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# A periodic boundary value problem for a generalized 2D Ginzburg-Landau equation

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**Abstract.** This article studies the periodic boundary value problem for a generalized Ginzburg-Landau equation with additional fifth order term and cubic terms containing spatial derivatives. We present sufficient condition for global existence. A blow-up of solutions is found via numerical simulation.

*Key words*: generalized Ginzburg-Landau equation, global solution, blow-up.

## 1. Introduction

The classical one-dimensional Ginzburg-Landau equation (GL)

$$u_t = (\nu + i\alpha)u_{xx} - (\kappa + i\beta)|u|^2 u + \gamma u \tag{1-1}$$

frequently occurs as the leading term in an asymptotic expansion of the slowly varying envelope of solutions for such "exact" models such as the Navier-Stocks equations [1]. If $\kappa < 0$ then as $\gamma$ increases, (1-1) with periodical boundary condition undergoes a subcritical bifurcation after which almost all solutions become unbounded in finite time. It is also of physical interest (see [2–4] for details) to carry the expansion to second order in case of small $\kappa$. This leads to the resulting generalized GL [5].

$$u_t = \alpha_0 u + \alpha_1 u_{xx} + \alpha_2 |u|^2 u + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^4 u \tag{1-2}$$

where $\alpha_j = a_j + ib_j$ are all complex parameters (though we note that $\alpha_0$ can be regarded as real since the complex part can be eliminated via a simple transformation). If $a_1 > 0 > a_5$ and $-4a_1a_5 > (b_3 - b_4)^2$ then (1-2) possesses a global classical solution $u(t) \in C([0,\infty); H^1_{per}[0,L]) \cap C^1((0,\infty); H^1_{per}[0,L])$ for every $u(0) \in H^1_{per}[0,L]$ [2]. It has been found that the cubic terms involving partial derivatives can significantly slow the

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propagation speed of moving fronts and pulses [6]. They can create problems and must be balanced by the fifth order term and the second derivative term. In general, the higher order terms in (1-2) when compared with (1-1) provide more opportunities for instability and blow-up. From physical point of view, the periodic boundary condition serves as a "natural" case to study the evolution of patterns and to interpret numerical results. However, the type of boundary data does not seem to be essential as far as the result of global existence is concerned. For example, if we impose the Dirichlet boundary condition, it appears that most of our results remain the same.

The GL equation in higher spatial dimensions

$$u_t = (\nu + i\alpha)\nabla^2 u - (\kappa + i\beta)|u|^{2q} + \gamma u \tag{1-3}$$

has been studied fairly extensively as a model for "turbulent" dynamics in nonlinear partial differential equations [7–12]. When $q = 1$ this equation has been found for a general class of nonlinear evolution problems including several classical problems from hydrodynamics and other fields of physics and chemistry. We note that the generalized complex Ginzburg-Landau equation (1-1) could be derived as a wave envelope or amplitude equation governing wave-packet solutions, for example, in the study of the Taylor-Couette flow, Benard convection and plane Poiseuille flow. There is significance difference in behavior of the hard and soft turbulence in the system when we move up from $D = 1$. This change is caused by the background role of the nonlinear Schrödinger equation (NLS) which is the dissipationless limit of the complex Ginzburg-Landau equation. For $D = 1$, the NLS is integrable and has infinitely many conserved quantities but when $D \geq 2$, solutions of the NLS fail to exist under certain conditions. For $D = 2, 3$, [13] gives some interesting estimates in their study of the possibility of soft and hard turbulence in the GL equation with periodic boundary value problem. Results on the existence of global solution for initial value problem of (1-3) with $D = 2, 3, 4$ were found in several cases [14]. For example, when $D = q = 2$, $\kappa = \nu = 1$, there is a global solution if $|\alpha| < \sqrt{3}$ or $\frac{-\alpha\beta}{|\beta-\alpha|} \leq \frac{\sqrt{5}}{2}$. Further [14] also gives various results on 3D or 4D GL with different nonlinearity. However, these are sufficient conditions for global solution. It appears that no necessary condition for global existence has been found in the literature. The lack of global solutions for 2D in several cases was demonstrated through numerical simulation in [15].

The objective of this article is to extend the global existence theorem in [2] to a 2D GL equation with additional third order term with spatial derivatives and fifth order term. We will present a sufficient condition for global solution and give an example of blow-up phenomenon.

## 2. Global existence theorem

We consider the following generalized Ginzburg-Landau equation in 2 spatial dimensions

$$\begin{aligned} u_t &= \alpha_0 u + \alpha_1 \Delta u + \alpha_2 |u|^2 u \\ &\quad + \alpha_3 |u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^2 u_y + \alpha_6 u^2 \bar{u}_y - \alpha_7 |u|^4 u \end{aligned} \tag{2-1}$$

where $\alpha_j = a_j + ib_j$, $j = 0, \ldots, 7$. For the sake of simplicity, we set $b_0 = 0$ (If $b_0 \neq 0$ then we can apply a transformation to eliminate it.) Also, both $\operatorname{Re}\alpha_1$ and $\operatorname{Re}\alpha_7$ must be positive, i.e. $a_1 > 0$, $a_7 > 0$, otherwise the solution either blows up at finite time or the equation is ill-posed. Periodic boundary condition is imposed as follows

$$u(x, y + L) = u(x, y - L), \;\; u(x + L, y) = u(x - L, y) \tag{2-2}$$

along with initial condition $u(x, y, 0) = f(x) \in H^2(\Omega)$ where $\Omega = [-L, L] \times [-L, L]$. For $D = 2$, it is more reasonable physically to study the periodic boundary value problem than Dirichlet or Neumann boundary value problem on a square domain. When taking $L$ large enough, one can get most of the behavior of the equation. A local existence theorem can be obtained from semigroup theory or using the Galerkin method (see §3.3 and §3.5 of [16]). To obtain a global existence theorem, we impose the following two conditions.

$$(|b_4 - b_3| + |b_6 - b_5|)^2 < 4a_1 a_7, \qquad a_1 > 0, \;\; a_7 > 0 \tag{I}$$

$$\sum_{j=3}^{6} |\alpha_j|^2 < a_1 \left( \frac{3}{2} - \sqrt{1 + \left(\frac{b_7}{a_7}\right)^2} \right) \tag{II}$$

We note that since $a_1 > 0$, the right hand side of (II) must be positive. Thus (II) would also imply that $\frac{3}{2} - \sqrt{1 + (\frac{b_7}{a_7})^2} > 0$ or $|\frac{b_7}{a_7}| < \frac{\sqrt{5}}{2}$.

**Lemma 1** *The $L^1$ norm of the solution to* (2-1) *is bounded for all times if* (I) *holds.*

*Proof.* Differentiate $|u|^2$ with respect to $t$ variable, substitute $u_t$ by (2-1) then integrate over $\Omega$:

$$\begin{aligned}
\partial\|u\|_2^2 &= 2\mathrm{Re}\iint_\Omega u_t\bar{u} \\
&= 2\mathrm{Re}\iint_\Omega (a_0|u|^2 + (a_1 + ib_1)\Delta u \cdot u + \alpha_2|u|^4)dxdy \\
&\quad + 2\mathrm{Re}\iint_\Omega (\alpha_3|u|^2 u_x\bar{u} + \alpha_4|u|^2 u\bar{u}_x + \alpha_5|u|^2 u_y\bar{u} \\
&\qquad\qquad + \alpha_6|u|^2 u\bar{u}_y - (a_7 + ib_7)|u|^6)dxdy \\
&= 2_0\|u\|_2^2 - 2a_1\|\nabla u\|_2^2 + 2a_2||u||_4^4 - 2a_7||u||_6^6 \\
&\quad + 2(b_4 - b_3)\mathrm{Im}\iint_\Omega |u|^2 u_x\bar{u}dxdy \\
&\quad + 2(b_6 - b_5)\mathrm{Im}\iint_\Omega |u|^2 u_y\bar{u}dxdy \qquad (2\text{-}3)
\end{aligned}$$

Write

$$|b_4 - b_3| = A, \qquad |b_6 - b_5| = B \qquad (2\text{-}4)$$

and estimate the following ($C$ is an appropriate positive number)

$$\begin{aligned}
\left|2(b_4 - b_3)\mathrm{Im}\iint_\Omega |u^2 u_x\bar{u}dxdy\right| &\le 2|b_4 - b_3|||u||_6^3||u_x||_2 \\
&\le AC||u||_6^6 + \frac{A}{C}||u_x||_2^2 \qquad (2\text{-}5)
\end{aligned}$$

$$\begin{aligned}
\left|2(b_6 - b_5)\mathrm{Im}\iint_\Omega |u^2 u_y\bar{u}dxdy\right| &\le 2|b_6 - b_5|||u||_6^3||u_y||_2 \\
&\le BC||u||_6^6 + \frac{B}{C}||u_y||_2^2 \qquad (2\text{-}6)
\end{aligned}$$

Add (2-5) and (2-6)

$$\begin{aligned}
&\left|2(b_4 - b_3)\mathrm{Im}\iint_\Omega |u^2 u_x\bar{u}dxdy\right| + \left|2(b_6 - b_5)\mathrm{Im}\iint_\Omega |u^2 u_y\bar{u}dxdy\right| \\
&\quad \le (A + B)C||u||_6^6 + \frac{A + B}{C}\|\nabla u\|_2^2 \qquad (2\text{-}7)
\end{aligned}$$

Since $(|b_4 - b_3| + |b_6 - b_5|)^2 < 4a_1a_7$, this implies that $(A + B)^2 < 4a_1a_7$.

Therefore there exists a positive number $C$ such that both

$$(A+B)C < 2a_7, \qquad \frac{A+B}{C} < 2a_1 \tag{2-8}$$

hold. Write $\epsilon = \min\{2a_7 - (A+B)C, 2a_1 - \frac{A+B}{C}\}$ and combine (2-2) thorough (2-8) to get

$$\begin{aligned}
\partial_t \|u\|_2^2 \le\ & 2a_0\|u\|_2^2 - 2a_1\|\nabla u\|_2^2 + 2a_2||u||_4^4 - 2a_7||u||_6^6 \\
& + 2\left|(b_4 - b_3)\mathrm{Im}\iint_\Omega |u|^2 u_x \bar{u}\, dxdy\right| \\
& + 2\left|(b_6 - b_5)\mathrm{Im}\iint_\Omega |u|^2 u_y \bar{u}\, dxdy\right| \\
\le\ & 2a_0\|u\|_2^2 - 2a_1\|\nabla u\|_2^2 + 2a_2||u||_4^4 - 2a_7||u||_6^6 \\
& + (A+B)C||u||_6^6 + \frac{A+B}{C}\|\nabla u\|_2^2 \\
\le\ & 2a_0\|u\|_2^2 - 2a_1\|\nabla u\|_2^2 + 2a_2||u||_4^4 - 2a_7||u||_6^6 \\
& + (2a_7 - \epsilon)||u||_6^6 + (2a_1 - \epsilon)\|\nabla u\|_2^2 \\
\le\ & 2a_0\|u\|_2^2 - \epsilon\|\nabla u\|_2^2 + 2a_2||u||_4^4 - \epsilon||u||_6^6 \\
\le\ & a_0\|u\|_2^2 - \epsilon\|\nabla u\|_2^2 + \frac{\epsilon}{2}||u||_6^6 + \frac{2a_2^2}{\epsilon}\|u\|_2^2 - \epsilon||u||_6^6 \\
=\ & c_0\|u\|_2^2 - \epsilon\|\nabla u\|_2^2 - \frac{\epsilon}{2}||u||_6^6
\end{aligned} \tag{2-9}$$

Integrate both sides of (2-9) in $t$ one has

$$\|u\|_2^2 \le c' + c_0 \int_0^t \|u\|_2^2 d\tau \tag{2-10}$$

An application of Gronwall's lemma yields that $\|u\|_2^2$ is bounded for all $t > 0$. □

**Lemma 2** *If in addition to* (I), *we assume that* (II) *holds, then the* $H^1$ *norm of the solution to* (2-1) *is bounded for all times.*

*Proof.* To establish $H^1$ estimates, differentiate $|u_x|^2$ with respect to $t$ variable, substitute the equation (2-1) then integrate over $\Omega$.

$$\begin{aligned}
&\partial_t ||u_x||_2^2 \\
&\quad = 2\mathrm{Re}\iint_\Omega u_{xt}\bar{u}_x\, dxdy = -2\mathrm{Re}\iint_\Omega u_t \bar{u}_{xx}\, dxdy
\end{aligned}$$

$$
\begin{aligned}
&= -2\mathrm{Re}\iint_\Omega \bar{u}_{xx}(a_0 u + \alpha_1 \Delta u + \alpha_2 |u|^2 u)dxdy \\
&\quad - 2\mathrm{Re}\iint_\Omega \bar{u}_{xx}(\alpha_3|u|^2 u_x + \alpha_4 u^2 \bar{u}_x + \alpha_5 |u|^2 u_y \\
&\qquad\qquad\qquad\qquad + \alpha_6 u^2 \bar{u}_y - \alpha_7 |u|^4 u)dxdy \\
&\le 2a_0||u_x||_2^2 - 2a_1||u_{xx}||_2^2 - 2a_1||u_{xy}||_2^2 \\
&\quad + 2\mathrm{Re}\iint_\Omega \alpha_2 \bar{u}_x (2|u|^2 u_x + u^2 \bar{u}_x)dxdy \\
&\quad + \sum_{j=3}^{6}\left(\frac{a_1(1-\epsilon)||u_{xx}||_2^2}{2} + \frac{2|\alpha_j|^2}{a_1(1-\epsilon)}\iint_\Omega |u|^4|\nabla u|^2 dxdy\right) \\
&\quad - 2\mathrm{Re}\iint_\Omega \alpha_7 |u|^2(3|u|^2|u_x|^2 + 2u^2\bar{u}_x^2)dxdy \\
&\le 2a_0||u_x||_2^2 - a_1\epsilon||u_{xx}||_2^2 + 6|\alpha_2|\iint_\Omega |uu_x|^2 dxdy \\
&\quad + \sum_{j=3}^{6}\left(\frac{2|\alpha_j|^2}{a_1(1-\epsilon)}\iint_\Omega |u|^4|\nabla u|^2 dxdy\right) \\
&\quad - 2\iint_\Omega |u|^2(3a_7|u|^2|u_x|^2 + 2a_7\mathrm{Re}\, u^2\bar{u}_x^2 - 2b_7\mathrm{Im}\, u^2\bar{u}_x^2)dxdy
\end{aligned}
\tag{2-11}
$$

Let $u\bar{u}_x = a + ib$, $k = \frac{|b_7|}{a_7}$ then

$$
\begin{aligned}
&|u|^2(3a_7|u|^2|u_x|^2 + 2a_7\mathrm{Re}\, u^2\bar{u}_x^2 - 2b_7\mathrm{Im}\, u^2\bar{u}_x^2) \\
&\quad = |u|^2(2a_7(a^2+b^2) + 2a_7(a^2-b^2) - 4b_7 ab) \\
&\quad = |u|^2(5a_7a^2 + a_7b^2 - 4b_7ab) \\
&\quad \ge a_7|u|^2(5a^2 + b^2 - 4k|ab|) \\
&\quad \ge a_7|u|^2\left(5a^2 + b^2 - 2kMa^2 - \frac{2k}{M}b^2\right) \\
&\quad = a_7|u|^2(5a^2 + b^2 - (5-\sigma)a^2 - (1-\sigma)b^2) \\
&\quad = a_7\sigma|u|^2(a^2+b^2)
\end{aligned}
\tag{2-12}
$$

Here the unknown $\sigma > 0$ is obtained from the equations

$$
2kM = 5 - \sigma, \qquad \frac{2k}{M} = 1 - \sigma \tag{2-13}
$$

thus $\sigma = 3 - 2\sqrt{1+k^2}$ and $M$ is determined. Therefore, the last line above becomes

$$a_7(3-2\sqrt{1+k^2})|u|^2|u\bar{u}_x|^2 = a_7(3-2\sqrt{1+k^2})|u|^4|u_x|^2 \tag{2-14}$$

Combine (2-11), (2-12) and (2-14)

$$\begin{aligned}\partial_t||u_x||_2^2 \leq\ & 2a_0||u_x||_2^2 - a_1\epsilon||u_{xx}||_2^2 + 6|\alpha_2|\iint_\Omega |uu_x|^2dxdy \\ & + \sum_{j=3}^{6}\frac{2|\alpha_j|^2}{a_1(1-\epsilon)}\iint_\Omega |u|^4|\nabla u|^2dxdy \\ & - 2a_7(3-2\sqrt{1+k^2})\iint_\Omega |u|^4|u_x|^2dxdy\end{aligned} \tag{2-15}$$

Similarly one has

$$\begin{aligned}\partial_t||u_y||_2^2 \leq\ & 2a_0||u_y||_2^2 - a_1\epsilon||u_{yy}||_2^2 + 6|\alpha_2|\iint_\Omega |uu_y|^2dxdy \\ & + \sum_{j=3}^{6}\frac{2|\alpha_j|^2}{a_1(1-\epsilon)}\iint_\Omega |u|^4|\nabla u|^2dxdy \\ & - 2a_7(3-2\sqrt{1+k^2})\iint_\Omega |u|^4|u_y|^2dxdy\end{aligned} \tag{2-16}$$

From (2-15) and (2-16)

$$\begin{aligned}\partial_t||\nabla u||_2^2 \leq\ & 2a_0||\nabla u||_2^2 - -a_1\epsilon||\nabla^2 u||_2^2 + 6|\alpha_2|\iint_\Omega |u|^2|\nabla u|^2dxdy \\ & + \sum_{j=3}^{6}\frac{4|\alpha_j|^2}{a_1(1-\epsilon)}\iint_\Omega |u|^4|\nabla u|^2dxdy \\ & - 2a_7(3-2\sqrt{1+k^2})\iint_\Omega |u|^4|\nabla u|^2dxdy\end{aligned} \tag{2-17}$$

Because of (II), there exist $\epsilon, \epsilon' > 0$ such that

$$\sum_{j=3}^{6}\frac{4|\alpha_j|^2}{a_1(1-\epsilon)} - -2a_7(3-2\sqrt{1+k^2}) < -\epsilon' \tag{2-18}$$

Therefore

$$\begin{aligned}&\partial_t||\nabla u||_2^2 \\ &\quad \leq 2a_0||\nabla u||_2^2 - a_1\epsilon||\nabla^2 u||_2^2 \\ &\qquad + 6|\alpha_2|\iint_\Omega |u|^2|\nabla u|^2dxdy - \epsilon'\iint_\Omega |u|^4|\nabla u|^2dxdy\end{aligned}$$

$$
\begin{aligned}
&\leq 2a_0||\nabla u||_2^2 - a_1\epsilon||\nabla^2 u||_2^2 \\
&\quad + \iint_\Omega \left(\epsilon'|u|^4 + \frac{3|\alpha_2|}{\epsilon'}\right)|\nabla u|^2 dxdy - \epsilon' \iint_\Omega |u|^4|\nabla u|^2 dxdy \\
&= c_0||\nabla u||_2^2 - a_1\epsilon||\nabla^2 u||_2^2 \qquad (2\text{-}19)
\end{aligned}
$$

Integrate (2-19) one has

$$
||\nabla u||_2^2 + a_1\epsilon \int_0^t ||\nabla^2 u||_2^2 d\tau \leq \hat{c} + c_0 \int_0^t ||\nabla u||_2^2 d\tau \qquad (2\text{-}20)
$$

By Gronwall's lemma, $||\nabla u||_2^2$ is bounded for all $t > 0$. In addition we can see that for each $T > 0$, there exists a constant $\delta > 0$ such that

$$
\int_0^t ||\nabla^2 u||_2^2 d\tau < \delta \qquad (2\text{-}21)
$$

for $0 \leq t \leq T$. This proves the lemma. □

Now we can prove the global existence theorem.

**Theorem** *Assume that both* (I) *and* (II) *hold and* $u(x, y, 0) \in H^3$. *Then the* (2-1) *has a unique global classical solution* $u \in L^0(H^2) \cap C^1(L^0)$.

*Proof.* To get $H^2$ estimates, we differentiate $|u_{xx}|^2$ in $t$ variable then integrate over $\Omega$.

$$
\begin{aligned}
&\partial_t||u_{xx}||_2^2 \\
&= 2\mathrm{Re}\iint_\Omega u_{xxt}\bar{u}_{xx}dxdy = -2\mathrm{Re}\iint_\Omega u_{xt}\bar{u}_{xxx}dxdy \\
&= -2\mathrm{Re}\iint_\Omega \bar{u}_{xxx}(\alpha_0 u + \alpha_1\Delta u + \alpha_2|u|^2u - \alpha_7|u|^4u)_x dxdy \\
&\quad - 2\mathrm{Re}\iint_\Omega \bar{u}_{xxx}(\alpha_3|u|^2u_x + \alpha_4u^2\bar{u}_x + \alpha_5|u|^2u_y + \alpha_6u^2\bar{u}_y)_x dxdy \\
&\leq -2a_1||u_{xxx}||_2^2 - 2a_1||u_{xxy}||_2^2 \\
&\quad - 2\mathrm{Re}\iint_\Omega (a_0u_x + \alpha_2(|u|^2u_x + u^2\bar{u}_x) \\
&\quad - \alpha_7(3|u|^4u_x + 2|u|^2u^2\bar{u}_x))\bar{u}_{xxx}dxdy \\
&\quad + c_2\iint_\Omega (|\nabla^2 u||u|^2 + |\nabla u|^2|u|)||u_{xxx}||_2 dxdy \\
&\leq -2a_1||u_{xxx}||_2^2 + 2a_0||u_{xx}||_2^2 \\
&\quad + c_1\iint_\Omega ((|u|^2| + |u^4|)|u_x||u_{xxx}|dxdy
\end{aligned}
$$

$$
\begin{aligned}
&+ c_2 \iint_\Omega (|\nabla^2 u||u|^2 + |\nabla u|^2|u|)||u_{xxx}||_2 dxdy \\
&\le -2a_1||u_{xxx}||_2^2 + 2a_0||u_{xx}||_2^2 + \frac{a_1}{4}||u_{xxx}||_2^2 \\
&+ c_1' \iint_\Omega |u|^8|u_x|^2 dxdy + \tilde{c} \\
&+ 2 \times \frac{a_1}{4}||u_{xxx}||_2^2 + c_2' \iint_\Omega (|\nabla^2 u|^2|u|^4 + |\nabla u|^4|u|^2) dxdy \\
&= -\frac{a_1}{4}||u_{xxx}||_2^2 + 2a_0||u_{xx}||_2^2 + \tilde{c} + c_1'||u||_\infty^8||u_x||_2^2 \\
&+ c_2'(||\nabla^2 u||_2^2||u||_\infty^4 + ||\nabla u||_2^2||\nabla u||_\infty^2||u||_\infty^2) \qquad (2\text{-}22)
\end{aligned}
$$

To proceed we need the following Gagliardo-Nirenberg estimates [17]

$$||u||_\infty \le \lambda_1||\nabla^2 u||_2^{\frac{1}{2}}||u||_2^{\frac{1}{2}} \qquad (2\text{-}23)$$

$$||\nabla u||_\infty \le \lambda_2||\nabla^3 u||_2^{\frac{2}{3}}||u||_2^{\frac{1}{3}} \qquad (2\text{-}24)$$

Apply these two inequalities on (2-22) and use the fact that $||\nabla u||_2$ is bounded on $[0, T]$ one has

$$
\begin{aligned}
\partial_t||u_{xx}||_2^2 \le\ & -\frac{a_1}{4}||u_{xxx}||_2^2 + 2a_0||u_{xx}||_2^2 + \tilde{c} \\
&+ c_1'\lambda_1^8||\nabla^2 u||_2^4||u||_2^4||u_x||_2^2 + c_2'(||\nabla^2 u||_2^2\lambda_1^4||\nabla^2 u||_2^2||u||_2^2 \\
&+ ||\nabla u||_2^2\lambda_2^2||\nabla^3 u||_2^{\frac{4}{3}}||u||_2^{\frac{2}{3}}\lambda_1^2||\nabla^2 u||_2||u||_2) \\
\le\ & -\frac{a_1}{4}||u_{xxx}||_2^2 + 2a_0||u_{xx}||_2^2 + \tilde{c} + \hat{c}||\nabla^2 u||_2^4 \\
&+ \bar{c}||\nabla^3 u||_2^{\frac{4}{3}}||\nabla^2 u||_2 \qquad (2\text{-}25)
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\partial_t||u_{yy}||_2^2 \le\ & -\frac{a_1}{4}||u_{yyy}||_2^2 + 2a_0||u_{yy}||_2^2 + \tilde{c} \\
&+ \hat{c}||\nabla^2 u||_2^4 + \bar{c}||\nabla^3 u||_2^{\frac{4}{3}}||\nabla^2 u||_2 \qquad (2\text{-}26)
\end{aligned}
$$

From (2-15) and (2-26).

$$
\begin{aligned}
\partial_t||\nabla^2 u||_2^2 \le\ & -\frac{a_1}{4}||\nabla^3 u||_2^2 + 2a_0||\nabla^2 u||_2^2 + 2\tilde{c} \\
&+ 2\hat{c}||\nabla^2 u||_2^4 + 2\bar{c}||\nabla^3 u||_2^{\frac{4}{3}}||\nabla^2 u||_2 \qquad (2\text{-}27)
\end{aligned}
$$

By Cauchy-Swartz inequality

$$|fg| \le \frac{f^p}{p} + \frac{g^q}{q} \tag{2-28}$$

one has ($p = \frac{3}{2}$, $q = 3$)

$$f = \epsilon ||\nabla^3 u||_2^{\frac{4}{3}}, \qquad g = \frac{2\bar{c}}{\epsilon}||\nabla^2 u||_2 \tag{2-29}$$

$$2\bar{c}||\nabla^3 u||_2^{\frac{4}{3}}||\nabla^2 u||_2 \le \frac{2\epsilon^p}{3}||\nabla^3 u||_2^2 + \frac{1}{3}\left(\frac{2\bar{c}}{\epsilon}\right)^q ||\nabla^2 u||_2^3 \tag{2-30}$$

Let $\epsilon$ be small enough then substitute (2-30) in (2-27)

$$\begin{aligned} \partial_t ||\nabla^2 u||_2^2 &\le 2a_0||\nabla^2 u||_2^2 + 2\tilde{c} + 2\hat{c}||\nabla^2 u||_2^4 + c_0||\nabla^2 u||_2^3 \\ &\le c' + c''||\nabla^2 u||_2^4 \end{aligned} \tag{2-31}$$

Write $h(t) = 1 + ||\nabla^2 u||_2^2$, divide both sides of (2-31) by $h$ and integrate from 0 to $t$, one obtains (using (2-21))

$$\begin{aligned} ||\nabla^2 u||_2^2 &\le g(t) \le g(0)\exp\left(c't + c''\int_0^t g(\tau)d\tau\right) \\ &\le g(0)\exp\left(c't + c''\delta\right) \end{aligned} \tag{2-32}$$

for all $0 \le t \le T$. Since $T$ is arbitrary, we conclude (along with Lemma 1 and Lemma 2) that the $H^2$ norm of $u$ is bounded for all finite time thus the theorem is proved. □

It remains to be seen whether global solution exists when (I) or (II) does not hold. We try to partially answer this question by numerical simulation, using a software named PDE2D. This is a widely used tool and has been shown fairly effective in dealing with nonlinear boundary value problems for 2-dimensional equations. PDE2D is a general-purpose two-dimensional time-dependent partial differential equation solver [18], using a finite element program. The main program is complied by a FORTRAN 77 compiler. On our IBM RISC/6000, we attempted to solve various generalized GL equation (2-1) using PDE2D (version 3.2). It appears that solutions will blow up for certain $b_1 > 0$, $b_7 < 0$. For example, this happens when $a_0 = b_0 = 0$, $a_1 = 1$, $b_1 = 1$, $a_2 = b_2 = 0$, $a_3 = a_4 = a_5 = a_6 = 0$, $b_3 = b_5 = 1$, $b_4 = b_6 = -1$, $a_7 = 1$, $b_7 = -10$. The correspondance equation is

$$\begin{aligned} u_t &= (1+i)\Delta u + i|u|^2 u_x - i|u|^2 \bar{u}_x + i|u|^2 u_y \\ &\quad - i|u|^2 \bar{u}_y - (1-10i)|u|^4 u \end{aligned} \tag{2-33}$$

If $u = U + iV$ then the above equation becomes

$$\begin{aligned} U_t &= (U_{xx} + U_{yy}) - (V_{xx} + V_{yy}) - (U^2+V^2)^2(U+10V) \\ &\quad - 2(U^2+V^2)(V_x + V_y) \end{aligned} \tag{2-34}$$

$$V_t = (U_{xx} + U_{yy}) + (V_{xx} + V_{yy}) - (U^2+V^2)^2(V-10U) \tag{2-35}$$

which are used for PDE2D numerical calculation. We first verify that equation (2-33) does not satisfy the necessary conditions (I) and (II) for global existence stated in Lemma 1 and Lemma 2. Evidently (I) can not hold because

$$(|b_4 - b_3| + |b_6 - b_5|)^2 = 16 > 4a_1a_7 = 4 \tag{2-36}$$

Meanwhile, from the remark under condition (II) we require that

$$\frac{3}{2} - \sqrt{1 + \left(\frac{b_7}{a_7}\right)^2} > 0 \tag{2-37}$$

or

$$\left|\frac{b_7}{a_7}\right| < \frac{\sqrt{5}}{2} \tag{2-38}$$

However, in our example,

$$\left|\frac{b_7}{a_7}\right| = 10 > \frac{\sqrt{5}}{2} \tag{2-39}$$

thus both (I) and (II) are not satisfied for global existence. Indeed, a blow-up is found in this case. (See Fig. 1). The final triaglization of the numerical scheme is shown in Fig. 2.

Numerical outputs for the norm of $\nabla u$ is given in Table 1. We note that different initial values only change the solutions to a certain degree, a blow-up will eventually occur (unless the initial value is identically zero). There are several methods in the PDE2D program (we used the band method here). But applying different methods only alters the numerical output of the solution slightly.

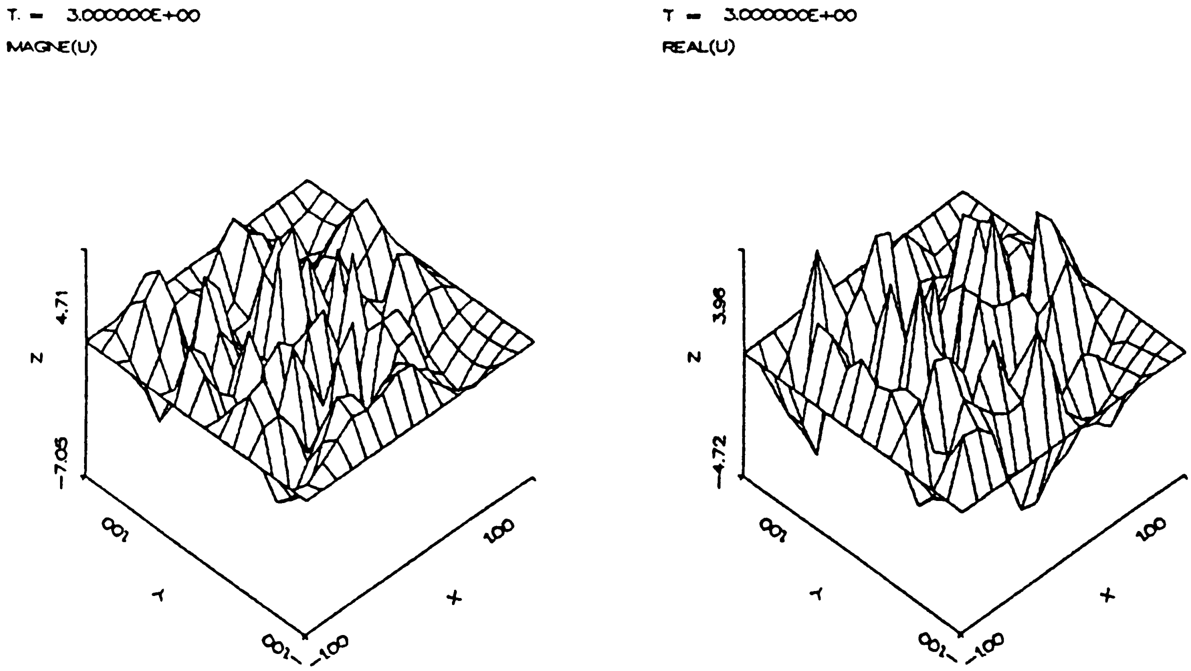


Fig. 1. Blow-up for the solution to the 2D GL equation. Produced by PDE2D on IBM RISC/6000 with band method (NSTEPS=100, $T = 10$, $u_0(x, y) = 1 - \sin^2 x + \sin^2 y$, dimensions of work arrays: IRWRK8Z=236188, IIWK8Z=2200), showing the graphs for $\mathrm{Re}\, u$ and $\mathrm{Im}\, u$ at $t = 3$.

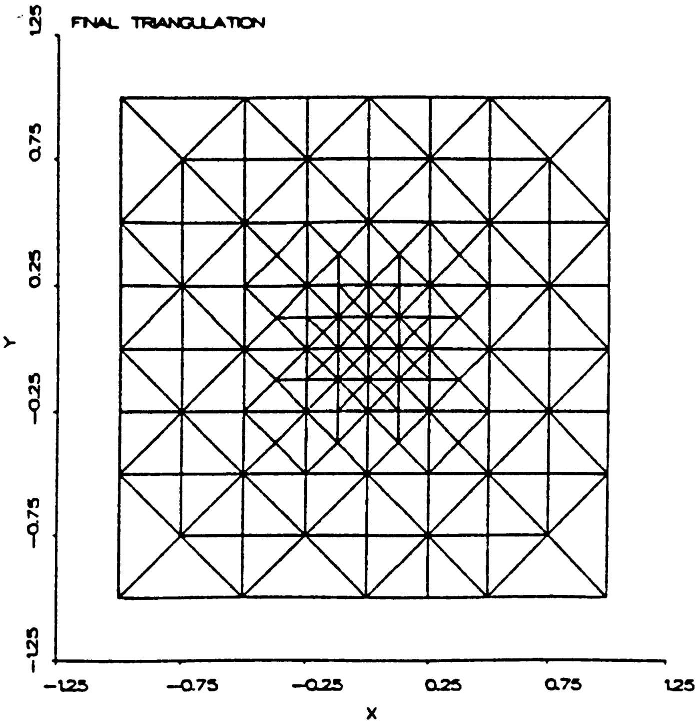


Fig. 2. Final Trianglization

| | | |
|---|---|---|
| T = 5.000000E−01 | Integral estimate = | 675.961345682602882 |
| T = 1.000000E+00 | Integral estimate = | 1209.88753719904275 |
| T = 1.500000E+00 | Integral estimate = | 2982.68347406797966 |
| T = 2.000000E+00 | Integral estimate = | 4425.90852545208963 |
| T = 2.500000E+00 | Integral estimate = | 6539.06775364662280 |
| T = 3.000000E+00 | Integral estimate = | 10809.0941266501450 |
| T = 3.500000E+00 | Integral estimate = | 18052.3402689260693 |
| T = 4.000000E+00 | Integral estimate = | 23930.4366167518820 |
| T = 4.500000E+00 | Integral estimate = | 33522.6579867473556 |
| T = 5.000000E+00 | Integral estimate = | 53013.7673756343574 |
| T = 5.500000E+00 | Integral estimate = | 80276.0150922616449 |
| T = 6.000000E+00 | Integral estimate = | 122501.539642156873 |
| T = 6.500000E+00 | Integral estimate = | 164016.626101799106 |
| T = 7.000000E+00 | Integral estimate = | 206970.842716049723 |
| T = 7.500000E+00 | Integral estimate = | 332312.663851631514 |
| T = 8.000000E+00 | Integral estimate = | 495083.882031293295 |
| T = 8.500000E+00 | Integral estimate = | 679797.603704631212 |
| T = 9.000000E+00 | Integral estimate = | 825050.795782348257 |
| T = 9.500000E+00 | Integral estimate = | 1139936.20926420321 |
| T = 1.000000E+01 | Integral estimate = | 1527942.92813526723 |

Table 4. Numerical Output for $\|\nabla u\|_2^2$

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