \right)^{1/s} \\ &\qquad \times \left(\int_{D_r(a)} w^{-1/(s-1)} dm \right)^{(s-1)/s} \end{aligned}$$

for all f in H and a in D. Since $m(D_r(a))$ is equivalent to $(1 - |a|^2)^2$, the function $(1 - |z|^2)^\alpha$ can be replaced by $(1 - |a|^2)^\alpha$ for z in $D_r(a)$, and w satisfies the (A_s)-condition, hence lemma 1 implies that $|f(a)|^q \leqq C(\int_{D_r(a)} |f|^p d\mu)^{q/p} \widehat{\mu}_r(a)^{-q/p}(1 - |a|^2)^{-2q/p}$ for all f in H and a in D. Integrating the inequality with respect to ν over D, and by lemma 4.3.6 in [14; p62], there is a positive integer $N = N_r$ such that there exists $\{\lambda_n\} \subset D$ satisfying that $D = \cup D_r(\lambda_n)$ and any z in D belongs to at most N of the sets $D_{2r}(\lambda_n)$, therefore lemma 1 implies that

$$\begin{aligned} &\int_D |f|^q d\nu \\ &\leqq C \sum \int_{D_r(\lambda_n)} \left(\int_{D_r(a)} |f|^p d\mu \, \widehat{\mu}_r(a)^{-1}(1 - |a|^2)^{-2} \right)^{q/p} d\nu(a) \\ &\leqq C \sum \left(\int_{D_{2r}(\lambda_n)} |f|^p d\mu \, \widehat{\mu}_r(\lambda_n)^{-1}(1 - |\lambda_n|^2)^{-2} \right)^{q/p} \nu(D_r(\lambda_n)), \end{aligned}$$

since $D_r(a) \subset D_{2r}(\lambda_n)$ for a in $D_r(\lambda_n)$ and $(1 - |z|^2)^\alpha$ can be replaced by $(1 - |a|^2)^\alpha$ for z in $D_r(a)$. Hence the hypothesis and the choice of Bergman disks imply that the (ν, μ)-Carleson inequality of (q, p) is satisfied, because $q/p \geqq 1$.

(2) Suppose that there exists $0 < r < \infty$ such that $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ is in $L^t(\mu)$, where $1/t + 1/(p/q) = 1$. At the first inequality in the proof of (1)

of this proposition, we replace p by q, then we have that

$$|f(a)|^q \leqq C\left(\frac{1}{m(D_r(a))}\right)^s \int_{D_r(a)} |f|^q w dm \times \left(\int_{D_r(a)} w^{-1/(s-1)} dm\right)^{s-1}$$

for all f in H and a in D. Moreover, similar arguments in the proof of (1) implies that $|f(a)|^q \leqq C(\int_{D_r(a)} |f|^q d\mu)\widehat{\mu}_r(a)^{-1}(1-|a|^2)^{-2}$ for all f in H and a in D. Integrating the inequality with respect to ν over D, and applying Fubini's theorem, then lemma 1 implies that

$$\begin{aligned}&\int_D |f(a)|^q d\nu(a)\\ &\leqq C\int_D |f(z)|^q \int_{D_r(z)} \widehat{\mu}_r(a)^{-1}(1-|a|^2)^{-2} d\nu(a) d\mu(z)\\ &\leqq C\int_D |f(z)|^q (\widehat{\nu}_r(z)\widehat{\mu}_r(z)^{-1}) d\mu(z),\end{aligned}$$

because $\chi_{D_r(a)}(z) = \chi_{D_r(z)}(a)$, and $(1-|z|^2)$ is equivalent to $(1-|a|^2)$ for a in $D_r(z)$. Hence the hypothesis and Hölder's inequality imply that the (ν,μ)-Carleson inequality of (q,p) is satisfied, because $p/q > 1$. $\square$

In the statements of (1) and (2) of proposition 1, if we replace $\widehat{\mu}_r(a)$ by $(w^{-1/(s-1)})\widehat{}_r(a)^{-(s-1)}$, then we can omit the hypothesis of the (A_s)-condition. Therefore, we can give more general sufficient conditions.

3. Interpolating sequences in $L^p{}_a(\mu)$

For any a in D, let $K_a(z) = (1-\bar{a}z)^{-2}$ and $k_a(z) = K_a(z)/K_a(a)^{1/2}$ $(z \in D)$. For $\alpha > -1$, put $\widetilde{\mu}_\alpha(a) = \int |k_a|^{2+\alpha} d\mu$ $(a \in D)$ which is called a Berezin transform of μ. If $d\mu = w dm_\alpha$, then we write it $\widetilde{w}_\alpha$ instead of $\widetilde{\mu}_\alpha$. For $1 < p < \infty$, we say that w satisfies an $(A_p(\alpha))_\partial$-condition if there exists $0 < C < \infty$ such that

$$\widetilde{w}_\alpha(a)(w^{-1/(p-1)})\widetilde{}_\alpha(a)^{p-1} \leqq C$$

for all a in D (The $(A_p(\alpha))_\partial$-condition is stronger than the (A_p)-condition.). For $0 < R < \infty$, put

$$\varepsilon_R(\mu,\alpha) = \sup_{a\in D}\left(\int_{D\setminus D_R(a)} |k_a|^{2+\alpha} d\mu\right)\widetilde{\mu}_\alpha(a)^{-1}.$$

Moreover, for $0 < R < \infty$ and θ is a real number, put

$$\delta_R(\mu, \alpha, \theta) = \sup_{a \in D} \left(\int_{D \setminus D_R(a)} |K_a|^{1+\alpha/2} (1 - |z|^2)^\theta d\mu \right) \{(1 - |a|^2)^\theta \widetilde{\mu}_\alpha(a)\}^{-1}.$$

The quantity $\varepsilon_R(\mu, 0)$ was defined in [9] and was used for an observation of a uniformly absolutely integrability for some measures, here we need it for our objects. Clearly, $0 < \varepsilon_R(\mu, \alpha) \leqq 1$ and if $\theta = 0$ then $\varepsilon_R(\mu, \alpha) \leqq 2^{2+\alpha} \delta_R(\mu, \alpha, 0)$ for all $0 < R < \infty$, because $|k_a|^{2+\alpha} = (1 - |a|^2)^{2+\alpha} \{|K_a|^{1+\alpha/2}\}^2 \leqq (1 - |a|^2)^{2+\alpha} |K_a|^{1+\alpha/2} / (1 - |a|)^{2+\alpha}$.

Firstly, we show that the converse implication of (1) of proposition 1 is true when $\varepsilon_R(\mu, \alpha) < 1$ for some $0 < R < \infty$. Moreover, we also find that the converse implication of (2) of proposition 1 is true when $\varepsilon_R(\mu, \alpha) \to 0$ $(R \to \infty)$ and $0 < p \leqq 1$. In order prove it, we need a notion of interpolating sequences in $L^p{}_a(\mu)$, which was studied by Amar [1] when $\mu = m$. When $d\mu = w dm_\alpha$ and w satisfies a condition $B_p(\alpha)$, Luecking [7] studied a sufficient condition for separated sequences in order to embed $L^p{}_a(\mu)$ isometrically as a closed subspace of l^p, and hence obtained a representation formula for $L^p{}_a(\mu)$-functions, which are closely related to interpolating sequences in that space (The condition $B_p(\alpha)$ is stronger than the (A_p)-condition, and weaker than the $(A_p(\alpha))_\partial$-condition, the definitions of separated and interpolating sequences are below.). Here, we give a sufficient condition for separated sequences in order that the embedding map from $L^p{}_a(\mu)$ to l^p is onto when $d\mu = w dm_\alpha$, w satisfies the (A_s)-condition for some $1 < s < \infty$, $\varepsilon_R(\mu, \alpha) \to 0$ $(R \to \infty)$, and $0 < p \leqq 1$. We also observe the interpolating sequences in $L^p{}_a(\mu)$, and obtain the characterization of the (ν, μ)-Carleson inequality of (q, p) when $1 < p < \infty$.

Theorem 2 *Suppose that* $0 < p \leqq q < \infty$, $d\mu = w dm_\alpha$, w *satisfies the* (A_s)*-condition for some* $0 < s < \infty$ *and there exists* $0 < R < \infty$ *such that* $\varepsilon_R(\mu, \alpha) < 1$. *Then* ν *and* μ *satisfy the* (ν, μ)*-Carleson inequality of* (q, p) *if and only if there exists* $0 < r < \infty$ *such that* $(1 - |a|^2)^{2(1-q/p)} \widehat{\nu}_r(a) / \widehat{\mu}_r(a)^{q/p}$ *is bounded for* $a \in D$.

Proof. By (1) of proposition 1, it is enough to prove the "only if" part. Therefore. we suppose that there exists a constant $0 < C < \infty$ such that $\int |f|^q d\nu \leqq C(\int |f|^p d\mu)^{q/p}$ for all f in H. Here, put $f = k_a{}^{(2+\alpha)/p}$, by lemma

4.3.3 in [14], we have that $(\int |k_a|^{2+\alpha} d\mu)^{q/p} \geqq C^{-1} \int_{D_r(a)} |k_a|^{(2+\alpha)q/p} d\nu \geqq C'(1-|a|^2)^{2(1-q/p)-\alpha q/p} \widehat{\nu}_r(a)$ for all $0 < r < \infty$ and a in D (Here, C' is depend only on ν, μ and r.). Moreover, a simple computation shows that $\varepsilon_R(\mu, \alpha) < 1$ if and only if there exists a constant $0 < C < \infty$ such that $\int |k_a|^{2+\alpha} d\mu \leqq C(1-|a|^2)^{-\alpha} \widehat{\mu}_R(a)$ for all a in D (See lemma 1 in [9].). Hence, the desired result follows from lemma 1. □

We observe interpolating sequences for $L^p{}_a(\mu)$. Let $A = \{a_j\}$ be an infinite sequence in D, and put $R_A = 1/2 \inf\{\beta(a_i, a_j);\ i \neq j\}$. A sequence A is said to be separated if $R_A > 0$. For $0 < p < \infty$, let $s(\mu, p, a) = s(a) = \inf\{\int |f|^p d\mu; f(a) = 1, f \in H\}$ $(a \in D)$, which is called a Riesz's function of μ and was studied in [10]. We define a map T_A from $L^p{}_a(\mu)$ to l^p by $T_A f = \{s(a_j)^{1/p} f(a_j)\}$, and a separated sequence A is called an interpolating sequence for $L^p{}_a(\mu)$ if T_A is onto. If $d\mu = dm_\alpha$, then $s(a) = (1-|a|^2)^{2+\alpha}$. When $\alpha = 0$, our definition of interpolating sequences is same to Amar's one. Hence, using the Riesz's function, a notion of interpolating sequences can be defined for a general weighted Bergman space $L^p{}_a(\mu)$.

Lemma 2 *Suppose that $0 < p < \infty$, $d\mu = w dm_\alpha$, w satisfies the (A_s)-condition for some $1 < s < \infty$ and there exists $0 < R < \infty$ such that $\varepsilon_R(\mu, \alpha) < 1$. If a sequence A is separated, then T_A is continuous.*

Proof. Clearly, a simple computation and the hypothesis in lemma 2 imply that there exists a constant $0 < C < \infty$ such that $s(a) \leqq (1-|a|^2)^{2+\alpha} \int |k_a|^{2+\alpha} d\mu \leqq C\mu(D_R(a))$ for all a in D (See lemma 1 in [9].). Hence a continuity of T_A follows from theorem 3.12 in [7]. □

When D is a symmetric domain in C^n, Rochberg [12], using results in [2] to avoid direct and complicated computations, gave a sufficient condition for $A = \{a_j\}$ in order that T_A is onto. In order to prove proposition 3, we use ideas in [1] and [12], but for general weighted Bergman space $L^p{}_a(\mu)$, it may be hard to consider estimations of reproducing kernels in that space which are used in [1] and [12]. We refer the problem to the quantity ε_R and δ_R. In order to prove theorem 4, the following proposition 3 is important and essential.

Proposition 3 *Suppose that $\mu = w dm_\alpha$.*

(1) *If $0 < p \leqq 1$, w satisfies the (A_s)-condition for some $1 < s < \infty$ and $\varepsilon_R(\mu, \alpha) \to 0$ $(R \to \infty)$. Then, there exist $0 < R_0$, $\gamma < \infty$ such that*

if a sequence $A = \{a_j\}$ *is in* D *and* $R_A \geqq R_0$, *then there is a map* S_A *from* l^p *to* $L^p{}_a(\mu)$ *so that* $T_A S_A = I$ *and* $\sup\{\int |S_A\{c_j\}|^p d\mu; \Sigma |c_j|^p \leqq 1, \{c_j\} \in l^p\} \leqq \gamma$.

(2) *If* $1 < p < \infty$, w *satisfies the* $(A_p(\alpha))_\partial$*-condition and there exists* θ *such that* $(1-p)(1+\alpha) < \theta < 0$, $\delta_R(\mu, \alpha, \theta) \to 0$ $(R \to \infty)$. *Then there exist* $0 < R_0$, $\gamma < \infty$ *such that they satisfy the same properties in* (1) *of proposition* 3.

Proof. (1) It is enough to prove that there exists a sequence of functions f_i in $L^p{}_a(\mu)$ such that $\int |f_i|^p d\mu \leqq C$ and $\Sigma_j |s(a_j)^{1/p} f_i(a_j) - \delta_{ij}|^p \leqq 1 - \eta$ $(i \geqq 1)$ for some $0 < C < \infty$ and $0 < \eta < 1$. In fact, suppose such a sequence of functions exists. Let σ_j be a point mass of a_j, and $L^p(D, \Sigma\sigma_j)$ be a usual measure space on D. Put $v_i(a_j) = \delta_{ij}$, and we define $Bv_i = f_i$, then B is continuously extendable to $L^p(D, \Sigma\sigma_j)$, because $\int |f_i|^p d\mu \leqq C$ for all $i \geqq 1$. Moreover, we have that

$$\|(T_A B - I)(\Sigma \lambda_i v_i)\|_p^p \leqq \Sigma |\lambda_i|^p \|T_A B v_i - v_i\|_p^p \leqq (1-\eta)\Sigma|\lambda_i|^p.$$

Therefore, put $S_A = B(T_A B)^{-1}$, then S_A satisfies the required property. We will prove the existence of f_i. Let $f_i(z) = s(a_i)^{-1/p}\{(1-|a_i|^2)k_{a_i}(z)\}^{(2+\alpha)/p}$, then f_i is analytic on D and $\int |f_i|^p d\mu = s(a_i)^{-1}(1-|a_i|^2)^{2+\alpha} \int |k_{a_i}|^{2+\alpha} d\mu$. By the definition of $s(a)$, making a change of variable, and Jensen's inequality implies that

$$\begin{aligned} s(a) &\geqq C(1-|a|^2)^\alpha \inf\left\{\int_{D_r(0)} |f \circ \phi_a|^p w \circ \phi_a |k_a|^2 dm; f(a) = 1\right\} \\ &\geqq C(1-|a|^2)^{2+\alpha} m(D_r(0)) \exp\left\{\int_{D_r(0)} w \circ \phi_a dm / m(D_r(0))\right\} \\ &\geqq C(1-|a|^2)^{2+\alpha} (w^{-1/(s-1)})\widehat{_r}(a)^{-(s-1)}, \end{aligned}$$

where $0 < r < \infty$ is arbitrary and C depends only on r. Since w satisfies the (A_s)-condition and $\varepsilon_R(\mu, \alpha) < 1$ for some R, lemma 1 implies that $\int |f_i|^p d\mu \leqq C$ $(i \geqq 1)$. Next, the above results imply that for any fixed $0 < r < \infty$, there exists a constant $0 < C = C_r < \infty$ so that

$$\begin{aligned} &\Sigma_j |s(a_j)^{1/p} f_i(a_j) - \delta_{ij}|^p \\ &\quad = \Sigma_{j \neq i} s(a_j) s(a_i)^{-1} (1-|a_i|^2)^{2+\alpha} |k_{a_i}(a_j)|^{2+\alpha} \\ &\quad \leqq C s(a_i)^{-1} (1-|a_i|^2)^{2+\alpha} \Sigma_{j \neq i} \int_{D_r(a_j)} |k_{a_i}|^{2+\alpha} d\mu \end{aligned}$$

$$\leqq s(a_i)^{-1}(1-|a_i|^2)^{2+\alpha}\int_{\bigcup_{j\neq i}D_r(a_j)}|k_{a_i}|^{2+\alpha}d\mu$$
$$\leqq C\widetilde{\mu}_\alpha(a_i)^{-1}\int_{D\setminus D_{(2R_A-r)}(a_i)}|k_{a_i}|^{2+\alpha}d\mu\quad(i\geqq 1)$$

for all $A=\{a_i\}$ such that $R_A>r/2$, since for any $j\neq i$ if $\zeta\in D_r(a_j)$ then $2R_A\leqq\beta(a_i,a_j)\leqq\beta(a_i,\zeta)+\beta(\zeta,a_j)\leqq\beta(a_i,\zeta)+r$. Therefore, the desired result follows from the hypothesis $\varepsilon_R(\mu,\alpha)\to 0$ $(R\to\infty)$.

(2) Let $g_i(z)=s(a_i)^{-1/p}\{(1-|a_i|^2)\ K_{a_i}(z)^{1/2}\}^{2+\alpha}$ and $\{e_i\}$ be the usual basis in l^p. We can define a mapping L from l^p to $L^p{}_a(\mu)$ by $L(\Sigma\lambda_ie_i)=\Sigma\lambda_ig_i$. We claim that L is a continuous mapping. In fact, the $(A_p(\alpha))_\partial$-condition implies the condition $B_p(\alpha)$, therefore by theorem 2.1 in [7], we have that the dual of $L^p{}_a(\mu)=L^p{}_a(wdm_\alpha)$ can be identified with $L^q{}_a(w^{-q/p}dm_\alpha)$, where $1/p+1/q=1$ and the pairing is given by $\langle g,h\rangle=\int g\bar{h}dm_\alpha$. For any h in $L^q{}_a(w^{-q/p}dm_\alpha)$, we have that

$$|\langle L(\Sigma\lambda_ie_i),h\rangle|$$
$$\leqq\Sigma|\lambda_i|s(a_i)^{-1/p}(1-|a_i|^2)^{2+\alpha}\left|\int h\overline{K}_{a_i}{}^{1+\alpha/2}dm_\alpha\right|$$
$$\leqq(\Sigma|\lambda_i|^p)^{1/p}\{\Sigma s(a_i)^{-q/p}(1-|a_i|^2)^{q(2+\alpha)}|h(a_i)|^q\}^{1/q}.$$

Since w satisfies the $(A_p(\alpha))_\partial$-condition, there exists a constant $0<C<\infty$ such that

$$\Sigma s(\mu,p,a_i)^{-q/p}(1-|a_i|^2)^{q(2+\alpha)}|h(a_i)|^q\leqq C\Sigma\int_{D_{R_A}(a_i)}|h|^qw^{-q/p}dm_\alpha,$$

here C depends only on R_A. Moreover, we may assume that $R_A>0$, we obtain that L is continuous. As in the proof of (1) of proposition 3, it is enough to prove that an operator norm of T_AL-I can be less than $1-\eta$ for some $0<\eta<1$. Let (a_{ij}) be a matrix of T_AL-I with respect to $\{e_i\}$, then we have that $a_{ii}=0$ and $a_{ij}=\{s(a_i)/s(a_j)\}^{1/p}\{(1-|a_j|^2)K_{a_j}(a_i)^{1/2}\}^{2+\alpha}$ $(i\neq j)$. By theorem 3.2.2 in [14; p42], we only prove that there exists a non-negative sequence $\{h_i\}$ such that $\Sigma_j|a_{ij}|h_j{}^q\leqq(1-\eta)h_i{}^q$ $(i\geqq 1)$ and $\Sigma_i|a_{ij}|h_i{}^p\leqq(1-\eta)h_j{}^p$ $(j\geqq 1)$. By the hypothesis, there exists θ such that $(1-p)(1+\alpha)<\theta<0$ and $\delta_R(\mu,\alpha,\theta)\to 0$ $(R\to\infty)$. Let $h_i=s(a_i)^{1/pq}(1-|a_i|^2)^{\theta/p}$, then for any fixed $0<r<\infty$ lemma 4.3.3 in [14; p60], proposition 4.3.8 in [14; p62], and above arguments in the proof of (1) of proposition 3 imply that there exists a constant $0<C=C_r<\infty$

such that

$$\begin{aligned}&\Sigma_j|a_{ij}|h_j{}^q h_i{}^q\\&\leqq C(1-|a_i|^2)^{-q\theta/p}\int_{D_{2R_A-r}(a_i)^c}(1-|z|^2)^{q\theta/p+\alpha}|1-\bar{a}_i z|^{-(2+\alpha)}dm\end{aligned}$$

and

$$\begin{aligned}&\Sigma_i|a_{ij}|h_i{}^p h_j{}^p\\&\leqq C(1-|a_j|^2)^{2+\alpha-\theta}s(a_j)^{-1}\int_{D_{2R_A-r}(a_j)^c}(1-|z|^2)^{\theta}|1-\bar{a}_j z|^{-(2+\alpha)}d\mu\\&\leqq C\{(1-|a_j|^2)^{\theta}\tilde{\mu}_\alpha(a_j)\}^{-1}\int_{D_{2R_A-r}(a_j)^c}|K_{a_j}|^{1+\alpha/2}(1-|z|^2)^{\theta}d\mu,\end{aligned}$$

because $1-|a|^2$ is equivalent to $1-|z|^2$ for $z\in D_r(a)$. In the first inequality, by making a change of variable, lemma 4.2.2 in [14; p53] and corollary 1.2 in [3; p121] imply that the right hand side of the inequality can be sufficiently small if $R_A\to\infty$. In the second inequality, the same assertion follows from the hypothesis $\delta_R(\mu,\alpha,\theta)\to 0$ $(R\to\infty)$. □

Using the results in proposition 3, we give a necessary and sufficient condition in order to satisfy the (ν,μ)-Carleson inequality of (q,p) when $q<p$. This condition is a generalization of Luecking's result [8]. The proof of theorem 4 is similar to that in [8]. But, in the proof of main theorem in [8], it seems that a result which is concerned with interpolating sequences is important. In weighted Bergman spaces, an interpolating theorem also plays an important role, and proposition 3 enables us to prove theorem 4. Moreover, we will show that the hypotheses $\varepsilon_R(\mu,\alpha)\to 0$ $(R\to\infty)$ and $\delta_R(\mu,\alpha,\theta)\to 0$ $(R\to\infty)$ in theorem 4 are valid for many functions which are modulus of polynomials.

Theorem 4 *Suppose that* $0<q<p<\infty$ *and* $d\mu=wdm_\alpha$.

(1) *If* $0<p\leqq 1$, w *satisfies the* (A_s)*-condition for some* $1<s<\infty$ *and* $\varepsilon_R(\mu,\alpha)\to 0$ $(R\to\infty)$, *then* ν *and* μ *satisfy the* (ν,μ)*-Carleson inequality of* (q,p) *if and only if there exists* $0<r<\infty$ *such that* $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ *is in* $L^t(\mu)$, *here* $1/t+1/(p/q)=1$.

(2) *If* $1<p<\infty$, w *satisfies the* $(A_p(\alpha))_\partial$*-condition and there exists* θ *such that* $(1-p)(1+\alpha)<\theta<0$, $\delta_R(\mu,\alpha,\theta)\to 0$ $(R\to\infty)$, *then the same equivalence in* (1) *of theorem 4 is valid.*

Proof. (1) The "if" part is a consequence of (2) of proposition 1 and hence we will show the "only if" part. For any $0 < r < \infty$, by lemma 4.3.6 in [14; p62], there exists $\{\lambda_n\} \subset D$ satisfying that $D = \cup D_{r/2}(\lambda_n)$ and $D_{r/8}(\lambda_n) \cap D_{r/8}(\lambda_m) = \phi$ $(n \neq m)$. Since $\{\lambda_n\}$ is separated, Amar's theorem (See [1].) implies that $\{\lambda_n\}$ is a finite union of $A(l)$ $(1 \leqq l \leqq N)$, where $A(l)$ is a separated sequence such that $R_{A(l)}$ is sufficiently large for $1 \leqq l \leqq N$. Hence, by the hypothesis and (1) of proposition 3, we can assume that $R_{A(l)} \geqq R_0 \geqq 4r$ and $\sup\{\int |S_{A(l)}\{c_j\}|^p d\mu; \Sigma|c_j|^p \leqq 1, \{c_j\} \in l^p\} \leqq \gamma$ for all $1 \leqq l \leqq N$. Put $R = R_0/2$. Here, it is enough to prove that $\Sigma(\widehat{\nu}_r(a_j)/\widehat{\mu}_R(a_j))^t \mu(D_R(a_j)) < \infty$, where $\{a_j\}$ is a one of the separated sequences $A(l)$ $(1 \leqq l \leqq N)$. In fact, since w satisfies the (A_s)-condition, lemma 1 implies that

$$\begin{aligned}\int (\widehat{\nu}_{r/2}(z)/\widehat{\mu}_{r/2}(z))^t \mu &\\ &\leqq C\Sigma(\widehat{\nu}_r(\lambda_n)/\widehat{\mu}_r/_4(\lambda_n))^t \int_{D_r/2(\lambda_n)} d\mu \\ &\leqq C\Sigma(\widehat{\nu}_r(\lambda_n)/\widehat{\mu}_R(\lambda_n))^t \mu(D_R(\lambda_n)).\end{aligned}$$

Therefore, we will prove it. We replace ν with $\chi_{D_K(0)}\nu$ and put $y_j = \{m(D_R(a_j))^{-1}\nu(D_r(a_j))/\widehat{\mu}_R(a_j)\}\mu(D_R(a_j))^{1/t}$, where $\chi_{D_K(0)}$ is a characteristic function of $D_K(0)$. By the corollary of lemma 4.3.3 in [14; p60], it is enough to prove that $\Sigma|y_j|^t \leqq C$, where C is independent of $0 < K < \infty$. The (ν, μ)-Carleson inequality of (q, p) implies that

$$\begin{aligned}\left\{C_{\nu,\mu}\left(\int_D |f|^p d\mu\right)^{1/p}\right\}^q &\\ &\geqq \Sigma \int_{D_r(a_j)} |f|^q d\nu \\ &\geqq \Sigma|f(a_j)|^q \nu(D_r(a_j)) - \Sigma \int_{D_r(a_j)} |f(a_j) - f|^q d\nu\end{aligned}$$

for all f in H. Here, normal families arguments, Hölder's inequality, and the (A_s)-condition imply that

$$|f(a) - f(z)|^{q/s} \leqq C_R \beta(a, z)\{m(D_R(a))^{-1} \int_{D_R(a)} |f|^q d\mu\}^{1/s} \widehat{\mu}_R(a)^{-1/s}$$

for all f in H and a, $z \in D$ such that $\beta(a, z) < r$ (See [8].). Hence, two

Hölder's inequalities imply that

$$\begin{aligned}
&\Sigma\int_{D_r(a_j)}|f(a_j)-f|^q d\nu \\
&\leqq C_R r^s \Sigma\left(\int_{D_R(a_j)}|f|^p d\mu\right)^{q/p}\mu(D_R(a_j))^{1/t}\nu(D_r(a_j)) \\
&\quad\times m(D_R(a_j))^{-1}/\widehat{\mu}_R(a_j) \\
&\leqq C_R r^s\left(\Sigma\int_{D_R(a_j)}|f|^p d\mu\right)^{q/p}\{\Sigma\mu(D_R(a_j))\nu(D_r(a_j))^t \\
&\quad\times m(D_R(a_j))^{-t}/\widehat{\mu}_R(a_j)^t\}^{1/t} \\
&\leqq C_R r^s\left(\int_D|f|^p d\mu\right)^{q/p}(\Sigma|y_j|^t)^{1/t}
\end{aligned}$$

for all $f\in H$. Moreover, Hahn-Banach's theorem shows that there exists a sequence $\{d_j\}$ such that $\Sigma y_j d_j=(\Sigma|y_j|^t)^{1/t}$ and $(\Sigma|d_j|^{p/q})^{q/p}=1$. Therefore, put $|c_j|^q=|d_j|s(a_j)^{q/p}\mu(D_R(a_j))^{1/t}m(D_R(a_j))^{-1}\widehat{\mu}_R(a_j)^{-1}$, then we have that $\Sigma|c_j|^p\leqq C_{R,\mu}$. Hence, by (1) of proposition 3, there exist $0<\gamma_R<\infty$ and f in $L^p{}_a(\mu)$ such that $f(a_j)=c_j$ and $\int|f|^p d\mu\leqq\gamma_R$, and they imply that $\Sigma|f(a_j)|^q\nu(D_r(a_j))\geqq(\Sigma|y_j|^t)^{1/t}$ and $\int|f|^p d\mu\leqq\gamma_R$ (γ_R may not be different from above γ.). Therefore, above inequalities and the choice of f imply that $C_{\nu,\mu}{}^q\gamma_R{}^{q/p}\geqq(1-C_R r^s\gamma_R{}^{q/p})(\Sigma|y_j|^t)^{1/t}$, hence let r be sufficiently small, the desired result follows.

(2) The proof is same to (1). □

We give some examples. Some results in example 1 and example 2 are more general than (1) of proposition 5, (5) of proposition 9 and (2) of proposition 5, (6) of proposition 9 respectively.

Example 1. Let $w(z)=(1-|z|^2)^l$ such that l is real and $d\mu=wdm_\alpha$. Then, clearly w satisfies the (A_p)-condition for all $1<p<\infty$. Moreover, by making a change of variable, lemma 4.2.2 in [14; p53] implies that $\widetilde{w}_\alpha(a)\leqq C(1-|a|^2)^l$, if $\alpha+l>-1$ and $l-\alpha-2<0$. Analogously, we have that $(w^{-1/(p-1)})\widetilde{\ }_\alpha(a)^{p-1}\leqq C(1-|a|^2)^{-l}$, if $-l/(p-1)+\alpha>-1$ and $-l/(p-1)-\alpha-2<0$. Hence, if $l\geqq0$, $|l-1/2|<\alpha+3/2$, and $1+l/(1+\alpha)<p$, then w satisfies the $(A_p(\alpha))_\partial$-condition. If $l<0$, $|l-1/2|<\alpha+3/2$, and $1-l/(2+\alpha)<p$, then w satisfies the $(A_p(\alpha))_\partial$-condition. Similar calculations, lemma 4.2.2 in [14; p53], and corollary 1.2 in [3; p121] show that $\varepsilon_R(\mu,\alpha)\to0$ $(R\to\infty)$ if $|l-1/2|<\alpha+3/2$, because for any fixed

$0 < r < \infty$ we have that $\widetilde{\mu}_\alpha(a)^{-1} \leqq C_r \widehat{w}_r(a)^{-1} \leqq C(1-|a|^2)^{-l}$ for all a in D. Similarly, we obtain that $\delta_R(\mu, \alpha, \theta) \to 0$ $(R \to \infty)$ if $\theta > -1-\alpha-l$ and $\theta < -l$. Since $0 > -1-\alpha$, for any $1 < p < \infty$ there is θ such that $(1-p)(1+\alpha) < \theta < 0$ and $\delta_R(\mu, \alpha, \theta) \to 0$ $(R \to \infty)$, if $-(1+\alpha) < l < (p-1)(1+\alpha)$.

Example 2. Let $\{b_j\}$ be a finite sequence of complex numbers with $b_i \neq b_j$ $(i \neq j)$ and $\{l(j)\}$ be a finite sequence of non-negative real numbers. Put $w(z) = \prod_j |z-b_j|^{l(j)}$ and $d\mu = w\,dm_\alpha$. Set $\Lambda = \{j; b_j \in \partial D\}$ and $\Gamma = \{j; b_j \in D\}$. For any $0 < r < \infty$, lemma 2 in [9] asserts that $C_r{}^{-1}\widehat{w}_r(a) \leqq \prod_{j\in\Lambda} |a-b_j|^{l(j)} \leqq C_r \widehat{w}_r(a)$ for all a in D when $l(j) > -2$ $(j \in \Gamma)$ even if $\{l(j)\}$ is not non-negative. Therefore, we have that $\widehat{w}_r(a) \leqq C_r \prod_{j\in\Lambda} |a-b_j|^{l(j)}$ and $(w^{-1/(p-1)})\widehat{\ }_r(a)^{p-1} \leqq C_r \prod_{j\in\Lambda} |a-b_j|^{-l(j)}$, if $-l(j)/(p-1) > -2$ $(j \in \Gamma)$. Hence, we obtain that w satisfies the (A_p)-condition for some $1 < p < \infty$. We claim that w satisfies the $(A_p(\alpha))_\partial$-condition if $l(j) < \alpha+2$ $(j \in \Lambda)$ and $l(j)\max\{1, 1/(1+\alpha)\}/2 + 1 < p$ $(j \in \Gamma \cup \Lambda)$. In fact, let $w(z) = |z-b_1|^{l(1)}|z-b_2|^{l(2)}$ such that b_1 is in D, b_2 is in ∂D, and $l(1)$, $l(2)$ are non-negative, it is enough to prove that the assertion is true for such a w. Since $|z-b_1|^{l(1)}$ is a bounded function, making a change of variable, and lemma 4.2.2 in [14; p53] imply that $\widetilde{w}_\alpha(a) \leqq 2^{l(1)}C|1-\bar{a}b_2|^{l(2)}\|\phi_a(b_2) - z\|_\infty{}^{l(2)} \leqq C|1-\bar{a}b_2|^{l(2)}$ if $l(2) < \alpha+2$. Moreover, let $U(1)$ and $U(2)$ be neighborhoods of b_1 and b_2 in $D \cup \partial D$ such that $U(1) \cap U(2) = \phi$. Then, we have that

$$
\begin{aligned}
&(w^{-1/(p-1)})\widetilde{\ }_\alpha(a)^{p-1} \\
&\quad = \left(\int_{U(1)} + \int_{U(2)} + \int_{(U(1)\cup U(2))^c}\right)^{p-1} \\
&\quad \leqq 2^{p-1}C(|z-b_1|^{-l(1)/(p-1)})\widetilde{\ }_\alpha(a)^{p-1} \\
&\qquad + 2^{2(p-1)}C(|z-b_2|^{-l(2)/(p-1)})\widetilde{\ }_\alpha(a)^{p-1} + 2^{2(p-1)}C^2,
\end{aligned}
$$

where C is a constant such that $|z-b_2|^{-l(2)} \leqq C$ on $U(1)$ and $|z-b_1|^{-l(1)} \leqq C$ on $U(2)$. Here, for b in $D \cup \partial D$ and $l \geqq 0$ the similar calculation above for $\widetilde{w}_\alpha$ shows that $(|z-b|^{-l/(p-1)})\widetilde{\ }_\alpha(a)^{p-1} \leqq |1-\bar{a}b|^{-l}\|1-\bar{a}z\|_\infty{}^l(\int |\phi_a(b) - z|^{-l/(p-1)}dm_\alpha)^{p-1}$. Hence, if $\alpha > 0$ and $l/(p-1) < 2$, then $\int |\phi_a(b) - z|^{-l/(p-1)}dm_\alpha \leqq \int_{2D} |z|^{-l/(p-1)}dm < \infty$, where $2D = \{2z; z \in D\}$. Moreover, if $0 > \alpha > -1$ and $l/\{(1+\alpha)(p-1)\} < 2$, then there exists β such that $1 < 1/(1+\alpha) < \beta < 2(p-1)/l$. Therefore, Hölder's inequality implies that

$$\int |\phi_a(b) - z|^{-l/(p-1)} dm_\alpha$$
$$\leqq \left(\int |\phi_a(b) - z|^{-\beta l/(p-1)} dm \right)^{1/\beta}$$
$$\times \left(\int (1 - |z|^2)^{\alpha\beta/(\beta-1)} dm \right)^{(\beta-1)/\beta} < \infty,$$

because $\beta l/(p-1) < 2$ and $\alpha\beta/(\beta-1) > -1$. Hence, the desired result follows for $b = b_j$ and $l = l(j)$ $(j = 1, 2)$. Next, we assert that $\varepsilon_R(\mu, \alpha) \to 0$ $(R \to \infty)$ if $l(j) < \alpha + 2$ $(j \in \Lambda)$, and for any $1 < p < \infty$ there is θ such that $(1-p)(1+\alpha) < \theta < 0$ and $\delta_R(\mu, \alpha, \theta) \to 0$ $(R \to \infty)$ if $l(j) < (1+\alpha)\min\{1, p-1\}$ $(j \in \Lambda)$. Let $w(z) = |z - b_1|^{l(1)}|z - b_2|^{l(2)}$ such that b_1 is in D and b_2 is in ∂D. It is enough to prove that the assertions are true for such a w. For any $1 < s < \infty$, we have that $\varepsilon_R(\mu, \alpha) \leqq \sup(\int_{D_R(a)^c} |k_a|^{2+\alpha} dm_\alpha)(w^{-1/(s-1)})\tilde{}_\alpha(a)^{s-1}$. Hence, if s is sufficiently large and $l(2) < \alpha + 2$, then the above arguments for $(A_p(\alpha))_\partial$-condition and corollary 1.2 in [3; p121] imply that $\varepsilon_R(\mu, \alpha) \to 0$ $(R \to \infty)$. Similarly, we have that

$$\delta_R(\mu, \alpha, \theta)$$
$$\leqq 2^{l(1)} \sup\Big\{ (1 - |a|^2)^\theta |1 - \bar{a}b_2|^{l(2)} \|\phi_a(b_2) - z\|_\infty{}^{l(2)}$$
$$\times \left(\int_{D_R(0)^c} (1 - |z|^2)^{\theta + l(2)} |1 - \bar{a}z|^{-\{2 + (\theta+\alpha) + (\theta + l(2))\}} dm \right) \Big\}$$
$$\times \{(1 - |a|^2)^{-\theta} (w^{-1/(s-1)})\tilde{}_\alpha(a)^{s-1}\}.$$

Hence, lemma 4.2.2 in [14; p53] and corollary 1.2 in [3; p121] imply that $\delta_R(\mu, \alpha, \theta) \to 0$ $(R \to \infty)$ if $\theta + \alpha > -1$ and $\theta + l(2) < 0$. Since $0 > -1 - \alpha$ and $l(2) > 0$, for any $1 < p < \infty$, there is θ such that $(1-p)(1+\alpha) < \theta < 0$ and $\delta_R(\mu, \alpha, \theta) \to 0$ $(R \to \infty)$ if $l(2) < (1+\alpha)\min\{1, p-1\}$.

Example 3. We will observe that the hypotheses of ε_R and δ_R in theorem 2 and theorem 4 are not sharp. We show that there are measures ν and μ such that they satisfy the (ν, μ)-Carleson inequality of (q, q), (ν, μ)-Carleson inequality of (q, p), w satisfies the (A_s)-condition for some $1 < s < \infty$, and w satisfies the $(A_p(0))_\partial$-condition, but $\varepsilon_R(\mu)$ and $\delta_R(\mu)$ do not converges to 0, where $d\mu = w dm$. Let $w(z) = |1 - z|^l$, $d\mu = w dm$ and $d\nu = |1 - z|^k dm$

such that $k > l > 0$. For $0 < q < p < \infty$, we have that $(\int |f|^q \nu)^{1/q} \leqq 2^{(k-l)/q}(\int |f|^q d\mu)^{1/q} \leqq 2^{(k-l)/q}\mu(D)^{1/q-1/p}(\int |f|^p d\mu)^{1/p}$ for all f in H. Here, the last inequality follows from Hölder's inequality. Therefore, ν and μ satisfy the (ν,μ)-Carleson inequality of (q,q) and (ν,μ)-Carleson inequality of (q,p). Moreover, since $\widehat{\nu}_r(a)$ and $\widehat{\mu}_r(a)$ are equivalent to $|1-a|^k$ and $|1-a|^l$ respectively, $\widehat{\nu}_r(a)/\widehat{\mu}_r(a)$ is bounded and $\widehat{\nu}_r/\widehat{\mu}_r$ is in $L^t(\mu)$, where $1/t + 1/(p/q) = 1$. By example 2, w satisfies the (A_s)-condition for some $1 < s < \infty$, but we can prove that $\varepsilon_R(\mu, 0) = 1$ for all $0 < R < \infty$ if $l \geqq 2$. Suppose that there exists $0 < R < \infty$ such that $\varepsilon_R(\mu,0) < 1$. Then, we have that $\widetilde{\mu}_0(a) \leqq C_R\widehat{\mu}_R(a) \leqq C_R|1-a|^l$ for all a in D. Hence, we obtain that $\infty > C_R \geqq \widetilde{\mu}_0(a)|1-a|^{-l} = \int |1+z|^l|1-az|^{-l}dm$ for all $0 < a < 1$. Let $D_+ = \{z \in D; \operatorname{Re} z \geqq 0\}$, then $C_R \geqq \int_{D+} |1+z|^l|1-az|^{-l}dm \geqq \int_{D+}|1-az|^{-l}dm$. And, hence we have that $\int_D |1-az|^{-l}dm \leqq C_R + \int_{D+^c}|1-az|^{-l}dm \leqq C$ for all $0 < a < 1$. This contradicts lemma 4.2.2 in [14; p53]. Furthermore, by example 2, w satisfies the $(A_p(0))_\partial$-condition if $l < 2$ and $l < 2(p-1)$. But, let $p = 2$, then we can prove that there is not θ such that $-1 = (1-p)(1+0) < \theta < 0$ and $\delta_R(\mu,0,\theta) \to 0$ $(R \to \infty)$ if $1 \leqq l < 2$. We suppose that there is θ such that $-1 < \theta < 0$ and $\delta_R(\mu,0,\theta) \to 0$ $(R\to\infty)$. Since $1 \leqq l < 2$, example 2 implies that there exists $0 < r < \infty$ such that $\varepsilon_r(\mu,0) < 1$. Hence, we have that $\widetilde{\mu}_0(a) \leqq C_r\widehat{\mu}_r(a) \leqq C_r|1-a|^l$ for all a in D. Therefore, there exists $0 < R < \infty$ such that $\infty > \delta_R(\mu,0,\theta) \geqq C_R{}^{-1}\int_{D_R(0)^c}|1+z|^l(1-|z|^2)^\theta|1-az|^{-(2+l+2\theta)}dm$ for all $0 < a < 1$, because $\delta_R(\mu,0,\theta)\to 0$ $(R\to\infty)$. Hence, similar arguments imply that

$$\begin{aligned}
&\int_D (1-|z|^2)^\theta|1-az|^{-(2+l+2\theta)}dm\\
&\leqq C_r\delta_R(\mu,0,\theta) + \int_{D_R(0)}(1-|z|^2)^\theta|1-az|^{-(2+l+2\theta)}dm\\
&\quad + \int_{(D+^c)\setminus D_R(0)}(1-|z|^2)^\theta|1-az|^{-(2+l+2\theta)}dm\\
&\leqq C_r(\mu,0,\theta) + C_R\int_D(1-|z|^2)^\theta dm \leqq C < \infty
\end{aligned}$$

for all $0 < a < 1$. Therefore, this contradicts lemma 4.2.2 in [14; p53], because $\theta + l > 0$.

References

[1] Amar E., *Suites d'interpolation pour les class de Bergman de la boule et du polydisque*

de C^n, Canad. J Math. **30** (1978), 711–737.

[2] Coifman R. and Rochberg R., *Representation theorems for holomorphic and harmonic functions*, Astérisque **77** (1980), 11–65.

[3] Gamelin T.W., *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, New Jersey, 1969.

[4] Hunt R., Muckenhoupt B. and Wheeden R., *Weighted norm inequalities for the conjugate function and Hilbert transform*, Trans. Amer. Math. Soc. **176** (1973), 227–251.

[5] Luecking D., *Inequalities in Bergman spaces*, Ill. J. Math. **25** (1981), 1–11.

[6] Luecking D., *Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives*, Amer. J. Math. **107** (1985), 85–111.

[7] Luecking D,, *Representation and duality in weighted spaces of analytic functions*, Indiana Univ. Math. J. **34** (1985), 319–336.

[8] Luecking D., *Multipliers of Bergman spaces into Lebesgue spaces*, Proc. Edinburgh Math. Soc. **29** (1986), 125–131.

[9] Nakazi T. and Yamada M., (A_2)*-conditions and Carleson inequalities in Bergman spaces*, to appear in Pacific J. Math.

[10] Nakazi T. and Yamada M., *Riesz's functions in weighted Hardy and Bergman spaces*, in preprint.

[11] Oleinik V.L. and Pavlov B.S., *Embedding theorems for weighted class of harmonic and analytic functions*, J. Soviet Math. **2** (1974), 135–142.

[12] Rochberg R., *Interpolation by functions in the Bergman spaces*, Mich. Math. J. **29** (1982), 229–236.

[13] Rudin W., *Functional Analysis*, McGraw-Hill Book Company.

[14] Zhu K., *Operator Theory in Function Spaces*, Marcel Dekker, New York, 1990.

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 739, Japan