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Local solutions of fully nonlinear weakly hyperbolic differential equations in Sobolev spaces

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Abstract. The goal of the present paper is to study fully nonlinear weakly hyperbolic equations of second order with space- and time degeneracy. A local existence result in Sobolev spaces under sharp Levi conditions of C^{∞} type is proved. These Levi conditions and the behaviour of the nonlinearities determine the required smoothness of the data and the loss of Sobolev regularity.

Key words: fully nonlinear weakly hyperbolic equations, local solutions in Sobolev spaces, energy method.

Introduction

In this paper we want to study fully nonlinear weakly hyperbolic differential equations in one space dimension with time- and spatial degeneracy. We will prove a local existence result in Sobolev spaces for the Cauchy problem

$$F(u_{tt}, \sigma(x)\lambda(t)u_{xt}, \sigma(x)^2\lambda(t)^2u_{xx}, \sigma(x)\lambda'(t)u_x, u_t, u, x, t) = 0, \quad (0.1)$$

$$u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x). \quad (0.2)$$

We assume that this Cauchy problem is strictly hyperbolic if $\sigma(x) \equiv 1$ and $\lambda(t) \equiv 1$. The functions $\sigma(x)$ and $\lambda(t)$ describe the degeneration, which occurs for $\sigma(x) = 0$ (spatial degeneracy) and $\lambda(t) = 0$ (time degeneracy). The first question is that for classes of well-posedness with respect to x. If we restrict ourselves to Gevrey classes of order ≤ 2, then we can use ideas of [Kaj83] to prove a local existence result for

$$F(u_{tt}, \sigma(x)\lambda(t)u_{xt}, \sigma(x)^2\lambda(t)^2u_{xx}, u_x, u_t, u, x, t) = 0,$$
$$u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x).$$

To overcome the critical order 2 we need so-called Levi conditions. In [RY96] a local existence result in all Gevrey spaces could be proved under special

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assumptions for

$$u_{tt} - (a(x,t)u_x)_x = f(x,t,u,u_x),$$

where the nonlinear Levi condition of C^∞-type

$$\left|\partial_p^l f(x,t,u,p)\right| \le C_K M_K^{ls} l!^{s'} \sqrt{a(x,t)} \tag{0.3}$$

is satisfied for all $l \ge 1$ and all compact sets $K \subset \mathbb{R}_x \times [0,T] \times \mathbb{R}_u \times \mathbb{R}_p$. There are different results for local existence in C^∞ for special quasilinear weakly hyperbolic model equations under quite different assumptions [D'A93], [DT95], [Man96]. But in all these model equations the nonlinearities depend at most on u and u_t. In the case of spatial degeneracy ($\lambda'(t)$ and $\lambda(t)$ are absent in (0.1)), the C^∞- and Gevrey well posedness is proved in [DR]. What about results for local existence of Sobolev solutions? In [Ner66] one can find one of the first results for the time degeneracy case

$$\begin{aligned}
&u_{tt} - \lambda(t)^2 u_{xx} - a(x,t)u_x - b(x,t)u_t - c(x,t)u = f(x,t),\\
&u(x,0) = \varphi_0(x), \quad u_t(x,0) = \varphi_1(x)
\end{aligned}$$

under the Levi condition

$$\limsup_{t\to+0} \frac{|a(x,t)|}{\lambda'(t)} \le q < \infty.$$

The loss of regularity depends on q and the Levi condition is sharp. Later weakly hyperbolic Cauchy problems of the form

$$\begin{aligned}
u_{tt} &- \sum_{i,j=1}^N (a_{ij}(x,t)u_{x_i})_{x_j} + \sum_{i=1}^N b_i(x,t)u_{x_i} + b_0(x,t)u_t + c(x,t)u\\
&= f(x,t),\\
u(x,0) &= \varphi_0(x), \quad u_t(x,0) = \varphi_1(x)
\end{aligned}$$

were studied in [Ole70] under the Levi condition

$$ct\left(\sum_{i=1}^N b_i(x,t)\xi_i\right)^2 \le A\sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j + \partial_t\left(\sum_{i,j=1}^N a_{ij}(x,t)\xi_i\xi_j\right).$$

In the case of time degeneracy this Levi condition is only sharp if we have a degeneracy of finite order (compare with the Levi condition of Nersesian).

In [Rei96] it was shown for the model problem

$$u_{tt} - \lambda(t)^2 \triangle u = f(x, t, u, u_t, \lambda'(t)\nabla_x u),$$
$$u(x, 0) = \varphi_0(x), \quad u_t(x, 0) = \varphi_1(x)$$

how to prove the local existence of Sobolev solutions. The difficulty is to show how to overcome the quasilinear structure with the loss of Sobolev regularity which appears even in the linear theory. In [KY] a local existence result could be derived for a general quasilinear weakly hyperbolic equation of higher order in the case of time degeneracy. To prove local existence of Sobolev solutions we need C^∞-type Levi conditions. The special structure of the arguments in (0.1) ensures the fulfilment of these conditions. There are other ways to describe Levi conditions. The investigations in this paper serve as a preparation for further studies which will be devoted to the question for global existence. As usual, in those Cauchy problems the function F depends only on the solution and some of its derivatives. In this case it is necessary to use the functions σ and λ to formulate the C^∞-Levi conditions. The study of a model equation

$$u_{tt} - \sigma(x)^2\lambda(t)^2 \triangle u = f(\nabla_x u)$$

under the assumption

$$|\partial_p f(p)| \leq C|\sigma(x)\lambda'(t)|$$

(similar to (0.3)) leads to spaces of solutions with asymptotics, which seem to be extremely difficult to handle.

The aim of this paper is to prove

Theorem 0.1 *We suppose:*

A1 *The function $F(u_{11}, \sigma\lambda u_{12}, \sigma^2\lambda^2 u_{22}, \sigma\lambda' u_2, u_1, u, x, t)$ is defined on the set $M \times P \times I$, where M is an open set in $\mathbb{R}^6$, $P \subset \mathbb{R}$ is a compact interval and $I = [0, T]$.*

A2 *The functions F, σ, φ_0, φ_1 are P-periodic in x.*

A3 *The function $\lambda \in C^3([0, T])$ fulfils the conditions*

$$\lambda(0) = \lambda'(0) = 0, \quad \lambda(t) > 0, \quad \lambda'(t) > 0 \quad (t > 0). \tag{0.4}$$

A4 *It is assumed that $\sigma \in H^{N+2}_{\mathrm{per}}(P)$, $\varphi_0 \in H^{N+1}_{\mathrm{per}}(P)$, $\varphi_1 \in H^{N}_{\mathrm{per}}(P)$, where $N \geq 5$ and $H^{N}_{\mathrm{per}}(P)$ denotes the functions from $H^{N}_{\mathrm{loc}}(\mathbb{R})$ which are P-periodic.*

A5 *The derivatives* $F_{u_{11}}, \dots, F_t$ *belong to* $C^1([0,T], C^\infty(\mathbb{R}^6) \times H^N_{\text{per}}(P))$.
A6 *With a suitable constant* α *it holds* $|F_{u_{11}}| \geq \alpha > 0$ *on* $M \times P \times I$.
A7 *Let on* $M \times P \times I$ *be*

$$\left(\frac{F_{\sigma\lambda u_{12}}}{F_{u_{11}}}\right)^2 - 4\frac{F_{\sigma^2\lambda^2 u_{22}}}{F_{u_{11}}} \geq \gamma > 0.$$

From condition A6 *we conclude that the set in* $M \times P \times I$ *which is given by* $F = 0$ *can be represented in the form*

$$u_{11} = G(\sigma\lambda u_{12}, \sigma^2\lambda^2 u_{22}, \sigma\lambda' u_2, u_1, u, x, t).$$

A8 *Let* $\varphi_2(x)$ *be the function which is defined by*

$$\varphi_2(x) = G(0, 0, 0, \varphi_1(x), \varphi_0(x), x, 0).$$

We assume that the set

$$K := \{(\varphi_2(x), 0, 0, 0, \varphi_1(x), \varphi_0(x), x, 0) : x \in P\}$$

is contained in $M \times P \times \{0\}$.
Then there exist constants $r \in \mathbb{N}$ *and* $T^* \in (0, T]$ *such that the Cauchy problem* (0.1), (0.2) *has a solution* u,

$$u, u_t, \sigma\lambda u_x \in C^2([0,T], H^{N-r}_{\text{per}}(P))$$

if $N - r \geq 5$. *The constant* r *describes the loss of Sobolev regularity and may depend on* N. *One can show that* $N - r \geq 5$ *for sufficiently large* N. *This guarantees the existence of a solution for large* N.

For the convenience of the reader we give an overview of this paper: In the first section we provide some tools which will be used in later sections. The second and third section deal with problems which have no time degeneracy, whereas the fourth and fifth section include this degeneration.

We study *linear* equations in Section 2. At first we derive energy estimates. The existence of the solution is proved by applying a smoothing technique and the abstract Theorem of Cauchy–Kowalewskaja. We can prove convergence of a suitable sequence of solutions by our energy estimates.

Quasilinear equations are studied in Section 3. We prove the existence of a solution by linearization and standard iteration. The convergence of this sequence is again shown by energy estimates. Although we consider only scalar equations it is obvious that similar results can be proved for quasilinear systems with diagonal principal part.

In Section 4 we consider *quasilinear* equations with *both degeneracies.* We approximate the problem by problems without time degeneracy and apply the results of Section 3.

We show how one can reduce fully nonlinear equations to a quasilinear system with diagonal principal part in Section 5.

In Section 6 we study some examples. We can show that the loss of Sobolev regularity predicted in Section 4 occurs, indeed. Additionally we give some remarks which are of independent interest.

1. Evolution operators and energy estimates

In this section we assemble some tools which we need in later proofs.

We define the partial energies

$$e_j(u)(t) := \left(\int_P (\partial_x^j u(x,t))^2 \, dx \right)^{\frac{1}{2}}$$

and the energies of finite order

$$E_N(u)(t) := \sum_{j=0}^{N} e_j(u)(t).$$

It is worth to remark that $E_N(u)(t) = \|u(t)\|_{H^N(P)}$. These energies have the properties

$$\begin{aligned}
&E_N(uv)(t) \le C_{\mathrm{prod},N} E_N(u)(t) E_N(v)(t), \quad N \ge 1, \\
&E_0(uv)(t) \le C_{\mathrm{prod},0} E_1(u)(t) E_0(v)(t), \\
&\|\partial_x u(x,t)\|_\infty \le C_P e_2(u)(t), \quad u \in H^2_{\mathrm{per}}(P).
\end{aligned}$$

Proposition 1.1 *Let $N \ge 1$, $N' = \max(2, N)$ and*

$$\begin{aligned}
&\lambda \in C([0,T], H^{N'}_{\mathrm{per}}(P)), \\
&f \in C([0,T], H^{N-1}_{\mathrm{per}}(P)) \cap L^1([0,T], H^N_{\mathrm{per}}(P)).
\end{aligned}$$

Let $u(x,t)$ be a solution of

$$u_t(x,t) - \lambda(x,t) u_x(x,t) = f(x,t), \quad (x,t) \in P \times [0,T].$$

(a) *If* $u \in L^\infty([0,T], H^{N+1}_{\text{per}}(P))$ *and* $u_t \in L^\infty([0,T], H^N_{\text{per}}(P))$, *then*

$$E_N(u)'(t) \le C_N E_N(u)(t) + E_N(f)(t). \tag{1.1}$$

(b) *If* $u \in L^\infty([0,T], H^N_{\text{per}}(P))$ *and* $u_t \in L^\infty([0,T], H^{N-1}_{\text{per}}(P))$, *then*

$$E_{N-1}(u)'(t) \le C_{N-1} E_{N-1}(u)(t) + E_{N-1}(f)(t) \tag{1.2}$$

and if $0 \le t_0 \le t \le T$, *then it holds*

$$E_N(u)(t) \le E_N(u)(t_0) e^{C_N(t-t_0)} + \int_{t_0}^t e^{C_N(t-\tau)} E_N(f)(\tau)\, d\tau. \tag{1.3}$$

The constants C_n *depend on* $\|\lambda\|_{C\left([0,T], H^{\max(2,n)}_{\text{per}}(P)\right)}$.

Proof of (a) Let $0 \le j \le N$. Then it holds

$$\begin{aligned}
e_j(u)(t) e_j(u)'(t) &= \int_P (\partial_x^j u)(\partial_x^j u_t)\, dx \\
&= \int_P (\partial_x^j u) \partial_x^j (f + \lambda u_x)\, dx \\
&\le e_j(u)(t) e_j(f)(t) + \int_P (\partial_x^j u) \lambda (\partial_x^{j+1} u)\, dx \\
&\quad + \sum_{n=1}^{j} \binom{j}{n} \int_P (\partial_x^j u)(\partial_x^n \lambda)(\partial_x^{j-n+1} u)\, dx.
\end{aligned}$$

The second integral of the right-hand side can be estimated after partial integration by $\|\lambda_x\|_\infty e_j(u)^2 \le C \|\lambda\|_{H^{N'}(P)} e_j(u)^2$. We get for $n = j$ in the last integral the inequality

$$\begin{aligned}
\int_P (\partial_x^j u)(\partial_x^j \lambda)(\partial_x u)\, dx &\le e_j(u) \|\lambda\|_{H^j(P)} \|u_x\|_\infty \\
&\le C \|\lambda\|_{H^j(P)} e_j(u) e_2(u),
\end{aligned}$$

and for $n \le j - 1$

$$\begin{aligned}
\int_P (\partial_x^j u)(\partial_x^n \lambda)(\partial_x^{j-n+1} u)\, dx &\le e_j(u) \|\lambda\|_{C^n} e_{j-n+1}(u) \\
&\le C \|\lambda\|_{H^{N'}(P)} e_j(u) e_{j-n+1}(u).
\end{aligned}$$

Summation ($j = 0, \ldots, N$) yields the assertion. □

Proof of (b) It remains to prove the estimate (1.3). We approximate

$$\begin{aligned}
\mathfrak{A}_{\mathrm{per}}(P) \ni u^n(t_0) &\xrightarrow{H^N_{\mathrm{per}}(P)} u(t_0),\\
C([0,T],\mathfrak{A}_{\mathrm{per}}(P)) \ni f^n &\xrightarrow{L^1([0,T],H^N_{\mathrm{per}}(P))} f,\\
C([0,T],\mathfrak{A}_{\mathrm{per}}(P)) \ni \lambda^n &\xrightarrow{C([0,T],H^{N'}_{\mathrm{per}}(P))} \lambda.
\end{aligned}$$

Here we denote by $\mathfrak{A}_{\mathrm{per}}(P)$ the space of functions which are analytic and P-periodic.

Lemma 4.1 from [DR] shows that there exists a solution $u^n \in C^1([t_0,T],\mathfrak{A}_{\mathrm{per}}(P))$ of

$$u^n_t - \lambda^n u^n_x = f^n.$$

By Gronwall's Lemma and (a) we obtain for every n

$$E_N(u^n)(t) \le E_N(u^n)(t_0)e^{C_N(t-t_0)} + \int_{t_0}^t e^{C_N(t-\tau)} E_N(f^n)(\tau)\,d\tau.$$

From (a) and

$$(u-u^n)_t - \lambda(u-u^n)_x = f - f^n - (\lambda^n - \lambda)u^n_x$$

we conclude that $\|u^n - u\|_{H^{N-1}(P)} \to 0$.

Furthermore, there exists a constant C with $\|u^n(t)\|_{H^N(P)} \le C$ for all n, t. We fix $t \ge t_0$. There exists a subsequence $u^n_*(t)$ weakly converging in $H^N_{\mathrm{per}}(P)$ to some function $w(t)$. The uniqueness of the limit yields $u(t) = w(t)$. Hence

$$\begin{aligned}
&E_N(u)(t)\\
&\quad\le \liminf_{n\to\infty} E_N(u^n)(t)\\
&\quad\le \liminf_{n\to\infty} \left(E_N(u^n)(t_0)e^{C_N(t-t_0)} + \int_{t_0}^t e^{C_N(t-\tau)} E_N(f^n)(\tau)\,d\tau \right).
\end{aligned}$$

From this and the approximations we have (1.3). □

We will use these results to estimate the solutions of the weakly hyperbolic Cauchy problem

$$u_{tt}(x,t) + \sigma(x)b(x,t)u_{xt}(x,t) - \sigma^2(x)a(x,t)u_{xx}(x,t) = f(x,t), \quad (1.4)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.5}$$

$$b^2(x,t) + 4a(x,t) \geq \gamma > 0, \quad (x,t) \in P \times [0,T]. \tag{1.6}$$

We factorize the differential operator:

$$\begin{aligned} l_{1,2}(x,t) &:= \sigma(x)\beta_{1,2}(x,t) \\ &:= \frac{1}{2}\sigma(x)\Big(-b(x,t) \pm \sqrt{b^2(x,t)+4a(x,t)}\Big), \\ \partial_{1,2} &:= \partial_t - l_{1,2}(x,t)\partial_x, \\ \partial_1\partial_2 u &= u_{tt} + \sigma b u_{xt} - \sigma^2 a u_{xx} + (l_1 l_{2,x} - l_{2,t})u_x \end{aligned}$$

and define the energy

$$\mathcal{E}_N(u)(t) := E_N(u)(t) + E_N(\partial_1 u)(t) + E_N(\partial_2 u)(t).$$

Hence we obtain

$$(\partial_1\partial_2 + A_1(\partial_1 - \partial_2))u = f, \tag{1.7}$$

$$(\partial_2\partial_1 + A_2(\partial_2 - \partial_1))u = f, \tag{1.8}$$

$$A_1(x,t) = \frac{l_1 l_{2,x} - l_{2,t}}{l_1 - l_2}, \quad A_2(x,t) = \frac{l_2 l_{1,x} - l_{1,t}}{l_1 - l_2}.$$

This factorization is the essential idea of the proof of the following

Proposition 1.2 *We assume with* $N \geq 2$

$$\sigma \in H^N_{\mathrm{per}}(P), \tag{1.9}$$

$$a, b \in C^1([0,T], H^N_{\mathrm{per}}(P)), \tag{1.10}$$

$$f \in C([0,T], H^N_{\mathrm{per}}(P)). \tag{1.11}$$

Let u *be a solution of* (1.4) *with* $u, \partial_1 u, \partial_2 u \in C([0,T], H^M_{\mathrm{per}}(P))$.
(a) *If* $0 \leq M \leq N-1$, *then*

$$\begin{aligned} \mathcal{E}_M(u)(t) \leq\ & e^{D_M(t-t_0)}\Big(\mathcal{E}_M(u)(t_0)e^{C_M(t-t_0)} \\ & + 2\int_{t_0}^t e^{C_M(t-\tau)} E_M(f)(\tau)\, d\tau\Big). \end{aligned} \tag{1.12}$$

(b) *If we assume additionally* $\sigma a, \sigma b \in C([0,T], H^{N+1}_{\mathrm{per}}(P))$ *and* $\sigma \in H^{N+1}_{\mathrm{per}}(P)$, *then the estimate* (1.12) *holds for* $M = N$, *too.*

The constants D_M, C_M depend on

$$E_{M'}(A_j), \quad \|l_j\|_{C([0,T],H^{M'}_{\mathrm{per}}(P))}, \quad j=1,2, \quad M'=\max(M,2).$$

Proof. The assumptions guarantee $A_j \in C([0,T], H^{N-1}_{\mathrm{per}}(P))$. The application of Proposition 1.1 to

$$\partial_1\partial_2 u = f - A_1(\partial_1 - \partial_2)u$$

shows that

$$\begin{aligned} E_M(\partial_2 u)(t) \;\le\; & E_M(\partial_2 u)(t_0)e^{C_M(t-t_0)} \\ & + \int_{t_0}^{t} e^{C_M(t-\tau)} E_M(f - A_1(\partial_1-\partial_2)u)(\tau)\,d\tau. \end{aligned}$$

We can derive a similar estimate for $E_M(\partial_1 u)(t)$. Furthermore,

$$E_M(u)(t) \le E_M(u)(t_0)e^{C_M(t-t_0)} + \int_{t_0}^{t} e^{C_M(t-\tau)} E_M(\partial_1 u)(\tau)\,d\tau.$$

We have

$$E_M(A_i\partial_j u) \le C'_M E_{M'}(A_i) E_M(\partial_j u),$$

hence

$$\begin{aligned} \mathcal{E}_M(u)(t) \;\le\; & \mathcal{E}_M(u)(t_0)e^{C_M(t-t_0)} + 2\int_{t_0}^{t} e^{C_M(t-\tau)} E_M(f)(\tau)\,d\tau \\ & + C''_M \int_{t_0}^{t} e^{C_M(t-\tau)} \mathcal{E}_M(u)(\tau)\,d\tau. \end{aligned}$$

By Gronwall's Lemma, it follows (1.12).

To prove (b), we only note that $A_j \in C([0,T], H^N_{\mathrm{per}}(P))$. □

Remark 1.1. We can prove a similar result for systems

$$u_{i,tt}(x,t) + \sigma b u_{i,xt}(x,t) - \sigma^2 a u_{i,xx}(x,t) = f_i(x,t), \quad i=1,\dots,n$$

and the energy $\mathcal{E}_M(\vec{u})(t) = \sum_{i=1}^n \mathcal{E}_M(u_i)(t)$. This is possible since the system has diagonal structure.

The next lemma will be needed in Section 4. We define the energy

$$F_N(u)(t) := \sum_{j=0}^{N} \left(\int_P \left|\partial_x^j u(x,t)\right|^2 + \left|\partial_x^j u_t(x,t)\right|^2 dx \right)^{\frac{1}{2}} = \sum_{j=0}^{N} f_j(t).$$

Lemma 1.1 *Let $u \in C^2([0,T], H^N_{\text{per}}(P))$ be a solution of the following ordinary differential equation with parameter*

$$u_{tt} + h_1(x,t)u_t + h_2(x,t)u = g(x,t),$$

where $h_1, h_2, g \in L^\infty([0,T], H^N_{\text{per}}(P))$.

Then it holds

$$F_N(u)(t) \le F_N(u)(0)e^{Ct} + \int_0^t e^{C(t-\tau)} E_N(g)(\tau)\, d\tau.$$

Proof. From the inequalities

$$\begin{aligned} f_j(u)'(t) f_j(u)(t) &= \int_P ((\partial_x^j u)(\partial_x^j u_t) + (\partial_x^j u_t)(\partial_x^j u_{tt}))dx \\ &\le f_j(u)(t)^2 + \int_P (\partial_x^j u_t)(\partial_x^j (g - h_1 u_t - h_2 u))dx \\ &\le f_j(u)(t)^2 + f_j(u)(t)(f_j(g)(t) \\ &\qquad + CF_N(u)(t)(F_N(h_1)(t) + F_N(h_2)(t))) \end{aligned}$$

we deduce that

$$F_N(u)'(t) \le E_N(g)(t) + (1 + C(E_N(h_1)(t) + E_N(h_2)(t)))F_N(u)(t).$$

Gronwall's Inequality implies the assertion. □

We will need the following generalization of the well-known Gronwall's Lemma.

Lemma 1.2 (Nersesjan) *Let $y(t) \in C([0,T]) \cap C^1(0,T)$ be a solution of the differential inequality*

$$y'(t) \le K(t)y(t) + f(t), \quad 0 < t < T,$$

where the functions $K(t)$ and $f(t)$ belong to $C(0,T)$. We assume for every $t \in (0,T)$ and every $\delta \in (0,t)$

$$\int_0^\delta K(\tau)\, d\tau = \infty, \quad \int_\delta^T K(\tau)\, d\tau < \infty,$$

$$\lim_{\delta \to +0} \int_\delta^t \exp\left(\int_s^t K(\tau)\, d\tau\right) f(s)\, ds \quad \textit{exists},$$

$$\lim_{\delta \to +0} y(\delta) \exp\left(\int_\delta^t K(\tau)\, d\tau\right) = 0.$$

Then it holds

$$y(t) \leq \int_0^t \exp\left(\int_s^t K(\tau)\,d\tau\right) f(s)\,ds.$$

2. Existence results for linear equations with spatial degeneracy

Lemma 2.1 *We assume* $N \geq 3$, (1.6), (1.9), (1.10), (1.11) *and*

$$u_0 \in H^{N+1}_{\text{per}}(P), \quad u_1 \in H^N_{\text{per}}(P). \tag{2.1}$$

Then the weakly hyperbolic Cauchy problem (1.4), (1.5) *has a uniquely determined solution* $u \in C^1([0,T], H^{N-1}_{\text{per}}(P))$ *with* $\partial_j u \in C([0,T], H^{N-1}_{\text{per}}(P))$ *and* $u_{tt} \in C([0,T], H^{N-2}_{\text{per}}(P))$.

Proof. We approximate

$$\begin{aligned}
C^1([0,T], \mathfrak{A}_{\text{per}}(P)) \ni a^n, b^n &\xrightarrow{C^1\left([0,T], H^N_{\text{per}}(P)\right)} a, b,\\
\mathfrak{A}_{\text{per}}(P) \ni \sigma^n &\xrightarrow{H^N_{\text{per}}(P)} \sigma,\\
C([0,T], \mathfrak{A}_{\text{per}}(P)) \ni f^n &\xrightarrow{C\left([0,T], H^N_{\text{per}}(P)\right)} f,\\
\mathfrak{A}_{\text{per}}(P) \ni u_0^n &\xrightarrow{H^{N+1}_{\text{per}}(P)} u_0,\\
\mathfrak{A}_{\text{per}}(P) \ni u_1^n &\xrightarrow{H^N_{\text{per}}(P)} u_1.
\end{aligned}$$

From Theorem 4.1 of [DR] we deduce that the Cauchy problem

$$\begin{aligned}
&u_{tt} + \sigma^n b^n u_{xt} - (\sigma^n)^2 a^n u_{xx} = f^n,\\
&u(x,0) = u_0^n(x), \quad u_t(x,0) = u_1^n(x)
\end{aligned} \tag{2.2}$$

has a solution $u^n \in C^2([0,T], \mathfrak{A}_{\text{per}}(P))$. The norms $\|l_j^n\|_{C\left([0,T], H^N_{\text{per}}(P)\right)}$ and $\|A_j^n\|_{C\left([0,T], H^{N-1}_{\text{per}}(P)\right)}$ are uniformly bounded with respect to n. The Proposition 1.2 gives

$$\mathcal{E}^n_{N-1}(u^n)(t) \leq C, \quad \forall n, t. \tag{2.3}$$

We show that (u^n) is a Cauchy-sequence in suitable Banach spaces. Obviously,

$$
\begin{aligned}
&(u^n - u^m)_{tt} + \sigma^n b^n (u^n - u^m)_{xt} - (\sigma^n)^2 a^n (u^n - u^m)_{xx} \\
&\quad = f^n - f^m - (\sigma^n b^n - \sigma^m b^m) u^m_{xt} + (\sigma^n a^n - \sigma^m a^m) u^m_{xx}.
\end{aligned} \tag{2.4}
$$

Without loss of generality we may assume that

$$
\|a^n - a\| \leq \frac{1}{n}, \quad \dots, \quad \|u_1^n - u_1\| \leq \frac{1}{n}.
$$

From this it follows that we can estimate the $L^2(P)$-Norm of the right-hand side of (2.4) by $C(\frac{1}{n} + \frac{1}{m})$. Here we used $N \geq 3$ and the uniform estimates (2.3). Proposition 1.2 leads to

$$
\mathcal{E}_0^n(u^n - u^m)(t) \leq C\left(\frac{1}{n} + \frac{1}{m}\right). \tag{2.5}
$$

Nirenberg–Gagliardo interpolation and (2.3), (2.5) imply

$$
\mathcal{E}_{N-2}^n(u^n - u^m)(t) \leq C\left(\frac{1}{n} + \frac{1}{m}\right)^\theta, \quad 0 < \theta < 1.
$$

Consequently, there exists a function $u \in C([0,T], H^{N-2}_{\mathrm{per}}(P))$ with $\partial_j u \in C([0,T], H^{N-2}_{\mathrm{per}}(P))$ and

$$
(u^n, \partial_1^n u^n, \partial_2^n u^n) \xrightarrow{C([0,T], H^{N-2}_{\mathrm{per}}(P))} (u, \partial_1 u, \partial_2 u).
$$

It is easy to show that u is a solution of (1.4), (1.5).

Now we show the better regularity of u. We fix $t_0 \in [0,T]$. From $\|u^n(t_0)\|_{H^{N-1}(P)} \leq C$ we gain the existence of a subsequence $u_*^n(t_0)$ with $u_*^n(t_0) \rightharpoonup w_{t_0}$ in $H^{N-1}_{\mathrm{per}}(P)$. The embedding of the dual spaces $(H^{N-2}_{\mathrm{per}}(P))' \subset (H^{N-1}_{\mathrm{per}}(P))'$ is continuous and dense, hence $u_*^n(t_0) \rightharpoonup w_{t_0}$ in $H^{N-2}_{\mathrm{per}}(P)$. On the other hand, we have $u^n(t_0) \to u(t_0)$ in $H^{N-2}_{\mathrm{per}}(P)$. This yields the weak convergence of the whole sequence and $w_{t_0} = u(t_0)$, hence $u \in L^\infty([0,T], H^{N-1}_{\mathrm{per}}(P))$ (we do not study the question whether u is Bochner-measurable). Similar arguments apply to $\partial_j u$. It follows that

$$
u(t) \rightharpoonup u(t_0), \quad \partial_j u(t) \rightharpoonup \partial_j u(t_0) \ \text{ in } \ H^{N-1}_{\mathrm{per}}(P), \quad t \to t_0.
$$

We consider the evolution equation

$$
\partial_1 \partial_2 u = f - A_1(\partial_1 - \partial_2) u =: \tilde{f}.
$$

The right-hand side belongs to $L^\infty([0,T],H^{N-1}_{\text{per}}(P))\cap C([0,T],H^{N-2}_{\text{per}}(P))$, and the "solution" $\partial_2 u$ of $\partial_1 v=\tilde f$ belongs to $L^\infty([0,T],H^{N-1}_{\text{per}}(P))$, hence $(\partial_2 u)_t\in L^\infty([0,T],H^{N-2}_{\text{per}}(P))$. Proposition 1.1 now shows that

$$\begin{aligned} E_{N-1}(\partial_2 u)(t) \le\ & E_{N-1}(\partial_2 u)(t_0)e^{C_{N-1}(t-t_0)} \\ & + \int_{t_0}^t e^{C_{N-1}(t-\tau)}E_{N-1}(f-A_1(\partial_1-\partial_2)u)(\tau)d\tau. \end{aligned}$$

This gives

$$\limsup_{t\to t_0+0} E_{N-1}(\partial_2 u)(t)\le E_{N-1}(\partial_2 u)(t_0)\le \liminf_{t\to t_0} E_{N-1}(\partial_2 u)(t),$$

and, in consequence, $\lim_{t\to t_0+0}\|\partial_2 u(t)\|_{H^{N-1}(P)}=\|\partial_2 u(t_0)\|_{H^{N-1}(P)}$. It follows that $\partial_2 u(t)$ is $H^{N-1}_{\text{per}}(P)$-continuous from the right. Changing the time direction gives continuity from the left. By the proof of Lemma 3.1 we have $u_t\in C([0,T],H^{N+1}_{\text{per}}(P))$. □

The following proposition sharpens this result.

Proposition 2.1 *Let the assumptions of the previous lemma be satisfied. Additionally, we suppose*

$$\sigma\in H^{N+1}_{\text{per}}(P), \tag{2.6}$$

$$\sigma a,\sigma b\in C([0,T],H^{N+1}_{\text{per}}(P)). \tag{2.7}$$

Then the solution u of (1.4), (1.5) *satisfies*

$$\begin{aligned} & u\in C^1([0,T],H^N_{\text{per}}(P))\cap C^2([0,T],H^{N-1}_{\text{per}}(P)),\\ & \partial_j u\in C([0,T],H^N_{\text{per}}(P)). \end{aligned}$$

Proof. Let h_ε be a Friedrichs Mollifier with support $[-\varepsilon,\varepsilon]$ and define $a_\varepsilon(x,t):=(a(.,t)*h_\varepsilon(.))(x)$, $b_\varepsilon(x,t):=(b(.,t)*h_\varepsilon(.))(x)$. Then we have

$$a_\varepsilon,b_\varepsilon \xrightarrow{C^1([0,T],H^N_{\text{per}}(P))} a,b$$

and

$$\|\sigma a_\varepsilon\|_{C([0,T],H^{N+1}_{\text{per}}(P))},\|\sigma b_\varepsilon\|_{C([0,T],H^{N+1}_{\text{per}}(P))}\le C\quad\forall\varepsilon>0,$$

see Lemma A.2. It follows that

$$\|l_j^\varepsilon\|_{C([0,T],H^{N+1}_{\mathrm{per}}(P))} \le C, \quad \|A_j^\varepsilon\|_{C([0,T],H^N_{\mathrm{per}}(P))} \le C$$

for all $\varepsilon > 0$. We set $u_0^\varepsilon := u_0 * h_\varepsilon$, $u_1^\varepsilon := u_1 * h_\varepsilon$, $f^\varepsilon := f * h_\varepsilon$ and consider the Cauchy problem

$$\begin{aligned} &u_{tt} + \sigma b_\varepsilon u_{xt} - \sigma^2 a_\varepsilon u_{xx} = f^\varepsilon, \\ &u(x,0) = u_0^\varepsilon(x), \quad u_t(x,0) = u_1^\varepsilon(x). \end{aligned}$$

From the previous lemma we know that there exists a solution $u^\varepsilon \in C^1([0,T], H^N_{\mathrm{per}}(P))$ with $\partial_j^\varepsilon u^\varepsilon \in C([0,T], H^N_{\mathrm{per}}(P))$. We have

$$\mathcal{E}_N^\varepsilon(u^\varepsilon)(t) \le C \quad \forall \varepsilon > 0.$$

The same arguments as in the previous lemma give strong convergence of $u^\varepsilon, \partial_j^\varepsilon u^\varepsilon$ in $H^{N-1}_{\mathrm{per}}(P)$, weak convergence in $H^N_{\mathrm{per}}(P)$ and regularity of the limit. $\square$

Remark 2.1. These results can be generalized to systems, see Remark 1.1.

3. Quasilinear weakly hyperbolic equations with spatial degeneracy

We study the Cauchy problem (1.5),

$$u_{tt} + \sigma(x) b(x,t,u) u_{xt} - \sigma^2(x) a(x,t,u) u_{xx} = f(x,t,u,u_t,\sigma(x)u_x) \tag{3.1}$$

under the assumptions $N \ge 3$, (1.6), (2.6), (2.1) and

$$a, b \in C^1([0,T], C^{N+1}(K_\delta)), \tag{3.2}$$

$$f \in C([0,T], C^N(K'_\delta)), \tag{3.3}$$

$$a,\ b,\ f \text{ are } P\text{-periodic with respect to } x, \tag{3.4}$$

with

$$\begin{aligned} K_\delta &:= \left\{ (x,v) \in \mathbb{R}^2 : x \in P, |v - u_0(x)| \le \delta \right\}, \\ K'_\delta &:= \{ (x, v_1, v_2, v_3) \in \mathbb{R}^4 : (x, v_1) \in K_\delta, |v_2 - u_1(x)| \le \delta, \\ &\qquad |v_3 - \sigma(x) u_{0,x}(x)| \le \delta \}. \end{aligned}$$

To simplify notation we define

$$\begin{aligned}S(T^*) := \{&v \in C^1([0,T^*], H^N_{\mathrm{per}}(P)) : \sigma v_x \in C([0,T^*], H^N_{\mathrm{per}}(P)),\\ &(x, v(x,t), v_t(x,t), \sigma(x)v_x(x,t)) \in K'_\delta \quad \forall (x,t) \in P \times [0,T^*],\\ &v \text{ fulfils the initial conditions (1.5)}\}.\end{aligned}$$

The aim of this section is to prove the

Theorem 3.1 *We assume* (1.6), (2.6), (2.1), (3.2)–(3.4) *and* $N \geq 3$.
Then there exists a T^*, $0 < T^* \leq T$, *such that* (3.1), (1.5) *has a solution* $u \in S(T^*)$. *This solution is unique in the set of* P*-periodic functions.*

The uniqueness follows from Hadamard's Formula and the energy estimate (1.12).

We consider the linearized problem

$$\begin{aligned}L^{(v)}u &= f_{(v)},\\ L^{(v)} &= \partial_{tt} + \sigma b(x,t,v)\partial_{xt} - \sigma^2 a(x,t,v)\partial_{xx},\\ f_{(v)} &= f(x,t,v(x,t),v_t(x,t),\sigma(x)v_x(x,t))\end{aligned}$$

with initial data u_0, u_1 and study the mapping $v \mapsto u$.

The following, rather technical, lemma provides the equivalence between some norms.

Lemma 3.1 *Write*

$$S_N(v)(t) := \|v(t)\|_{H^N(P)} + \|v_t(t)\|_{H^N(P)} + \|\sigma v_x(t)\|_{H^N(P)}\,.$$

Let $h \in S(T)$ *and* $E_N(h)(t) \leq D$ *for all* t. *Then there exists a constant* $C_{S,N} = C_{S,N}(D)$ *such that*

$$\frac{1}{C_{S,N}} S_N(v)(t) \leq \mathcal{E}_N^{(h)}(v)(t) \leq C_{S,N} S_N(v)(t) \quad \forall v \in S(T).$$

Here we used the notation

$$\begin{aligned}\mathcal{E}_N^{(h)}(v)(t) &= E_N(v)(t) + E_N(\partial_1^{(h)}v)(t) + E_N(\partial_2^{(h)}v)(t),\\ \partial_j^{(h)}(v) &= (\partial_t - l_j(x,t,h(x,t))\partial_x)v.\end{aligned}$$

A proof can be found in the appendix.

Proposition 3.1 *There exist constants* C^*, $0 < T^* \leq T$, *such that:*

If $v \in S(T^*)$ *and* $\mathcal{E}_N^{(v)}(v) \leq C^*$, *then there exists a solution of* (3.1), (1.5) *with* $u \in S(T^*)$ *and* $\mathcal{E}_N^{(u)}(u) \leq C^*$.

Proof. We set $C^* = 2\mathcal{E}_N^{(u_0+tu_1)}(u_0 + tu_1)(t = 0)$. From $v \in S(T^*)$ we get

$$a(x, t, v(x, t)), \quad b(x, t, v(x, t)) \in C^1([0, T^*], H^N_{\text{per}}(P)),$$
$$\sigma(x)\frac{d}{dx}a(x, t, v(x, t)), \quad \sigma(x)\frac{d}{dx}b(x, t, v(x, t)) \in C([0, T^*], H^N_{\text{per}}(P)),$$
$$f_{(v)} \in C([0, T^*], H^N_{\text{per}}(P))$$

and

$$\|A_j^{(v)}(x, t)\|_{H^N(P)} \leq C_{A,N}(\|v\|_\infty, \|v_t\|_\infty, \|\sigma v_x\|_\infty)(S_N(v)(t) + 1),$$
$$\|l_j^{(v)}(x, t)\|_{H^N(P)} \leq C_{l,N}(\|v\|_\infty, \|v_t\|_\infty, \|\sigma v_x\|_\infty)(S_N(v)(t) + 1),$$
$$\|f_{(v)}(x, t)\|_{H^N(P)} \leq C_{f,N}(\|v\|_\infty, \|v_t\|_\infty, \|\sigma v_x\|_\infty)(S_N(v)(t) + 1),$$

see Lemma A.1. The norms $\|v\|_\infty$, $\|v_t\|_\infty$, $\|\sigma v_x\|_\infty$ are bounded, since $v \in S(T^*)$. Therefore we may assume that $C_{A,N}$, $C_{l,N}$, $C_{f,N}$ are constants depending on δ and $\|u_0\|_\infty$, $\|u_1\|_\infty$, $\|\sigma u_{0,x}\|_\infty$.

Proposition 2.1 guarantees a solution $u \in C^1([0, T^*], H^N_{\text{per}}(P))$ with $\partial_j u \in C([0, T^*], H^N_{\text{per}}(P))$.

We next show that $\mathcal{E}_N^{(v)}(u)(t) \leq \frac{2}{3}C^*$ if $t \leq T^*$ and T^* is sufficiently small. We have

$$\mathcal{E}_N^{(v)}(u)(t) \leq e^{D_N t}(\mathcal{E}_N^{(v)}(u)(0)e^{C_N t} + 2\int_0^t e^{C_N(t-\tau)}E_N(f_{(v)})(\tau)\,d\tau),$$

where D_N, C_N depend only on $C_{A,N}$, $C_{l,N}$ and C^*. We deduce that

$$\mathcal{E}_N^{(v)}(u)(t) \leq e^{(D_N+C_N)t}(\frac{1}{2}C^* + 2\int_0^t C_{f,N}(C_S C^* + 1)\,d\tau) \leq \frac{2}{3}C^*$$

if T^* is sufficiently small. From $\mathcal{E}_3^{(v)}(u) \leq \frac{2}{3}C^*$ we conclude that

$$\|u_t\|_\infty + \|u_{tx}\|_\infty + \|u_{xx}\|_\infty \leq C(\|u_t\|_{H^2(P)} + \|u\|_{H^3(P)}) \leq \frac{2}{3}C'C^*,$$

hence

$$\|u_{tt}\|_\infty \leq (\|\sigma\|_\infty \|b\|_\infty + \|\sigma\|_\infty^2 \|a\|_\infty)\frac{2}{3}C'C^* + \|f_{(v)}\|_\infty \leq C.$$

The result is

$$\begin{aligned}&|u(x,t)-u_0(x)| \le tC, \quad |u_t(x,t)-u_1(x)| \le tC,\\ &|\sigma(x)u_x(x,t)-\sigma(x)u_{0,x}(x)| \le tC.\end{aligned}$$

It follows that $u \in S(T^*)$ if T^* is sufficiently small.

It remains to show that $\mathcal{E}_N^{(u)}(u) \le C^*$. Therefore we prove that

$$\|(l_j(x,t,v(x,t)) - l_j(x,t,u(x,t)))u_x\|_{H^N(P)} \le \frac{1}{6}C^*. \tag{3.5}$$

We denote the left-hand side of (3.5) by $c_j(x,t)$ and have

$$\begin{aligned}c_j(x,t) &\le C_{\mathrm{prod},N}\,\|\alpha_j(x,t)\|_{H^N(P)}\,\|\sigma u_x\|_{H^N(P)}\\ &\le \frac{2}{3}C_{\mathrm{prod},N}C_S C^*\,\|\alpha_j(x,t)\|_{H^N(P)},\end{aligned}$$

where $\alpha_j(x,t) = \beta_j(x,t,v(x,t)) - \beta_j(x,t,u(x,t))$. From $d_t\,\|\alpha(.,t)\|_{H^N(P)} \le \|\alpha_t(.,t)\|_{H^N(P)}$, $\alpha(x,0)=0$,

$$\begin{aligned}\alpha_t(x,t) &= \beta_{j,t}(x,t,v(x,t)) - \beta_{j,t}(x,t,u(x,t))\\ &\quad + \beta_{j,v}(x,t,v(x,t))v_t(x,t) - \beta_{j,v}(x,t,u(x,t))u_t(x,t)\end{aligned}$$

and $\|v_t\|_{H^N(P)}$, $\|u_t\|_{H^N(P)} \le C_S C^*$ we obtain

$$\|\alpha(.,t)\|_{H^N(P)} \le \int_0^t \|\alpha_\tau(.,\tau)\|_{H^N(P)}\,d\tau \le Ct.$$

Thus we have (3.5). □

From this lemma it may be concluded that there exists a sequence $(u^n) \subset S(T^*)$ with

$$L^{(u^{n-1})}u^n = f_{(u^{n-1})}.$$

Lemma 3.2 *The sequences (u^n), (u_t^n), (σu_x^n) are Cauchy sequences in the space $C([0,T^*],H_{\mathrm{per}}^{N-1}(P))$ if T^* is sufficiently small.*

Proof. It holds

$$L^{(u^{n-1})}(u^n - u^{n+1}) = f_{(u^{n-1})} - f_{(u^n)} + (L^{(u^n)} - L^{(u^{n-1})})u^{n+1}.$$

Using Hadamard's Formula one can estimate the $L^2(P)$-norm of the right-hand side by $CS_0(u^n - u^{n-1})(t)$. It follows that

$$\mathcal{E}_0^{(u^{n-1})}(u^n - u^{n+1})(t) \le C' \int_0^t S_0(u^n - u^{n-1})(\tau)\, d\tau.$$

Hence we obtain

$$\max_{t\in[0,T^*]} S_0(u^n - u^{n+1})(t) \le C''T^* \max_{t\in[0,T^*]} S_0(u^n - u^{n-1})(t).$$

We apply the Interpolation Theorem of Gagliardo-Nirenberg and the proof is complete. □

Consequently, there exists a limit $u \in C([0,T^*], H^{N-1}_{\text{per}}(P))$ with $\partial_j u \in C([0,T^*], H^{N-1}_{\text{per}}(P))$. It is immediate that u is a solution. The boundedness in $H^N_{\text{per}}(P)$ implies

$$u^n(.,t) \rightharpoonup u(.,t), \quad u^n_t(.,t) \rightharpoonup u_t(.,t), \quad \sigma u^n_x(.,t) \rightharpoonup \sigma u_x(.,t)$$

in $H^N_{\text{per}}(P)$. This clearly forces $u, u_t, \partial_j u \in L^\infty([0,T^*], H^N_{\text{per}}(P))$, even $u \in C([0,T^*], H^N_{\text{per}}(P))$. It remains to prove that $\partial_j u \in C([0,T^*], H^N_{\text{per}}(P))$. We have

$$\partial_1^{(u)}\partial_2^{(u)} u = f_{(u)} - A_1^{(u)}(\partial_1^{(u)} - \partial_2^{(u)})u =: \tilde{f}.$$

The right-hand side belongs to $L^\infty([0,T^*], H^N_{\text{per}}(P)) \cap C([0,T^*], H^{N-1}_{\text{per}}(P))$, the "solution" $\partial_2^{(u)}u$ of $\partial_1^{(u)}v = \tilde{f}$ belongs to $L^\infty([0,T^*], H^N_{\text{per}}(P))$ and we have $l_1^{(u)}(x,t) \in C([0,T^*], H^N_{\text{per}}(P))$. Hence we can apply Proposition 1.1 and obtain

$$\begin{aligned} E_N(\partial_2^{(u)}u)(t) \;\le\; & E_N(\partial_2^{(u)}u)(t_0)e^{C_N(t-t_0)} \\ & + \int_{t_0}^t e^{C_N(t-\tau)} E_N(f_{(u)} - A_1^{(u)}(\partial_1^{(u)} - \partial_2^{(u)})u)(\tau)\, d\tau. \end{aligned}$$

This gives

$$\limsup_{t\to t_0+0} E_N(\partial_2^{(u)}u)(t) \le E_N(\partial_2^u u)(t_0) \le \liminf_{t\to t_0} E_N(\partial_2^{(u)}u)(t),$$

which implies the $H^N_{\text{per}}(P)$-continuity from the right of $\partial_2^{(u)}u$. Changing the time direction completes the proof of Theorem 3.1.

Remark 3.1. One can prove a similar result for the system

$$\begin{aligned}
&u_{i,tt} + \sigma(x)b(x,t,\vec{u})u_{i,xt} - \sigma^2(x)a(x,t,\vec{u})u_{i,xx} \\
&\qquad = f(x,t,\vec{u},\vec{u}_t,\sigma(x)\vec{u}_x), \quad i = 1,\dots,n \\
&\vec{u}(x,0) = \vec{u}_0(x), \quad \vec{u}_t(x,0) = \vec{u}_1(x),
\end{aligned}$$

if one uses the energy

$$\mathcal{E}_N^{(\vec{v})}(\vec{u}) := \sum_{i=1}^{n} \mathcal{E}_N^{(\vec{v})}(u_i).$$

4. Quasilinear weakly hyperbolic equations with time- and spatial degeneracy

In this section we will derive an existence result for the quasilinear Cauchy problem with spatial- and time-degeneracy (1.5),

$$\begin{aligned}
&u_{tt}(x,t) + \lambda(t)\sigma(x)b(x,t,u(x,t))u_{xt}(x,t) \\
&\qquad - \lambda(t)^2\sigma(x)^2a(x,t,u(x,t))u_{xx}(x,t) \\
&\quad = f(x,t,u(x,t),u_t(x,t),\lambda'(t)\sigma(x)u_x(x,t)).
\end{aligned} \tag{4.1}$$

Let $\lambda(t) \in C^1([0,T])$ be a function satisfying (0.4).

The aim of this section is to prove the

Theorem 4.1 *We suppose* (2.1) *and*

$$b^2(x,t,u) + 4a(x,t,u) \geq \gamma > 0, \quad (x,t,u) \in K_\delta, \tag{4.2}$$

$$\sigma \in H^{N+2}_{\text{per}}(P), \tag{4.3}$$

$$a, b \in C^1([0,T], C^\infty(\mathbb{R}_u) \times H^N_{\text{per}}(P)), \tag{4.4}$$

$$\sigma a, \sigma b \in C([0,T], C^\infty(\mathbb{R}_u) \times H^{N+1}_{\text{per}}(P)), \tag{4.5}$$

$$f \in C([0,T], C^\infty(\mathbb{R}^3_{u,u_t,\lambda'\sigma u_x}) \times H^N_{\text{per}}(P)). \tag{4.6}$$

Then there exist constants $T^* > 0$ and $r \in \mathbb{N}$, such that: If

$$N \geq r + 3, \tag{4.7}$$

then there exists a solution u with $u, u_t, \sigma u_x \in C([0,T^*], H^{N-r}_{\text{per}}(P))$. The number r describes the loss of Sobolev regularity and may depend on N. If

N is sufficiently large, then (4.7) *holds.*

We divide the proof into three steps. At first we present a special technique to transform the quasilinear problem to another problem whose right-hand side has a suitable asymptotic. Then we consider *linear* weakly hyperbolic problems with special right-hand side and show an existence result and an a priori estimate. After that we study a *linearized* version of the new *quasilinear* problem and construct a mapping of functions. Using the results of the second step we construct a sequence of such functions and prove the convergence to a solution.

4.1. The reduction process

Let u be a solution of (4.1), (1.5). We study the system of ordinary differential equations with parameter x

$$\begin{aligned}
u^{(0)}_{tt}(x,t) &= f(x,t,u^{(0)}(x,t),u^{(0)}_t(x,t),0),\\
u^{(i)}_{tt}(x,t) &= g_i(x,t,u^{(i)}(x,t),u^{(i)}_t(x,t))\\
&:= f\Big(x,t,\sum_{j=0}^{i}u^{(j)}(x,t),\sum_{j=0}^{i}u^{(j)}_t(x,t),\sigma(x)\lambda'(t)\sum_{j=0}^{i-1}u^{(j)}_x(x,t)\Big)\\
&\quad - f\Big(x,t,\sum_{j=0}^{i-1}u^{(j)}(x,t),\sum_{j=0}^{i-1}u^{(j)}_t(x,t),\sigma(x)\lambda'(t)\sum_{j=0}^{i-2}u^{(j)}_x(x,t)\Big)\\
&\quad - \lambda(t)\sigma(x)b_{i-1}(x,t)\sum_{j=0}^{i-1}u^{(j)}_{xt}(x,t)\\
&\quad + \lambda(t)\sigma(x)b_{i-2}(x,t)\sum_{j=0}^{i-2}u^{(j)}_{xt}(x,t)\\
&\quad + \lambda(t)^2\sigma(x)^2a_{i-1}(x,t)\sum_{j=0}^{i-1}u^{(j)}_{xx}(x,t)\\
&\quad - \lambda(t)^2\sigma(x)^2a_{i-2}(x,t)\sum_{j=0}^{i-2}u^{(j)}_{xx}(x,t),\\
&\qquad i=1,\dots,p,\quad N-2p\ge 3,\quad \sum_{j=0}^{-1}=0,
\end{aligned}$$

with the initial data

$$u^{(0)}(x,0) = u_0(x), \quad u_t^{(0)}(x,0) = u_1(x),$$
$$u^{(i)}(x,0) = u_t^{(i)}(x,0) = 0.$$

Here we used the notation

$$b_k(x,t) = b\Big(x,t,\sum_{j=0}^{k} u^{(j)}(x,t)\Big), \; a_k(x,t) = a\Big(x,t,\sum_{j=0}^{k} u^{(j)}(x,t)\Big).$$

This system may be interpreted as a system of weakly hyperbolic equations with spatial degeneracy, whose degenerating function vanishes identically. Theorem 3.1 implies, that these equations have solutions $u^{(i)} \in C^2([0,T_i], H_{\mathrm{per}}^{N-2i}(P))$ with

$$\Big|u^{(0)}(x,t) - u_0(x)\Big| + \Big|u_t^{(0)}(x,t) - u_1(x)\Big|$$
$$+ \Big|\sigma(x)u_x^{(0)}(x,t) - \sigma(x)u_{0,x}(x)\Big| \leq \varepsilon_0,$$
$$\Big|u^{(i)}(x,t)\Big| + \Big|u_t^{(i)}(x,t)\Big| + \Big|\sigma(x)u_x^{(i)}(x,t)\Big| \leq \varepsilon_i \quad (i \geq 1).$$

Furthermore, there are constants C_i with $\|u^{(i)}\|_{C^2\left([0,T_i],H_{\mathrm{per}}^{N-2i}(P)\right)} \leq C_i$.

We want to show the following

Lemma 4.1 *It holds*

$$F_{N-2i}(u^{(i)})(t) \leq C_i\lambda(t)^i.$$

Proof. The assertion is true for $i = 0$.

Let $F_{N-2(i-1)}(u^{(i-1)})(t) \leq C_{i-1}\lambda(t)^{i-1}$. We want to apply Lemma 1.1 and for this purpose we study the right-hand side g_i: From Hadamard's Formula we see that

$$\begin{aligned} &g_i(x,t,u^{(i)},u_t^{(i)}) \\ &\quad = f_{1i}(x,t)u^{(i)} + f_{2i}(x,t)u_t^{(i)} + \sigma(x)\lambda'(t)f_{3i}(x,t)u_x^{(i-1)} \\ &\qquad + \sigma(x)\lambda(t)b_{1i}(x,t)u_{xt}^{(i-1)} + \sigma(x)\lambda(t)b_{2i}(x,t)u^{(i-1)} \\ &\qquad + \sigma(x)^2\lambda(t)^2a_{1i}(x,t)u_{xx}^{(i-1)} + \sigma(x)^2\lambda(t)^2a_{2i}(x,t)u^{(i-1)}. \end{aligned}$$

Hence we obtain

$$
\begin{aligned}
& F_{N-2i}(u^{(i)})(t) \\
& \le C\int_0^t \lambda'(\tau)\|u^{(i-1)}(\tau)\|_{H^{N+1-2i}(P)} \\
& \quad + \lambda(\tau)\|u_\tau^{(i-1)}(\tau)\|_{H^{N+1-2i}(P)} + \lambda(\tau)\|u^{(i-1)}(\tau)\|_{H^{N-2i}(P)} \\
& \quad + \lambda(\tau)^2\|u^{(i-1)}(\tau)\|_{H^{N+2-2i}(P)}\,d\tau \\
& \le C\int_0^t \lambda'(\tau)\lambda(\tau)^{i-1} + \lambda(\tau)\lambda(\tau)^{i-1} + \lambda(\tau)^2\lambda(\tau)^{i-1}\,d\tau \\
& \le C_i\lambda(t)^i.
\end{aligned}
$$

□

We define $u =: \sum_{j=0}^p u^{(j)} + v$ and

$$
b_k(x,t,v) = b\Big(x,t,\sum_{j=0}^k u^{(j)} + v\Big), \quad a_k(x,t,v) = a\Big(x,t,\sum_{j=0}^k u^{(j)} + v\Big).
$$

From (4.1) we see that

$$
\begin{aligned}
& v_{tt} + \lambda(t)\sigma(x)b_p(x,t,v)v_{xt} - \lambda(t)^2\sigma(x)^2 a_p(x,t,v)v_{xx} \\
& = f\bigg(x,t,\sum_{j=0}^p u^{(j)} + v, \sum_{j=0}^p u_t^{(j)} + v_t, \lambda'(t)\sigma(x)\Big(\sum_{j=0}^p u_x^{(j)} + v_x\Big)\bigg) \\
& \quad - f\bigg(x,t,\sum_{j=0}^p u^{(j)}, \sum_{j=0}^p u_t^{(j)}, \lambda'(t)\sigma(x)\sum_{j=0}^{p-1} u_x^{(j)}\bigg) \\
& \quad - \lambda(t)\sigma(x)b_p(x,t,v)\sum_{j=0}^p u_{xt}^{(j)} + \lambda(t)\sigma(x)b_{p-1}(x,t,0)\sum_{j=0}^{p-1} u_{xt}^{(j)} \\
& \quad + \lambda(t)^2\sigma(x)^2 a_p(x,t,v)\sum_{j=0}^p u_{xx}^{(j)} - \lambda(t)^2\sigma(x)^2 a_{p-1}(x,t,0)\sum_{j=0}^{p-1} u_{xx}^{(j)} \\
& =: F_p(x,t,v,v_t,\lambda'(t)\sigma(x)v_x).
\end{aligned}
$$

The purpose for these considerations is the following

Proposition 4.1 *It holds*

$$
\|F_p(x,t,0,0,0)\|_{H^{N-2-2p}(P)} \le C\lambda(t)^p.
$$

Proof. We write $F_p(x,t,0,0,0) = D_1 + D_2 + D_3$, with

$$\begin{aligned}
D_1 &:= f\Big(x,t,\sum^p,\sum^p,\sum^p\Big) - f\Big(x,t,\sum^p,\sum^p,\sum^{p-1}\Big),\\
D_2 &:= -\lambda\sigma b_p(x,t,0)\Big(\sum^p\cdots\Big)_{xt} + \lambda\sigma b_{p-1}(x,t,0)\Big(\sum^{p-1}\cdots\Big)_{xt}\\
D_3 &:= \lambda^2\sigma^2 a_p(x,t,0)\Big(\sum^p\cdots\Big)_{xx} - \lambda^2\sigma^2 a_{p-1}(x,t,0)\Big(\sum^{p-1}\cdots\Big)_{xx}.
\end{aligned}$$

From Hadamard's Formula, Lemma 4.1 and

$$\begin{aligned}
D_2 &= -\lambda\sigma b_p(x,t,0)u^{(p)}_{xt}\\
&\quad - \lambda\sigma(b_p(x,t,0) - b_{p-1}(x,t,0))\Big(\sum^{p-1}\cdots\Big)_{xt},\\
D_3 &= \lambda^2\sigma^2 a_p(x,t,0)u^{(p)}_{xx}\\
&\quad + \lambda^2\sigma^2(a_p(x,t,0) - a_{p-1}(x,t,0))\Big(\sum^{(p-1)}\cdots\Big)_{xx}
\end{aligned}$$

one has the assertion. □

4.2. Linear theory for equations with special right-hand side

Now we are able to derive an existence result and an a priori estimate for linear equations with spatial- and time-degeneracy.

Proposition 4.2 *Let $\lambda(t) \in C^1([0,T])$ be a function satisfying* (0.4). *We assume $N \geq 3$,* (1.6), (4.3), (1.10), (2.7),

$$\begin{aligned}
&\frac{f}{\lambda^{d-1}\lambda'} \in C([0,T], H^N_{\mathrm{per}}(P)),\\
&d > Q := \sup_{(x,t)\in P\times[0,T]}\left|\frac{b(x,t)}{\sqrt{b^2(x,t)+4a(x,t)}}\right| + 1.
\end{aligned}$$

Then there exists a solution u of

$$u_{tt} + \sigma(x)\lambda(t)b(x,t)u_{xt} - \sigma^2(x)\lambda^2(t)a(x,t)u_{xx} = f(x,t), \tag{4.8}$$

$$u(x,0) = u_t(x,0) = 0 \tag{4.9}$$

with $u \in C^1([0,T], H^N_{\mathrm{per}}(P))$ and $\lambda^{-d}u, \sigma u_x \in C([0,T], H^N_{\mathrm{per}}(P))$. The fol-

lowing estimates hold:

$$\mathcal{E}_M(u)'(t) \le \left(C_M + Q'\frac{\lambda'(t)}{\lambda(t)}\right)\mathcal{E}_M(u)(t) + 2E_M(f)(t), \qquad M = 0, \dots, N-1,$$

$$\mathcal{E}_N(u)(t) \le 2\int_0^t e^{C_N(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{Q'} E_N(f)(\tau)d\tau,$$

where $Q < Q' < d$. *The constants* C_M *depend on*

$$\begin{aligned}
&\|l_j\|_{C\left([0,T],H_{\text{per}}^{\max(2,M)}(P)\right)}, \\
&\|a\|_{C^1\left([0,T],H_{\text{per}}^{\max(1,M)}(P)\right)}, \quad \|b\|_{C^1\left([0,T],H_{\text{per}}^{\max(1,M)}(P)\right)}, \\
&\|\sigma a_x\|_{C\left([0,T],H_{\text{per}}^{\max(1,M)}(P)\right)}, \quad \|\sigma b_x\|_{C\left([0,T],H_{\text{per}}^{\max(1,M)}(P)\right)}.
\end{aligned}$$

Proof. We approximate

$$C^1([0,T],H_{\text{per}}^{N+2}(P)) \ni a_\varepsilon, b_\varepsilon \xrightarrow{C^1\left([0,T],H_{\text{per}}^N(P)\right)} a, b,$$

$$C([0,T],H_{\text{per}}^{N+1}(P)) \ni \tilde{f}_\varepsilon \xrightarrow{C\left([0,T],H_{\text{per}}^N(P)\right)} \frac{f}{\lambda^{d-1}\lambda'},$$

such that

$$\|\sigma a_{\varepsilon,x}\|_{H^N(P)}, \|\sigma b_{\varepsilon,x}\|_{H^N(P)} \le C, \quad b_\varepsilon^2 + 4a_\varepsilon \ge \frac{\gamma}{2} \quad \forall \quad 0 < \varepsilon \le \varepsilon_0,$$

see Lemma A.2. We set $\tilde{b}_\varepsilon := b_\varepsilon(\lambda+\varepsilon)$, $\tilde{a}_\varepsilon := a_\varepsilon(\lambda+\varepsilon)^2$ and obtain $\tilde{b}_\varepsilon^2 + 4\tilde{a}_\varepsilon \ge \frac{1}{2}\varepsilon^2\gamma > 0$. Due to Proposition 2.1 there exists a solution u^ε of

$$\begin{aligned}
&u_{tt}^\varepsilon + \sigma\tilde{b}_\varepsilon u_{xt}^\varepsilon - \sigma^2\tilde{a}_\varepsilon u_{xx}^\varepsilon = \lambda^{d-1}\lambda'\tilde{f}_\varepsilon, \\
&u^\varepsilon(x,0) = u_t^\varepsilon(x,0) = 0
\end{aligned}$$

with $u^\varepsilon, u_t^\varepsilon, \sigma u_x^\varepsilon \in C([0,T],H_{\text{per}}^{N+1}(P))$. It follows that

$$\begin{aligned}
&\partial_j^\varepsilon = \partial_t - l_j^\varepsilon(x,t)\partial_x, \\
&l_{1,2}^\varepsilon(x,t) = \frac{1}{2}\sigma(x)(\lambda(t)+\varepsilon)(-b_\varepsilon(x,t) \pm \sqrt{b_\varepsilon^2(x,t) + 4a_\varepsilon(x,t)}), \\
&(\partial_1^\varepsilon\partial_2^\varepsilon + A_1^\varepsilon(\partial_1^\varepsilon - \partial_2^\varepsilon))u^\varepsilon = \lambda^{d-1}\lambda'\tilde{f}_\varepsilon, \\
&A_1^\varepsilon(x,t) = c_{1,\varepsilon}(x,t) + \frac{\lambda'(t)}{\lambda(t)+\varepsilon}c_{2,\varepsilon}(x,t),
\end{aligned}$$

$$\begin{aligned} c_{1,\varepsilon} &= \frac{-b_\varepsilon + S_\varepsilon}{4S_\varepsilon}(\sigma(-b_\varepsilon - S_\varepsilon))_x(\lambda(t)+\varepsilon) + \frac{(b_\varepsilon + S_\varepsilon)_t}{2S_\varepsilon}, \\ c_{2,\varepsilon} &= \frac{b_\varepsilon + S_\varepsilon}{2S_\varepsilon} \end{aligned}$$

with $S_\varepsilon = \sqrt{b_\varepsilon^2 + 4a_\varepsilon}$. We have

$$l_j^\varepsilon, c_{2,\varepsilon} \in C^1([0,T], H^{N+2}_{\mathrm{per}}(P)), \quad c_{1,\varepsilon} \in C([0,T], H^{N+1}_{\mathrm{per}}(P))$$

and

$$\begin{aligned} &\|l_j^\varepsilon\|_{C^1([0,T],H^N_{\mathrm{per}}(P))}, \|c_{1,\varepsilon}\|_{C([0,T],H^N_{\mathrm{per}}(P))}, \|c_{2,\varepsilon}\|_{C^1([0,T],H^N_{\mathrm{per}}(P))} \\ &\quad \le C \quad \forall \varepsilon. \end{aligned}$$

Now we can derive estimates for u^ε. Obviously,

$$\partial_1^\varepsilon \partial_2^\varepsilon u^\varepsilon = \lambda^{d-1}\lambda' \tilde{f}_\varepsilon - A_1^\varepsilon(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon,$$

where the right-hand side belongs to $C([0,T], H^{N+1}_{\mathrm{per}}(P))$ and the solution $\partial_2^\varepsilon u^\varepsilon$ belongs to $C([0,T], H^{N+1}_{\mathrm{per}}(P))$. From Proposition 1.1 we get for $0 \le M \le N$

$$\begin{aligned} E_M(\partial_2^\varepsilon u^\varepsilon)'(t) \le\; & C_M E_M(\partial_2^\varepsilon u^\varepsilon)(t) \\ & + E_M(\lambda^{d-1}\lambda' \tilde{f}_\varepsilon - A_1^\varepsilon(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon)(t). \end{aligned}$$

We estimate the second energy on the right:

$$\begin{aligned} &\|\partial_x^M(A_1^\varepsilon(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon)\|_2 \\ &\le \|\partial_x^M(c_{1,\varepsilon}(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon)\|_2 + \frac{\lambda'(t)}{\lambda(t)+\varepsilon}\|c_{2,\varepsilon}\partial_x^M(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon\|_2 \\ &\quad + C\sum_{j=0}^{M-1} \frac{\lambda'(t)}{\lambda(t)+\varepsilon}\|(\partial_x^{M-j}c_{2,\varepsilon})\partial_x^j(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon\|_2 \\ &\le C\mathcal{E}_M^\varepsilon(u^\varepsilon)(t) + \frac{\lambda'(t)}{\lambda(t)+\varepsilon}\frac{Q+\varepsilon'}{2}(\|\partial_x^M(\partial_1^\varepsilon u^\varepsilon)\|_2 + \|\partial_x^M(\partial_2^\varepsilon u^\varepsilon)\|_2) \\ &\quad + C\frac{\lambda'(t)}{\lambda(t)+\varepsilon}\|(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon\|_{H^{M-1}(P)} \\ &\le C\mathcal{E}_M^\varepsilon(u^\varepsilon)(t) + \frac{\lambda'(t)}{\lambda(t)+\varepsilon}\frac{Q+\varepsilon'}{2}(\|\partial_x^M(\partial_1^\varepsilon u^\varepsilon)\|_2 + \|\partial_x^M(\partial_2^\varepsilon u^\varepsilon)\|_2). \end{aligned}$$

The result is

$$E_M(\partial_2^\varepsilon u^\varepsilon)'(t) \le C_M \mathcal{E}_M^\varepsilon(\partial_2^\varepsilon u^\varepsilon)(t) + \lambda^{d-1}(t)\lambda'(t)E_N(\tilde{f}_\varepsilon)(t) + \frac{Q+\varepsilon'}{2}\frac{\lambda'(t)}{\lambda(t)+\varepsilon}\mathcal{E}_M^\varepsilon(u^\varepsilon)(t).$$

We can prove a similar inequality for $E_M(\partial_1^\varepsilon u^\varepsilon)'(t)$, if we use

$$\partial_2^\varepsilon \partial_1^\varepsilon u^\varepsilon = \lambda^{d-1}\lambda'\tilde{f}_\varepsilon - A_2^\varepsilon(\partial_1^\varepsilon - \partial_2^\varepsilon)u^\varepsilon.$$

We have to replace $c_{2,\varepsilon} = \frac{b_\varepsilon + S_\varepsilon}{2S_\varepsilon}$ by $\frac{b_\varepsilon - S_\varepsilon}{2S_\varepsilon}$. By standard technique one shows an estimate for $E_M(u)'(t)$. Combining these inequalities we obtain

$$\mathcal{E}_M^\varepsilon(u^\varepsilon)'(t) \le C_M\mathcal{E}_M^\varepsilon(u^\varepsilon)(t) + 2\lambda^{d-1}(t)\lambda'(t)E_M(\tilde{f}_\varepsilon)(t) + (Q+\varepsilon')\frac{\lambda'(t)}{\lambda(t)+\varepsilon}\mathcal{E}_M^\varepsilon(u^\varepsilon)(t).$$

We may assume that ε_0 is so small that $Q+\varepsilon' \le Q' < d$ for $0 < \varepsilon \le \varepsilon_0$. We want to apply the Lemma of Nersesjan. Before we can do this, we have to check whether $\mathcal{E}_M^\varepsilon(u^\varepsilon)(t) \le C_\varepsilon \lambda(t)^d$. Therefore, we apply Gronwall's Lemma to

$$\mathcal{E}_M^\varepsilon(u^\varepsilon)'(t) \le \frac{C}{\varepsilon}\mathcal{E}_M^\varepsilon(u^\varepsilon)(t) + C\lambda^{d-1}(t)\lambda'(t)$$

and get

$$\mathcal{E}_M^\varepsilon(u^\varepsilon)(t) \le \frac{C}{\varepsilon}\int_0^t e^{\frac{C}{\varepsilon}(t-\tau)}\lambda^{d-1}(\tau)\lambda'(\tau)d\tau \le C_\varepsilon\lambda(t)^d. \tag{4.10}$$

So we can apply Nersesjan's Lemma with

$$y(t) := \mathcal{E}_M^\varepsilon(u^\varepsilon)(t), \quad K(t) := Q'\frac{\lambda'(t)}{\lambda(t)} + C_M,$$
$$f := 2\lambda^{d-1}(t)\lambda'(t)E_M(\tilde{f}_\varepsilon)(t)$$

and deduce that

$$\mathcal{E}_M^\varepsilon(u^\varepsilon)(t) \le 2\int_0^t \left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{Q'} e^{C_M(t-\tau)}\lambda^{d-1}(\tau)\lambda'(\tau)E_M(\tilde{f}_\varepsilon)(\tau)d\tau \le C\lambda(t)^d. \tag{4.11}$$

This inequality holds for all ε in contradistinction to (4.10). We conclude

$$\|u^\varepsilon(t)\|_{H^N(P)} + \|u_t^\varepsilon(t)\|_{H^N(P)} + \|(\lambda(t)+\varepsilon)\sigma u_x^\varepsilon(t)\|_{H^N(P)} \le C\lambda(t)^d,$$

hence

$$\|u^\varepsilon(t)\|_{H^N(P)} + \|u_t^\varepsilon(t)\|_{H^N(P)} + \|(\lambda(t)+\varepsilon')\sigma u_x^\varepsilon(t)\|_{H^N(P)} \le C\lambda(t)^d$$

for $0 \le \varepsilon' < \varepsilon$, which implies $\mathcal{E}_N^{\varepsilon'}(u^\varepsilon)(t) \le C\lambda(t)^d$.

We next claim that the sequence (u^ε) converges in suitable Sobolev spaces. By definition of u^ε,

$$\begin{aligned} &L^\varepsilon u^\varepsilon = \lambda^{d-1}\lambda' \tilde f_\varepsilon,\\ &L^{\varepsilon'}(u^{\varepsilon'} - u^\varepsilon) = \lambda^{d-1}\lambda'(\tilde f_{\varepsilon'} - \tilde f_\varepsilon) + (L^\varepsilon - L^{\varepsilon'})u^\varepsilon =: g_{\varepsilon,\varepsilon'}(x,t). \end{aligned}$$

From $\|u_{xt}^\varepsilon\|_{H^{N-1}(P)} \le C\lambda(t)^d$ and $\|u_{xx}^\varepsilon\|_{H^{N-2}(P)} \le C\lambda(t)^d$ we deduce that $\|g_{\varepsilon,\varepsilon'}\|_{H^{N-2}(P)} \le C\lambda(t)^d + C\lambda(t)^{d-1}\lambda'(t)$. Without loss of generality we obtain for $0 < \varepsilon' < \varepsilon$ the estimate $\mathcal{E}_{N-2}^{\varepsilon'}(u^\varepsilon - u^{\varepsilon'})(t) \le C(\varepsilon + \varepsilon')\lambda(t)^d$.

It follows that the sequences (u^ε), (u_t^ε), $(\lambda\sigma u_x^\varepsilon)$ converge to limits u, u_t, $\lambda\sigma u_x$ in $C([0,T], H_{\mathrm{per}}^{N-2}(P))$. By the Interpolation Theorem of Nirenberg–Gagliardo we have convergence even in $C([0,T], H_{\mathrm{per}}^{N-1}(P))$. The weak convergence of this sequences in $H_{\mathrm{per}}^N(P)$ implies

$$u, u_t, \sigma u_x \in L^\infty([0,T], H_{\mathrm{per}}^N(P)), \quad \mathcal{E}_N(u)(t) \le C\lambda(t)^d.$$

By standard arguments one obtains for $t_0 > 0$

$$\begin{aligned} \limsup_{t\to t_0+0} \|(\partial_j u)(t)\|_{H^N(P)} &\le \|(\partial_j u)(t_0)\|_{H^N(P)}\\ &\le \liminf_{t\to t_0} \|(\partial_j u)(t)\|_{H^N(P)}, \end{aligned}$$

hence $H^N(P)$-continuity from the right. The continuity from the left is proved by the same technique after changing the time-direction.

The continuity for $t_0 = 0$ is trivial, since $\|u(t)\|_{H^N(P)} + \|u_t(t)\|_{H^N(P)} + \|\sigma u_x(t)\|_{H^N(P)}$ is obviously continuous for $t_0 = 0$.

Finally, from (4.11) it follows that

$$\begin{aligned} \mathcal{E}_N(u)(t) &\le \liminf_{\varepsilon\to 0} \mathcal{E}_N^\varepsilon(u^\varepsilon)(t)\\ &\le 2\int_0^t \left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{Q'} e^{C_N(t-\tau)} E_N(f)(\tau)\,d\tau. \end{aligned}$$

□

4.3. Quasilinear equations

Proof of Theorem 4.1 First, we have to compute the number of required reduction steps. Set

$$Q := \sup_{(x,t,u)\in K_\delta\times[0,T]} \left| \frac{b(x,t,u)}{\sqrt{b(x,t,u)^2+4a(x,t,u)}} \right| + 1$$

and choose $Q' > Q$. We define

$$Q_1 := \sup_{(x,t,u_1,u_2,u_3)\in K'_\delta\times[0,T]} \left|\partial_5 f(x,t,u_1,u_2,u_3)\right|,$$

where ∂_5 denotes the derivative with respect to the 5th argument. It is worth pointing out that

$$\begin{aligned}&\partial_5 F_p(x,t,v,v_t,\sigma\lambda' v_x)\\ &\quad = \partial_5 f\Big(x,t,\sum^p u^{(j)}+v, \sum^p u_t^{(j)}+v_t, \sigma\lambda'\Big(\sum^p u_x^{(j)}+v_x\Big)\Big).\end{aligned}$$

Now we choose $p\in\mathbb{N}$, $p>Q'$ with

$$\frac{2Q_1 C_{S,N-2-2p}}{p-Q'} \le \frac{1}{4}. \tag{4.12}$$

It may happen that for given N such a number p does not exist. But it is possible to find p if N is large. Namely, we fix $q\in 2\mathbb{N}$, $q\ge 4$. For even N we define $p := \frac{1}{2}(N-2-q)$. The condition (4.12) is fulfilled, if N (and hence p) is large. The number p is the number of steps in the reduction process. We set $r := 2p+3$, $M := N-r+1\ge 4$, where N and r are the constants from Theorem 4.1.

We choose $\delta_0 > 0$ such that, if $S_K(v) = \|v\|_{H^K(P)} + \|v_t\|_{H^K(P)} + \|\sigma\lambda v_x\|_{H^K(P)} \le \delta_0$ and $K\ge 1$, then $\|v\|_\infty \le \frac{\delta}{2}$, $\|v_t\|_\infty\le\frac{\delta}{2}$, $\|\sigma\lambda v_x\|_\infty \le \frac{\delta}{2}$.

Let $T_p>0$ be a constant such that

$$\begin{aligned}&\left|\sum^p u^{(j)}(x,t)-u_0(x)\right|, \left|\sum^p u_t^{(j)}(x,t)-u_1(x)\right|,\\ &\qquad \left|\sigma\lambda'\Big(\sum^p u_x^{(j)}(x,t)-u_{0,x}(x)\Big)\right| \le \frac{\delta}{2}, \quad t\in[0,T_p].\end{aligned}$$

We study the reduced problem (4.9),

$$L_p^{(v)}u = u_{tt}+\sigma\lambda b_p(x,t,v)u_{xt}-\sigma^2\lambda^2 a_p(x,t,v)u_{xx} \tag{4.13}$$

$$= F_p(x, t, v, v_t, \sigma\lambda' v_x)$$

with $S_M(v)(t) \leq \delta_0$ for $t \in [0, T_p]$ and $v(x, 0) = v_t(x, 0) = 0$.

We will prove the

Lemma 4.2 *There exists a constant* $0 < T_0 \leq T_p$ *with:*
If $S_M(v)(t) \leq \lambda(t)^p$, *then* $S_M(u)(t) \leq \lambda(t)^p$ *for* $0 \leq t \leq T_0$.

Proof. Let C_M be the constant from Proposition 4.2, such that for $S_M(v) \leq \delta_0$ and $L_p^{(v)} u = g$ it holds

$$\mathcal{E}_M^{(v)}(u)(t) \leq 2 \int_0^t e^{C_M(t-\tau)} \left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{Q'} E_M(g)(\tau) d\tau.$$

We choose constants $C_{F,1}$ and $C_{F,2}$ with

$$\begin{aligned}
&\|F_p(x, t, 0, 0, 0)\|_{H^M(P)} \leq C_{F,1}\lambda(t)^p, \\
&\|F_p(x, t, v_1, v_2, v_3)\|_{H^M(P)} \\
&\quad \leq C_{F,2}(\|v_1\|_{H^M(P)} + \|v_2\|_{H^M(P)} + \|v_3\|_{H^{M-1}(P)}) \\
&\qquad + Q_1 \|\partial_x^M v_3\|_2 + C_{F,1}\lambda(t)^p
\end{aligned}$$

for $\|v_j\|_\infty \leq \frac{\delta}{2}$, see Proposition 4.1 and Lemma A.1. Let T_0 satisfy the following conditions

$$\begin{aligned}
&\lambda(T_0) \leq \delta_0, \\
&C_P \lambda(T_0)^p \|\sigma\|_\infty \|\lambda'\|_\infty \leq \frac{\delta}{2}, \\
&C_{P,M-1} \|\sigma\|_{H^{M-1}(P)} \|\lambda'\|_\infty \leq 1, \\
&e^{C_M T_0} \leq 2, \\
&2(2C_{F,2} + C_{F,1}) C_{S,M} T_0 \leq \frac{1}{4},
\end{aligned}$$

where C_P and $C_{P,M-1}$ are the imbedding constants from Section 1. Now we show that $S_M(u)(t) \leq \lambda(t)^p$ for $t \leq T_0$.

The vector $(x, t, v, v_t, \sigma\lambda' v_x)$ lies in the domain of F_p, since $\|v\|_{H^M(P)} \leq \delta_0$, $\|v_t\|_{H^M(P)} \leq \delta_0$ and

$$\begin{aligned}
\|\sigma\lambda' v_x\|_\infty &\leq \|\sigma\|_\infty \|\lambda'\|_\infty C_P \|v\|_{H^2(P)} \\
&\leq C_P \|\sigma\|_\infty \|\lambda'\|_\infty S_M(v) \leq \frac{\delta}{2}.
\end{aligned}$$

Hence we obtain

$$
\begin{aligned}
&\|F_p(x,t,v,v_t,\sigma\lambda' v_x)\|_{H^M(P)} \\
&\quad \le C_{F,2}(\|v\|_{H^M(P)} + \|v_t\|_{H^M(P)} + \|\sigma\lambda' v_x\|_{H^{M-1}(P)}) \\
&\qquad + Q_1\|\partial_x^M(\sigma\lambda' v_x)\|_2 + C_{F,1}\lambda(t)^p.
\end{aligned}
$$

From

$$
\begin{aligned}
\|\sigma\lambda' v_x\|_{H^{M-1}(P)} &\le C_{P,M-1}\|\sigma\|_{H^{M-1}(P)}\|\lambda'\|_\infty\|v\|_{H^M(P)} \\
&\le \|v\|_{H^M(P)}, \\
\|\partial_x^M(\sigma\lambda' v_x)\|_2 &\le \frac{\lambda'(t)}{\lambda(t)}\|\sigma\lambda v_x\|_{H^M(P)} \le \frac{\lambda'(t)}{\lambda(t)} S_M(v)(t)
\end{aligned}
$$

we deduce that

$$
\begin{aligned}
&\|F_p(x,t,v,v_t,\sigma\lambda' v_x)\|_{H^M(P)} \\
&\quad \le (2C_{F,2} + C_{F,1})\lambda(t)^p + Q_1\lambda'(t)\lambda(t)^{p-1}.
\end{aligned}
$$

By $E_M(\cdots) = \|\cdots\|_{H^M(P)}$ and the choice of p and T_0 we can assert that

$$
\begin{aligned}
&C_{S,M}\mathcal{E}_M^{(v)}(u)(t) \\
&\le 2C_{S,M}\int_0^t e^{C_M(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{Q'} E_M(F_p(x,\tau,v,v_t,\sigma\lambda' v_x))d\tau \\
&\le C_{S,M}e^{C_M T_0}\lambda(t)^{Q'}\int_0^t \lambda(\tau)^{-Q'}2((2C_{F,2}+C_{F,1})\lambda(\tau)^p \\
&\quad + Q_1\lambda'(\tau)\lambda(\tau)^{p-1})d\tau \\
&\le C_{S,M}e^{C_M T_0}\lambda(t)^p 2(2C_{F,2}+C_{F,1})t + C_{S,M}e^{C_M T_0}\lambda(t)^p\frac{2Q_1}{p-Q'} \\
&\le \lambda(t)^p.
\end{aligned}
$$

The assertion follows from $S_M(u) \le C_S\mathcal{E}_M^{(v)}(u)$. The Lemma is proved.

$\square$

Thus, we can define a sequence $(v_i) \subset C^1([0,T_0], H^M_{\text{per}}(P))$ with $\partial_j v_i \in C([0,T_0], H^M_{\text{per}}(P))$ such that

$$
\begin{aligned}
v_0 &\equiv 0, \\
L_p^{(v_i)} v_{i+1} &= F_p(x,t,v_i,v_{i,t},\sigma\lambda' v_{i,x}), \\
v_i(x,0) &= v_{i,t}(x,0) = 0.
\end{aligned}
$$

These functions satisfy $S_M(v_i)(t) \le \lambda(t)^p$. We will prove a convergence result.

Lemma 4.3 *The sequence (v_i) is a Cauchy sequence in the Banach space $C^1([0,T_0], H^{M-1}_{\text{per}}(P))$.*

Proof. We define $w_i = v_{i+1} - v_i$ and have

$$\begin{aligned}
w_{i,tt} &+ \lambda\sigma b(x,t,v_{i-1})w_{i,xt} - \lambda^2\sigma^2 a(x,t,v_{i-1})w_{i,xx} \\
&= F_p(x,t,v_i,v_{i,t},\sigma\lambda' v_{i,x}) - F_p(x,t,v_{i-1},v_{i-1,t},\sigma\lambda' v_{i-1,x}) \\
&\quad + \lambda\sigma(b_p(x,t,v_{i-1}) - b_p(x,t,v_i))v_{i+1,xt} \\
&\quad - \lambda^2\sigma^2(a_p(x,t,v_{i-1}) - a_p(x,t,v_i))v_{i+1,xx} \\
&= F_p(x,t,v_i,v_{i,t},\sigma\lambda' v_{i,x}) - F_p(x,t,v_{i-1},v_{i-1,t},\sigma\lambda' v_{i-1,x}) \\
&\quad + \lambda\sigma b_i(x,t)w_{i-1}v_{i+1,xt} + \lambda^2\sigma^2 a_i(x,t)w_{i-1}v_{i+1,xx} \\
&=: g_i(x,t).
\end{aligned}$$

By Hadamard's Formula and the choice of Q_1 we have

$$\begin{aligned}
&\|F_p(x,t,v_i,v_{i-1,t},\sigma\lambda' v_{i-1,x}) - F_p(x,t,v_{i-1},v_{i-1,t},\sigma\lambda' v_{i-1,x})\|_2 \\
&\quad + \|F_p(x,t,v_i,v_{i,t},\sigma\lambda' v_{i-1,x}) - F_p(x,t,v_i,v_{i-1,t},\sigma\lambda' v_{i-1,x})\|_2 \\
&\quad \le C_f S_0(w_{i-1}), \\
&\|F_p(x,t,v_i,v_{i,t},\sigma\lambda' v_{i,x}) - F_p(x,t,v_i,v_{i,t},\sigma\lambda' v_{i-1,x})\|_2 \\
&\quad \le Q_1 \frac{\lambda'(t)}{\lambda(t)} S_0(w_{i-1}), \\
&\|\lambda\sigma b_i(x,t)w_{i-1}v_{i+1,xt}\|_2 \le C_P C_b \|\sigma\|_\infty \lambda(t)^{p+1} S_0(w_{i-1}), \\
&\|\lambda^2\sigma^2 a_i(x,t)w_{i-1}v_{i+1,xx}\|_2 \le C_P C_a \|\sigma\|_\infty^2 \lambda(t)^{p+2} S_0(w_{i-1}).
\end{aligned}$$

With the assumption

$$\begin{aligned}
2e^{C_0T_0}C_{S,0}C_KT_0 := 2e^{C_0T_0}C_{S,0}(C_f &+ C_PC_b\|\sigma\|_\infty \lambda(T_0)^{p+1} \\
&+ C_PC_a\|\sigma\|_\infty^2 \lambda(T_0)^{p+2})T_0 \le \frac{1}{4}
\end{aligned}$$

we obtain

$$\|g_i\|_2 \le C_K S_0(w_{i-1}) + Q_1 \frac{\lambda'(t)}{\lambda(t)} S_0(w_{i-1}).$$

We suppose $S_0(w_{i-1})(t) \le C^w_{i-1}\lambda(t)^p$. Without loss of generality we may assume that the sequences (C_q), $(C_{S,q})$ are monotonically increasing with

q. From this and Proposition 4.2 it follows that

$$\begin{aligned} C_{S,0}\mathcal{E}_0(w_i)(t) &\le 2C_{S,0}\int_0^t e^{C_0(t-\tau)}\left(\frac{\lambda(t)}{\lambda(\tau)}\right)^{Q'} \|g_i\|_2(\tau)d\tau \\ &\le C_{i-1}^w\left(2e^{C_0T_0}C_{S,0}C_Kt+\frac{2Q_1C_{S,0}}{p-Q'}e^{C_0T_0}\right)\lambda(t)^p \\ &\le \frac{3}{4}C_{i-1}^w\lambda(t)^p. \end{aligned}$$

Hence we obtain $S_0(w_{i-1})(t) \le C_0^w(\frac{3}{4})^i\lambda(t)^p$. Nirenberg–Gagliardo Interpolation and $S_M(w_i)(t) \le 2\lambda(t)^p$ give the assertion. The Lemma is proved. $\square$

By standard arguments one can show that the limit v is a solution of the equation

$$\begin{aligned} L_p^{(v)}v &= v_{tt}+\sigma\lambda b_p(x,t,v)v_{xt}-\sigma^2\lambda^2a_p(x,t,v)v_{xx} \\ &= F_p(x,t,v,v_t,\sigma\lambda' v_x). \end{aligned}$$

It follows that $u=\sum^p u^{(j)}+v$ solves (4.1) and (1.5). The Theorem 4.1 is proved.

Remark 4.1. Obviously, one can show a similar theorem for systems with diagonal principal part.

5. Fully nonlinear weakly hyperbolic equations

It is sufficient to show that a fully nonlinear weakly hyperbolic Cauchy problem is equivalent to a suitably chosen weakly hyperbolic quasilinear Cauchy system, see Remarks 1.1, 2.1, 3.1, 4.1. We will divide this proof into two parts.

Theorem 5.1 *Let*

$$\begin{aligned} &F_j \in C^1([0,T],C^\infty(\mathbb{R}^6)\times H^5_{\mathrm{per}}(P)), \\ &u,u_t,\sigma\lambda u_x \in C^2([0,T],H^5_{\mathrm{per}}(P)), \\ &\lambda\in C^2([0,T]), \quad \sigma\in H^7_{\mathrm{per}}(P). \end{aligned}$$

The function u is a solution of the Cauchy problem (0.1), (0.2), *if and only if* $(u_0,u_1,u_2):=(u,u_t,u_x)$ *is a solution of the system*

$$F_1u_{1,tt}+\sigma\lambda F_2u_{1,xt}+\sigma^2\lambda^2F_3u_{1,xx}+\sigma\lambda' F_4u_{1,x}+F_5u_{1,t}+F_6u_1+F_8$$

$$+\sigma\lambda' F_2 u_{2,t} + \sigma^2(\lambda^2)' F_3 u_{2,x} + \sigma\lambda'' F_4 u_2 = 0, \tag{5.1}$$

$$\begin{aligned} F_1 u_{2,tt} &+ \sigma\lambda F_2 u_{2,xt} + \sigma^2\lambda^2 F_3 u_{2,xx} + \sigma\lambda' F_4 u_{2,x} + F_5 u_{2,t} + F_6 u_2 + F_7 \\ &+ \sigma'\lambda F_2 u_{2,t} + (\sigma^2)'\lambda^2 F_3 u_{2,x} + \sigma'\lambda' F_4 u_2 = 0, \end{aligned} \tag{5.2}$$

$$\begin{aligned} F_1 u_{0,tt} &+ \sigma\lambda F_2 u_{0,xt} + \sigma^2\lambda^2 F_3 u_{0,xx} + \sigma\lambda' F_4 u_{0,x} \\ &- F_1 u_{1,t} - \sigma\lambda F_2 u_{2,t} - \sigma^2\lambda^2 F_3 u_{2,x} - \sigma\lambda' F_4 u_2 + F = 0, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &u_1(x,0) = \varphi_1(x), \quad u_{1,t}(x,0) = \varphi_2(x), \\ &u_2(x,0) = \varphi_{0,x}(x), \quad u_{1,t}(x,0) = \varphi_{1,x}(x), \\ &u_0(x,0) = \varphi_0(x), \quad u_{1,t}(x,0) = \varphi_1(x), \end{aligned}$$

where the function F and the derivatives F_j depend on

$$(u_{1,t}, \sigma\lambda u_{2,t}, \sigma^2\lambda^2 u_{2,x}, \sigma\lambda' u_2, u_{0,t}, u_0, x, t).$$

For the definition of φ_2 see the assumption A8.

Proof. We restrict us to show the $\Leftarrow$-direction.

Differentiating (5.3) with respect to t and subtraction from (5.1) gives

$$\begin{aligned} F_1(u_1 - u_{0,t})_{tt} &+ F_2\sigma\lambda(u_1 - u_{0,t})_{xt} + F_3\sigma^2\lambda^2(u_1 - u_{0,t})_{xx} \\ &+ F_4\sigma\lambda'(u_1 - u_{0,t})_x - (F_{1,t}(u_{0,t} - u_1)_t + F_{2,t}\sigma\lambda(u_{0,x} - u_2)_t \\ &+ F_{3,t}\sigma^2\lambda^2(u_{0,x} - u_2)_x) - (F_{4,t}\sigma\lambda'(u_{0,x} - u_2) \\ &+ F_2\sigma\lambda'(u_{0,x} - u_2)_t + F_3\sigma^2(\lambda^2)'(u_{0,x} - u_2)_x) - F_4\sigma\lambda''(u_{0,x} - u_2) \\ &+ F_5(u_1 - u_{0,t})_t + F_6(u_1 - u_{0,t}) = 0. \end{aligned} \tag{5.4}$$

Differentiating (5.3) with respect to x and subtracting from (5.2) yields

$$\begin{aligned} F_1(u_2 - u_{0,x})_{tt} &+ F_2\sigma\lambda(u_2 - u_{0,x})_{xt} + F_3\sigma^2\lambda^2(u_2 - u_{0,x})_{xx} \\ &+ F_4\sigma\lambda'(u_2 - u_{0,x})_x - (F_{1,x}(u_{0,t} - u_1)_t + F_{2,x}\sigma\lambda(u_{0,x} - u_2)_t \\ &+ F_{3,x}\sigma^2\lambda^2(u_{0,x} - u_2)_x) - (F_{4,x}\sigma\lambda'(u_{0,x} - u_2) \\ &+ F_2\sigma'\lambda(u_{0,x} - u_2)_t + F_3(\sigma^2)'\lambda^2(u_{0,x} - u_2)_x) - F_4\sigma'\lambda'(u_{0,x} - u_2) \\ &+ F_5(u_2 - u_{0,x})_t + F_6(u_2 - u_{0,x}) = 0. \end{aligned} \tag{5.5}$$

The system (5.4), (5.5) is a linear homogeneous weakly hyperbolic system for the functions $v_1 = u_{0,t} - u_1$, $v_2 = u_{0,x} - u_2$. The Levi conditions are satisfied. From the uniqueness of periodic solutions we get $v_1 \equiv v_2 \equiv 0$. Combining this result with (5.3) gives the assertion. $\square$

The next theorem will reduce the quasilinear principal part to a semilinear one.

Theorem 5.2 *We assume*

$$\begin{aligned}
&a, b \in C^1([0,T], C^\infty(\mathbb{R}_u) \times H^4_{\text{per}}(P)),\\
&f \in C([0,T], C^\infty(\mathbb{R}_u) \times H^4_{\text{per}}(P)),\\
&\vec{u}, \vec{u}_t, \sigma\lambda\vec{u}_x \in C^1([0,T], H^4_{\text{per}}(P)),\\
&\lambda \in C^2([0,T]).
\end{aligned}$$

The vector $\vec{u}$ is a solution of

$$\begin{aligned}
&\vec{u}_{tt} + \sigma\lambda b(\vec{u}, \vec{u}_t, \sigma\lambda'\vec{u}_x, x, t)\vec{u}_{xt} - \sigma^2\lambda^2 a(\vec{u}, \vec{u}_t, \sigma\lambda'\vec{u}_x, x, t)\vec{u}_{xx}\\
&\quad = \vec{f}(\vec{u}, \vec{u}_t, \sigma\lambda'\vec{u}_x, x, t),\\
&\vec{u}(x,0) = \vec{\varphi}_0(x), \quad \vec{u}_t(x,0) = \vec{\varphi}_1(x)
\end{aligned}$$

if and only if the vector $(\vec{u}_0, \vec{u}_1, \vec{u}_2) := (\vec{u}, \vec{u}_t, \vec{u}_x)$ is a solution of

$$\begin{aligned}
&\vec{u}_{1,tt} + \sigma\lambda b\vec{u}_{1,xt} - \sigma^2\lambda^2 a\vec{u}_{1,xx} + \sigma\lambda b_t\vec{u}_{2,t} + \sigma\lambda' b\vec{u}_{2,t}\\
&\quad - \sigma^2\lambda^2 a_t\vec{u}_{2,x} - \sigma^2(\lambda^2)' a\vec{u}_{2,x} = \vec{f}_t,
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
&\vec{u}_{2,tt} + \sigma\lambda b\vec{u}_{2,xt} - \sigma^2\lambda^2 a\vec{u}_{2,xx} + \sigma\lambda b_x\vec{u}_{2,t} + \sigma'\lambda b\vec{u}_{2,t}\\
&\quad - \sigma^2\lambda^2 a_x\vec{u}_{2,x} - (\sigma^2)'\lambda^2 a\vec{u}_{2,x} = \vec{f}_1\vec{u}_2 + \vec{f}_2\vec{u}_{2,t} + \vec{f}_3(\sigma\lambda'\vec{u}_2)_x + \vec{f}_4,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
&b_x = b_1\vec{u}_2 + b_2\vec{u}_{2,t} + b_3(\sigma\lambda'\vec{u}_2)_x + b_4,\\
&a_x = a_1\vec{u}_2 + a_2\vec{u}_{2,t} + a_3(\sigma\lambda'\vec{u}_2)_x + a_4,
\end{aligned}$$

$$\vec{u}_{0,tt} + \sigma\lambda b\vec{u}_{0,xt} - \sigma^2\lambda^2 a\vec{u}_{0,xx} = \vec{f} \tag{5.8}$$

with the initial conditions

$$\begin{aligned}
&\vec{u}_1(x,0) = \vec{\varphi}_1(x), \quad \vec{u}_{1,t}(x,0) = \vec{\varphi}_2(x),\\
&\vec{u}_2(x,0) = \vec{\varphi}_{0,x}(x), \quad \vec{u}_{2,t}(x,0) = \vec{\varphi}_{1,x}(x),\\
&\vec{u}_0(x,0) = \vec{\varphi}_0(x), \quad \vec{u}_{0,t}(x,0) = \vec{\varphi}_1(x).
\end{aligned}$$

The functions b, a, b_t, a_t depend on $(\vec{u}_0, \vec{u}_1, \sigma\lambda'\vec{u}_2, x, t)$ and $\vec{f}$, $\vec{f}_t$, $\vec{f}_j$ depend on $(\vec{u}_0, \vec{u}_{0,t}, \sigma\lambda'\vec{u}_2, x, t)$. The vector $\vec{\varphi}_2$ is defined similar to φ_2 in assumption A8.

Proof. We consider only the $\Leftarrow$-direction.

We differentiate (5.8) with respect to x, subtract the equation from (5.7) and obtain

$$
\begin{aligned}
&(\vec{u}_2-\vec{u}_{0,x})_{tt}+\sigma\lambda b(\vec{u}_2-\vec{u}_{0,x})_{xt}-\sigma^2\lambda^2 a(\vec{u}_2-\vec{u}_{0,x})_{xx}\\
&\quad+\sigma'\lambda b(\vec{u}_2-\vec{u}_{0,x})_t+\sigma\lambda(b_1\vec{u}_2+b_2\vec{u}_{2,t}+b_3(\sigma\lambda'\vec{u}_2)_x+b_4)\vec{u}_{2,t}\\
&\quad-\sigma\lambda(b_1\vec{u}_{0,x}+b_2\vec{u}_{1,x}+b_3(\sigma\lambda'\vec{u}_2)_x+b_4)\vec{u}_{0,xt}\\
&\quad-(\sigma^2)'\lambda^2 a(\vec{u}_2-\vec{u}_{0,x})_x-\sigma^2\lambda^2(a_1\vec{u}_2+a_2\vec{u}_{2,t}\\
&\qquad\qquad+a_3(\sigma\lambda'\vec{u}_2)_x+a_4)\vec{u}_{2,x}\\
&\quad+\sigma^2\lambda^2(a_1\vec{u}_{0,x}+a_2\vec{u}_{1,x}+a_3(\sigma\lambda'\vec{u}_2)_x+a_4)\vec{u}_{0,xx}\\
&=\vec{f}_1(\vec{u}_2-\vec{u}_{0,x})+\vec{f}_2(\vec{u}_2-\vec{u}_{0,x})_t.
\end{aligned}
\tag{5.9}
$$

Differentiating (5.8) with respect to t and subtraction from (5.6) implies

$$
\begin{aligned}
&(\vec{u}_1-\vec{u}_{0,t})_{tt}+\sigma\lambda b(\vec{u}_1-\vec{u}_{0,t})_{xt}-\sigma^2\lambda^2 a(\vec{u}_1-\vec{u}_{0,t})_{xx}\\
&\quad+\sigma\lambda' b(\vec{u}_2-\vec{u}_{0,x})_t+\sigma\lambda b_t(\vec{u}_2-\vec{u}_{0,x})_t-\sigma^2(\lambda^2)' a(\vec{u}_2-\vec{u}_{0,x})_x\\
&\quad-\sigma^2\lambda^2 a_t(\vec{u}_2-\vec{u}_{0,x})_x=0.
\end{aligned}
\tag{5.10}
$$

The equations (5.9), (5.10) are a weakly hyperbolic linear homogeneous system for the functions $\vec{v}_1=\vec{u}_{0,t}-\vec{u}_1$, $\vec{v}_2=\vec{u}_{0,x}-\vec{u}_2$. We leave it to the reader to verify that the Levi conditions are satisfied. Hence one obtains $\vec{v}_1\equiv\vec{v}_2\equiv 0$. □

6. Examples

Example 1. The methods presented in this paper enable us to study equations with, e.g.,

$$\lambda(t)=t^l\exp\left(-\frac{1}{t^\alpha}\right),\quad \sigma(x)=(\sin x)^k\sin\left(\frac{1}{\sin x}\right),$$

where $l\in\mathbb{Z}$, $\alpha\in\mathbb{R}^+$, $k\in\mathbb{N}$ large. The degeneration occurs in the set

$$
\begin{aligned}
&(\mathbb{R}\times\{0\})\cup(\{m\pi:m\in\mathbb{Z}\}\times[0,T])\\
&\qquad\bigcup\left(\left\{\arcsin\left(\frac{1}{m\pi}\right):m\in\mathbb{Z}\backslash\{0\}\right\}\times[0,T]\right).
\end{aligned}
$$

The following example goes back to Qi Min-you [Qi58].

Example 2. We consider

$$u_{tt}-t^2u_{xx}=au_x,\quad u(x,0)=\varphi(x),\quad u_t(x,0)=0.$$

If $a = const.$, $a = 4n+1$, where $n \geq 0$ is an integer, then the unique solution has the representation

$$u(x,t) = \sum_{k=0}^{n} \frac{\sqrt{\pi} t^{2k}}{k!(n-k)!\Gamma(k+\frac{1}{2})} \varphi^{(k)}\left(x + \frac{1}{2}t^2\right).$$

One can observe two interesting effects. First, a loss of regularity occurs, which depends on the value of the coefficient of the lower order term. Second, singularities of the initial data propagate along the characteristic $x + \frac{1}{2}t^2 = const.$ Propagation along the other characteristic $x - \frac{1}{2}t^2 = const.$ does not happen.

We will generalize this example.

Example 3. We consider the Cauchy problem

$$u_{tt} + bt^l u_{xt} - at^{2l} u_{xx} - dt^{l-1} u_x = 0, \tag{6.1}$$

$$u(x,0) = \varphi(x), \quad u_t(x,0) = 0, \tag{6.2}$$

where a, b, $d \in \mathbb{R}$, $l \in \mathbb{N}$, $l \geq 1$.

The ansatz $u(x,t) = \sum_{k=0}^{n} c_k t^{(l+1)k} \varphi^{(k)}(x + \kappa t^{l+1})$ leads to

$$\begin{aligned}
c_0 &= 1, \\
c_k &= -\frac{\kappa(l+1)(2(l+1)(k-1)+l) + b(l+1)(k-1) - d}{(l+1)k((l+1)k-1)} c_{k-1}, \\
\kappa_{1,2} &= -\frac{1}{2(l+1)}(b \mp \sqrt{b^2+4a}), \\
n &= \frac{-l(l+1)\kappa + d}{2(l+1)^2\kappa + (l+1)b}.
\end{aligned}$$

This gives for $\kappa = \kappa_1$ or $\kappa = \kappa_2$

$$\begin{aligned}
n_1 &= \frac{l}{2(l+1)}\left(\frac{b}{\sqrt{b^2+4a}} - 1\right) + \frac{d}{(l+1)\sqrt{b^2+4a}}, \\
n_2 &= -\frac{l}{2(l+1)}\left(\frac{b}{\sqrt{b^2+4a}} + 1\right) - \frac{d}{(l+1)\sqrt{b^2+4a}},
\end{aligned}$$

respectively. The natural number n describes the loss of Sobolev regularity.

It is worth to point out that $x + \kappa t^{l+1} = x + \tau_{1,2}\Lambda(t)$, where $\tau_{1,2}$ are the characteristic roots and $\Lambda(t) = \int_0^t \lambda(s)\, ds$, where $\lambda(t) = t^l$. Additionally, we emphasize that it is not possible to construct a solution as a linear combination of $\sum_{j=0}^{n_1} \cdots$ and $\sum_{j=0}^{n_2} \cdots$, since $n_1 + n_2 < 0$.

The question arises of whether the solution suffers from a loss of generality if the expressions for n_1 and n_2 are no integers, too. Here one can not express the solution by the above finite sum. But there are other explicit representations. The following ideas go back to Taniguchi and Tozaki [TT80], who studied equations (6.1) with $b = 0$ and $a = 1$.

We apply partial Fourier transform with respect to x and obtain with $y(t;\xi) = F_{x\to\xi}u(x,t)$ the ordinary differential equation with parameter ξ

$$y''(t) + ibt^l\xi y'(t) + at^{2l}\xi^2 y(t) - idt^{l-1}\xi y(t) = 0.$$

We transform the time variable,

$$\tau := \Lambda(t)\xi, \quad v(\tau) := y(t;\xi)$$

and conclude that

$$v''(\tau) + \left(\frac{1}{\tau}\frac{l}{l+1} + ib\right) v'(\tau) + \left(a - id\frac{1}{l+1}\frac{1}{\tau}\right) v = 0.$$

We change the variables again,

$$z := ri\tau, \quad w(z) := v\left(\frac{z}{ri}\right) e^{\frac{z}{s}},$$

which gives

$$\begin{aligned} w''(z) &+ \left(\frac{b}{r} - \frac{2}{s} + \frac{l}{l+1}\frac{1}{z}\right) w'(z) \\ &+ \left(\frac{1}{s^2} - \frac{b}{rs} - \frac{a}{r^2} - \frac{l}{l+1}\frac{1}{sz} - \frac{1}{l+1}\frac{d}{rz}\right) w(z) = 0. \end{aligned}$$

We choose

$$\frac{1}{s^2} - \frac{b}{rs} - \frac{a}{r^2} = 0, \quad \frac{2}{s} - \frac{b}{r} = 1, \quad |r| = \sqrt{b^2 + 4a}.$$

The result is

$$\begin{aligned} &zw''(z) + (\gamma - z)w'(z) - \alpha w(z) = 0, \\ &\gamma = \frac{l}{l+1}, \quad \alpha = \frac{l}{2(l+1)}\left(\frac{b}{\sqrt{b^2+4a}} + 1\right) + \frac{d}{(l+1)\sqrt{b^2+4a}}. \end{aligned}$$

This is a confluent hypergeometric differential equation. Two linearly independent solutions are

$$w_1(z) = {}_1\mathrm{F}_1(\alpha, \gamma, z), \quad w_2(z) = z^{1-\gamma}{}_1\mathrm{F}_1(1+\alpha-\gamma, 2-\gamma, z).$$

These series are polynomials if $-\alpha$ or $\alpha - \gamma$ are nonnegative integers, respectively. We have for $|z| \to \infty$ the asymptotic expansion

$$\begin{aligned} &\frac{{}_1\mathrm{F}_1(\alpha,\gamma,z)}{\Gamma(\gamma)} \\ &\quad = \frac{e^{\pm i\pi\alpha} z^{-\alpha}}{\Gamma(\gamma-\alpha)} \left(\sum_{n=0}^{R-1} \frac{(\alpha)_n (1+\alpha-\gamma)_n}{n!} (-z)^{-n} + O(|z|^{-R}) \right) \\ &\quad + \frac{e^z z^{\alpha-\gamma}}{\Gamma(\alpha)} \left(\sum_{n=0}^{S-1} \frac{(\gamma-\alpha)_n (1-\alpha)_n}{n!} z^{-n} + O(|z|^{-S}) \right), \end{aligned}$$

the upper sign been taken if $-\frac{\pi}{2} < \arg z < \frac{3}{2}\pi$, the lower sign if $-\frac{3}{2}\pi < \arg z \leq -\frac{1}{2}\pi$ ([AS84], 13.5.1). Here one can see the special role of the exponents $-\alpha = n_2$ and $\alpha - \gamma = n_1$. At least one of this exponents is negative, since $n_1 + n_2 = -\gamma < 0$. This negative exponent gives no loss of regularity.

After some computations one obtains

$$u(x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} p_0(t,\xi) \hat{\varphi}(\xi)\, d\xi,$$

where the function p_0 has the asymptotic behaviour $|\xi|^{-\alpha}$ or $|\xi|^{\alpha-\gamma}$ for $|\xi| \to \infty$.

Summary Let $\varphi \in H^s(\mathbb{R})$, $s \in \mathbb{R}$ and $n := \max(-\alpha, \alpha - \gamma)$. Then there exists a solution $u \in H^{s-n}(\mathbb{R})$.

Remark 6.1. We consider the Cauchy problem (0.2),

$$u_{tt} + b\lambda(t) u_{xt} - c\lambda(t)^2 u_{xx} - d\lambda'(t) u_x = 0,$$

where b, d, c are real constants and $b^2 + 4c > 0$. We define

$$\Lambda(t) := \int_0^t \lambda(\tau)\, d\tau, \quad y := x - \frac{b}{2}\Lambda(t), \quad v(y,t) := u(x,t)$$

and get

$$v_{tt} - \left(\frac{b^2}{4} + c\right)\lambda^2 v_{yy} - \left(\frac{b}{2} + d\right)\lambda' v_y = 0.$$

The transformation

$$y =: z\sqrt{\frac{b^2}{4} + c}, \quad w(z,t) := v(y,t)$$

yields

$$w_{tt} - \lambda(t)^2 w_{zz} - \frac{b+2d}{\sqrt{b^2+4c}}\lambda'(t) w_z = 0.$$

This reduction makes the results of Aleksandrian [Ale84] and Taniguchi–Tozaki [TT80] applicable to equations which have a term u_{xt}.

A. Appendix

Proof of Lemma 3.1 We use Lemma A.1 and get

$$\begin{aligned} E_N(\partial_j^{(h)} v) &= E_N(v_t - \beta_j(x,t,h)\sigma v_x) \\ &\le \|v_t\|_{H^N(P)} + C_{\mathrm{prod},N} \|\beta_j(x,t,h)\|_{H^N(P)} \|\sigma v_x\|_{H^N(P)} \\ &\le C(1 + \|h\|_{H^N(P)}) S_N(v). \end{aligned}$$

Similarly, we conclude that

$$\begin{aligned} \|\sigma v_x\|_{H^N(P)} &= \left\| \frac{(\partial_1^{(h)} - \partial_2^{(h)})v}{\beta_1^{(h)} - \beta_2^{(h)}} \right\|_{H^N(P)} \\ &\le C_{\mathrm{prod},N} \left\| \frac{1}{\beta_1^{(h)} - \beta_2^{(h)}} \right\|_{H^N(P)} \mathcal{E}_N^{(h)}(v) \\ &\le C(1 + \|h\|_{H^N(P)}) \mathcal{E}_N^{(h)}(v), \\ \|v_t\|_{H^N(P)} &\le \left\| \frac{\beta_2^{(h)}}{\beta_2^{(h)} - \beta_1^{(h)}} \partial_1^{(h)} v \right\|_{H^N(P)} + \left\| \frac{\beta_1^{(h)}}{\beta_2^{(h)} - \beta_1^{(h)}} \partial_2^{(h)} v \right\|_{H^N(P)} \\ &\le C(1 + \|h\|_{H^N(P)}) \mathcal{E}_N^{(h)}(v). \end{aligned}$$

□

Lemma A.1 *Let $f \in C^N(P \times K)$, where $K \subset \mathbb{R}^n$ and $P \subset \mathbb{R}$ are compact sets. Let $v_i \in H^N_{\mathrm{per}}(P)$ with $(x, v_1(x), \dots, v_n(x)) \in K$ for all $x \in P$.*
(a) *Then,*

$$E_N(f(., v_1(.), \dots, v_n(.)))$$

$$\leq \varphi_N(\|v_1\|_\infty, \ldots, \|v_n\|_\infty)(E_N(v_1) + \cdots + E_N(v_n) + 1).$$

(b) *More precisely, if $N \geq 3$, then*

$$\begin{aligned}
&E_N(f(.,v_1(.),\ldots,v_n(.))) \\
&\quad \leq \varphi_N(\|v_1\|_\infty, \|v_{1,x}\|_\infty, \ldots, \|v_n\|_\infty, \|v_{n,x}\|_\infty) \\
&\qquad\qquad \times (E_N(v_1) + \cdots + E_N(v_{n-1}) + E_{N-1}(v_n)) \\
&\qquad + \|f^{(0,\ldots,0,1)}(x, v_1(x), \ldots, v_n(x))\|_\infty \|\partial_x^N v_n\|_2 \\
&\qquad + \sum_{j=0}^{N} \|f^{(j,0,\ldots,0)}(x, v_1(x), \ldots, v_n(x))\|_2.
\end{aligned}$$

Proof. We make use of the Leibniz formula

$$\begin{aligned}
&d_x^j f(x, v_1(x), \ldots, v_n(x)) \\
&= \sum_{i+l=j} \frac{j!}{i!} \sum_{l_1+\ldots+l_n=l} \sum_{\nu_1=0}^{l_1} \cdots \sum_{\nu_n=0}^{l_n} \frac{1}{\nu_1! \cdots \nu_n!} f^{(i,\nu_1,\ldots,\nu_n)}(x, v_1, \ldots, v_n) \\
&\qquad \times \prod_{k=1}^{n} \left(\sum_{\substack{h_{k,1}+\cdots h_{k,\nu_k}=l_k \\ h_{k,m}\geq 1}} \frac{(\partial_x^{h_{k,1}} v_k) \cdots (\partial_x^{h_{k,\nu_k}} v_k)}{h_{k,1}! \cdots h_{k,\nu_k}!} \right),
\end{aligned}$$

where we use the following convention: If $\nu_k = 0$, then $l_k = 0$ and if all ν_k are 0, then $i = j$.

Another tool is the generalized Interpolation Inequality of Nirenberg-Gagliardo for periodical functions,

$$\begin{aligned}
&\|\partial_x^j v\|_r \leq C \|\partial_x^m v\|_p^{\frac{j-n}{m-n}} \|\partial_x^n v\|_q^{1-\frac{j-n}{m-n}}, \quad n \leq j \leq m, \qquad \text{(A.1)} \\
&\frac{1}{r} = \frac{j-n}{p(m-n)} + \frac{1}{q}\left(1 - \frac{j-n}{m-n}\right).
\end{aligned}$$

By Hölder's Inequality,

$$\begin{aligned}
&\|d_x^j f(x, v_1(x), \ldots, v_n(x))\|_2 \\
&\quad \leq C_j \sum_{i+l=j} \sum_{l_1+\ldots+l_n=l} \sum_{\nu_1=0}^{l_1} \cdots \sum_{\nu_n=0}^{l_n} \|f^{(i,\nu_1,\ldots,\nu_n)}(x, v_1, \ldots, v_n)\|_\infty
\end{aligned}$$

$$\times \prod_{k=1}^{n} \left(\sum_{\substack{h_{k,1}+\cdots h_{k,\nu_k}=l_k \\ h_{k,m}\geq 1}} \|(\partial_x^{h_{k,1}} v_k)^2\|_{\alpha_{k,1}}^{\frac{1}{2}} \cdots \|(\partial_x^{h_{k,\nu_k}} v_k)^2\|_{\alpha_{k,\nu_k}}^{\frac{1}{2}} \right),$$

where

$$\sum_{k=1}^{n} \sum_{s=1}^{\nu_k} \frac{1}{\alpha_{k,s}} \leq 1.$$

We choose $\alpha_{k,s} = \frac{l}{h_{k,s}}$ and apply (A.1) with $n = 0$, $r = 2\alpha_{k,s}$, $p = 2$, $q = \infty$, $m = l$, $j = h_{k,s}$. The result is

$$\|\partial_x^{h_{k,s}} v_k\|_{2\alpha_{k,s}} \leq C \|\partial_x^l v_k\|_2^{\frac{h_{k,s}}{l}} \|v_k\|_{\infty}^{1-\frac{h_{k,s}}{l}},$$

thus

$$\begin{aligned} &\|d_x^j f(x, v_1(x), \ldots, v_n(x))\|_2 \\ &\leq \tilde{\tilde{\varphi}}_N(\|v_1\|_\infty, \ldots, \|v_n\|_\infty) \left(\sum_{i+l=j} \sum_{l_1+\ldots+l_n=l} \prod_{k=1}^{n} \|\partial_x^l v_k\|_2^{\frac{l_k}{l}} + 1 \right) \\ &\leq \tilde{\varphi}_N(\|v_1\|_\infty, \ldots, \|v_n\|_\infty)(\|v_1\|_{H^N(P)} + \cdots + \|v_n\|_{H^N(P)} + 1). \end{aligned}$$

□

Proof of (b) We use the finer estimate

$$\begin{aligned} &\|d_x^N f(x, v_1(x), \ldots, v_n(x))\|_2 \\ &\leq C_j \sum_{i+l=N} \sum_{l_1+\cdots+l_n=l} \sum_{\nu_1=0}^{l_1} \cdots \sum_{\nu_n=0}^{l_n} {}^{\circ} \|f^{(i,\nu_1,\ldots,\nu_n)}(x, v_1, \ldots, v_n)\|_\infty \\ &\quad \times \prod_{k=1}^{n} \left(\sum_{\substack{h_{k,1}+\cdots h_{k,\nu_k}=l_k \\ h_{k,m}\geq 1}} \|(\partial_x^{h_{k,1}} v_k)^2\|_{\alpha_{k,1}}^{\frac{1}{2}} \cdots \|(\partial_x^{h_{k,\nu_k}} v_k)^2\|_{\alpha_{k,\nu_k}}^{\frac{1}{2}} \right) \\ &\quad + \|f^{(0,\ldots,0,1)}\|_\infty \|\partial_x^N v_n\|_2 + \|f^{(N,0,\ldots,0)}\|_2, \end{aligned}$$

where the "$\circ$" means that terms with $l_n = l = N$, $\nu_n = 1$ or $i = N$ do not occur. We apply (A.1) with

$$r = 2\alpha_{k,s} := \begin{cases} \dfrac{2(l-2)}{h_{k,s}-1} & : k = n, \\[2ex] \dfrac{2(l-1)}{h_{k,s}-1} & : k < n, \end{cases} \qquad m = \begin{cases} l-1 & : k = n, \\ l & : k < n, \end{cases}$$

and $p = 2$, $q = \infty$, $j = h_{k,s}$, $n = 1$. The result is

$$\begin{aligned} \|\partial_x^{h_{k,s}} v_k\|_{2\alpha_{k,s}} &\le C \|\partial_x^l v_k\|_2^{\frac{1}{\alpha_{k,s}}} \|v_{k,x}\|_\infty^{1-\frac{1}{\alpha_{k,s}}} \quad (k < n), \\ \|\partial_x^{h_{n,s}} v_n\|_{2\alpha_{n,s}} &\le C \|\partial_x^{l-1} v_n\|_2^{\frac{1}{\alpha_{n,s}}} \|v_{n,x}\|_\infty^{1-\frac{1}{\alpha_{n,s}}}. \end{aligned}$$

We have to check whether $\sum_{k,s} \frac{1}{\alpha_{k,s}} \le 1$. It holds

$$S := \sum_{k=1}^{n} \sum_{s=1}^{\nu_k} \frac{1}{\alpha_{k,s}} = \sum_{k=1}^{n-1} \frac{l_k - \nu_k}{l-1} + \frac{l_n - \nu_n}{l-2}.$$

We have to distinguish three cases:

$l_n = l$: This gives $l_1 = \cdots = l_{n-1} = 0$ and $\nu_n \ge 2$, hence $S \le 1$.

$1 \le l_n \le l-1$: At least one ν_k $(k < n)$ is positive, hence $\sum_{k=1}^{n-1} l_k - \nu_k \le l - l_n - 1$.

It follows that

$$S \le \frac{l - l_n - 1}{l-1} + \frac{l_n - \nu_n}{l-2} \le 1 - \frac{l_n}{l-1} + \frac{l_n - 1}{l-2} \le 1.$$

$l_n = 0$: (trivial)

We conclude that

$$\begin{aligned} &\|d_x^N f(x, v_1(x), \dots, v_n(x))\|_2 \\ &\quad \le \tilde{\tilde{\varphi}}_N(\|v_1\|_\infty, \dots, \|v_n\|_\infty) \\ &\qquad \times \left(\sum_{\beta \in B} \left(\prod_{k=1}^{n-1} \|\partial_x^N v_k\|_2^{\gamma_{\beta,k}} \|v_{k,x}\|_\infty^{\gamma'_{\beta,k}} \right) \|\partial_x^{N-1} v_n\|_2^{\gamma_{\beta,n}} \|v_{n,x}\|_\infty^{\gamma'_{\beta,n}} \right) \\ &\qquad + \|f^{(0,\dots,0,1)}\|_\infty \|\partial_x^N v_n\|_2 + \sum_{j=0}^{N} \|f^{(j,0,\dots,0)}\|_2, \end{aligned}$$

where B is some index set. It holds $\sum_{k=1}^{n} \gamma_{\beta,k} \le 1$. If this sum is strictly less than 1, we use the embeddings $H^{N-1}(P) \subset L^\infty(P)$ and $H^{N-2}(P) \subset L^\infty(P)$

to increase the exponents of $\|\partial_x^N v_k\|_2$ and $\|\partial_x^{N-1} v_n\|_2$ such that the sum becomes 1.

The application of Young's Inequality completes the proof. $\square$

Remark A.1. After obvious modifications one can prove for every $0 \le m < n$ the generalization

$$\begin{aligned} &E_N(f(.,v_1(.),\dots,v_n(.))) \\ &\le \varphi_N(\|v_1\|_\infty, \|v_{1,x}\|_\infty, \dots, \|v_n\|_\infty, \|v_{n,x}\|_\infty) \\ &\quad \times (E_N(v_1) + \cdots + E_N(v_m) + E_{N-1}(v_{m+1}) + \cdots + E_{N-1}(v_n)) \\ &\quad + \sum_{j=m}^{n} \|f^{(0,\dots,0,1,0,\dots,0)}(x, v_1(x), \dots, v_n(x))\|_\infty \|\partial_x^N v_j\|_2 \\ &\quad + \sum_{j=0}^{N} \|f^{(j,0,\dots,0)}(x, v_1(x), \dots, v_n(x))\|_2. \end{aligned}$$

Lemma A.2 *Let $\sigma \in H^{N+1}_{\mathrm{per}}(P)$, $a \in H^N_{\mathrm{per}}(P)$ and $\sigma a \in H^{N+1}_{\mathrm{per}}(P)$, where $N \ge 2$. Let $h \in C_0^\infty(\mathbb{R})$ be a function with*

$$\int_{\mathbb{R}} h(x)\,dx = 1, \quad \operatorname{supp} h = [-1,1]$$

and write $h_\varepsilon(x) := \varepsilon^{-1} h(x/\varepsilon)$, $a_\varepsilon := a * h_\varepsilon$. Then it holds

$$\|\sigma a_\varepsilon\|_{H^{N+1}(P)} \le C \quad \forall \varepsilon.$$

Proof. It is sufficient to prove that $\|\sigma \partial_x^{N+1} a_\varepsilon\|_2 \le C$. It holds

$$\begin{aligned} \sigma(x)(\partial_x^{N+1} a_\varepsilon)(x) &= \int_{\mathbb{R}} \sigma(x) a^{(N)}(z) h'_\varepsilon(x-z)\,dz = f_\varepsilon(x) + g_\varepsilon(x) \\ &:= \int_{\mathbb{R}} \sigma(z) a^{(N)}(z) h'_\varepsilon(x-z)\,dz \\ &\qquad + \int_{\mathbb{R}} (\sigma(x) - \sigma(z)) a^{(N)}(z) h'_\varepsilon(x-z)\,dz. \end{aligned}$$

The proof is divided into two parts. In the first part we show that $\|f_\varepsilon\|_2 \le C$, in the second we prove that $\|g_\varepsilon\|_2 \le C$. We have

$$f_\varepsilon(x) = \int_{\mathbb{R}} (\sigma a)^{(N)}(z) h'_\varepsilon(x-z) dz$$

$$-\sum_{j=1}^{N}\binom{N}{j}\int_{\mathbb{R}}(\sigma^{(j)}a^{(N-j)})(z)h'_\varepsilon(x-z)dz.$$

We observe that $(\sigma a)^{(N)} \in H^1_{\text{per}}(P)$, $\sigma^{(j)}a^{(N-j)} \in H^1_{\text{per}}(P)$, which results in

$$f_\varepsilon(x) = (\sigma a)^{(N+1)} * h_\varepsilon(x) - \sum_{j=1}^{N}\binom{N}{j}(\sigma^{(j)}a^{(N-j)})' * h_\varepsilon(x).$$

This implies immediately $\|f_\varepsilon\|_2 \le C$ for all ε. Now we consider $g_\varepsilon(x)$. Obviously, we have $|\sigma(x)-\sigma(z)| \le C|x-z| \le C\varepsilon$ and $|h'_\varepsilon(x-z)| \le C\varepsilon^{-2}$. This gives $|g_\varepsilon(x)| \le C(|a^{(N)}| * k_\varepsilon)(x)$, where

$$k_\varepsilon(x) = \begin{cases} (2\varepsilon)^{-1} & : |x| \le \varepsilon, \\ 0 & : |x| > \varepsilon, \end{cases} \qquad \|k_\varepsilon\|_{L^1(\mathbb{R})} = 1.$$

By standard arguments one shows that

$$\|g_\varepsilon\|_{L^2(P)} \le C\|\,|a^{(N)}|\,\|_{L^2(3P)}\|k_\varepsilon\|_{L^1(\mathbb{R})} \le C\|a^{(N)}\|_{L^2(P)} \le C.$$

$\square$

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