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Mod p cohomology algebras of finite groups with extraspecial Sylow p-subgroups

(Dedicated to Professor Yukio Tsushima on his sixtieth birthday)

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Abstract. We analyze mod p cohomology algebras of finite groups with extraspecial Sylow p-subgroups by applying the theory of relative projectivity of modules, which is of fundamental importance in the modular representation theory of finite groups, to the cohomology theory. Especially we shall calculate the mod p cohomology algebras of the general linear group $GL(3, \mathbf{F}_p)$.

Key words: cohomology of finite groups, relative projectivity of modules, Carlson modules, Green correspondence.

1. Introduction

Let p be a prime greater than three. In this paper we consider cohomology algebras of finite groups with extraspecial Sylow p-subgroup

$$P = \langle a, b \mid a^p = b^p = [a, b]^p = 1, [[a, b], a] = [[a, b], b] = 1\rangle$$

of order p^3 and exponent p with coefficients in fields of characteristic p.

Integral cohomology rings of these finite groups have been investigated by some people. Among them we should mention D. J. Green [6] and Tezuka-Yagita [11]. Green's work would be the first one dealing with such finite groups and contains a useful proposition that can be applied to modular case. Tezuka and Yagita's work is a comprehensive one considering finite simple groups with P as Sylow p-subgroups and gave universally stable classes. Some of these results and methods are valid for modular cases. The present paper is partly inspired by their works.

We should also mention Milgram-Tezuka [8]. There they calculated the mod 3 cohomology algebra of the Mathieu group M_{12}, whose Sylow 3-subgroup is extraspecial of order 27 and exponent 3; and they showed

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that the cohomology algebra is isomorphic with that of the general linear group $\mathrm{GL}(3, \mathbf{F}_3)$. They used the theory of geometry of subgroups, as the title suggests.

However, our aim in this paper is to understand mod p cohomology algebras from a view point of modular representation theory of finite groups. Our main tools include the theory of relative projectivity of modules and theory of cohomology varieties of modules.

In Okuyama-Sasaki [10] we studied some applications of theory of relative projectivity of modules to the cohomology theory of finite groups; and we calculated the mod 2 cohomology algebras of finite groups with wreathed Sylow 2-subgroups. The crucial was to analyze a Carlson module. To do that we used Green correspondence and the theory of projectivity of modules relative to modules. In this paper we apply our theory to finite groups with extraspecial Sylow p-subgroups for a prime $p > 3$; as an example we shall calculate the mod p cohomology algebra of the general linear group $\mathrm{GL}(3, \mathbf{F}_p)$.

In Section 2 we quote some facts from Okuyama-Sasaki [10]. In Section 3, following Leary [7], we state the mod p cohomology algebra of the extraspecial p-group P and some properties. We also prove some facts on which our further investigation depends. Sections 4 and 5 are devoted to construct general framework to study mod p cohomology algebras of finite groups in question. In Section 4 we shall define a universally stable homogeneous system of parameters $\{\rho, \sigma\}$ of the cohomology algebra $H^*(P, k)$ such that ρ is regular. For a finite group G with P as a Sylow p-subgroup let $\widetilde{\rho}$ denote the cohomology class in $H^*(G, k)$ that restricts to ρ in P. Since the class $\widetilde{\rho}$ is regular in $H^*(G, k)$, we obtain

$$\dim H^{n+r}(G, k)/H^n(G, k)\widetilde{\rho} = \dim \mathrm{Ext}^n_{kG}(L_{\widetilde{\rho}}, k),$$

where $r = \deg \widetilde{\rho}$. See Lemma 2.7. Here $L_{\widetilde{\rho}}$ is the Carlson module of the element $\widetilde{\rho}$, whose definition will be given at the beginning of Section 2. Therefore it would be useful to examine the Carlson module $L_{\widetilde{\rho}}$. To do that we have to know vertices and sources of indecomposable direct summands of the module $L_{\widetilde{\rho}}$. See Theorem 4.5. To obtain the extension groups $\mathrm{Ext}^n_{kG}(L_{\widetilde{\rho}}, k)$ we shall use Green correspondence. See Corollary 4.7. Next we have to describe explicitly the Green correspondents of indecomposable direct summands of $L_{\widetilde{\rho}}$. This will be done in Section 5. A property of projectivity of modules relative to modules will be used there. In Section 6 we shall calculate the mod p cohomology algebra of the general linear group $\mathrm{GL}(3, \mathbf{F}_p)$. Main results are Theorems 6.9, 6.10 and 6.11.

Mod p cohomology algebras of other finite groups in question will be investigated in another paper.

Here we fix notation. Let k be a field. Let G be a finite group. All kG-modules are finitely generated. Let H be a subgroup of G. For a class ζ in $H^*(G,k)$ we shall sometimes write ζ_H or $\zeta_{|H}$ for the restriction $\mathrm{res}_H\,\zeta$. For a class η in $H^*(H,k)$ we shall write $\mathrm{tr}^G\,\eta$ for the corestriction $\mathrm{cor}^G\,\eta$. For a homogeneous element η in $H^n(H,k)$, where the degree n is even, we shall denote by $\mathrm{norm}^G\,\eta$ the image of Evens' norm map $\mathrm{norm} : H^n(H,k) \longrightarrow H^{|G:H|n}(G,k)$. For a and b elements in G we let $a^b = b^{-1}ab$, the conjugate of a by b. For g an element in G we let $H^g = g^{-1}Hg$, the conjugate of H by g. We shall write η^g for the conjugate $\mathrm{con}^{g^{-1}}\,\eta$ in $H^*(H^g,k)$. For φ an automorphism of G the induced isomorphism $(\varphi^{-1})^*$ of the cohomology algebra $H^*(H,k) \longrightarrow H^*(H^\varphi,k)$ will be written on the right with the convention of writing composition; the image of a class η under this isomorphism will be written as η^φ. For kG-modules U and V we shall write $(U,V)_G$ for the space of the kG-homomorphisms $\mathrm{Hom}_{kG}(U,V)$. We shall often write U_H or $U_{|H}$ for the restiction of U to H. For other notation and terminology we follow Benson [1], Carlson [3], Evens [4], Gorenstein [5], or Nagao-Tsushima [9].

2. Preliminaries

In this section we quote some results from Okuyama-Sasaki [10]. Let p be an arbitrary prime and let k be a field of characteristic p. Let G be a finite group of order divisible by the prime p.

Definition 2.1 The nth cohomology group $H^n(G,k)$ is isomorphic with the vector space $(\Omega^n(k),k)_G$. For an element α in $H^n(G,k)$ we denote by $\widehat{\alpha}$ the kG-homomorphism of $\Omega^n(k)$ to k that corresponds to α. If the element α is not the zero element, then we denote by L_α the kernel of $\widehat{\alpha} : \Omega^n(k) \longrightarrow k$. While if $\alpha = 0$, then we define $L_\alpha = \Omega^n(k) \oplus \Omega(k)$. We call such a module a Carlson module.

Note by dimension shifting that $H^n(G,k) \simeq \mathrm{Ext}^1_{kG}(\Omega^{n-1}(k),k)$. An element α in $H^n(G,k) \setminus \{0\}$ corresponds to the extension

$$E_\alpha : 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\alpha) \longrightarrow \Omega^{n-1}(k) \longrightarrow 0.$$

2.1. Relative projectivity

The following theorem deals with Green correspondence of indecomposable direct summands of Carlson modules.

Theorem 2.1 *Let ρ in $H^n(G,k)$ be a homogeneous element. Let U be an indecomposable direct summand of the Carlson module L_ρ of ρ with vertex D. Let H be a subgroup of G containing the normalizer $N_G(D)$ and let V be a Green correspondent of U with respect to (G,D,H). Then the Green correspondent V is a direct summand of the Carlson module $L_{(\rho_H)}$ of the restriction $\rho_H = \operatorname{res}_H \rho$ of the element ρ to the subgroup H. Conversely, if an indecomposable kH-module V with vertex D is a direct summand of the Carlson module $L_{(\rho_H)}$, then the Green correspondent U of V with respect to (G,D,H) is a direct summand of the Carlson module L_ρ.*

Remark 2.1 Every indecomposable direct summand of the Carlson module of a homogeneous element that is not nilpotent has multiplicity one.

Next let us state briefly the theory of projectivity of modules relative to modules. Refer for example Carlson [3] in detail.

Definition 2.2 For V a kG-module let

$$\mathcal{P}(V) = \{R \mid R \text{ is a direct summand of } V \otimes A \text{ for a } kG\text{-module } A\}.$$

A kG-module belonging to $\mathcal{P}(V)$ above is said to be projective relative to $\mathcal{P}(V)$ or $\mathcal{P}(V)$-projective.

Definition 2.3 Let M be a kG-module. A short exact sequence $E: 0 \longrightarrow X \longrightarrow R \longrightarrow M \longrightarrow 0$ is called a $\mathcal{P}(V)$-projective cover of M if

(1) R is $\mathcal{P}(V)$-projective;

(2) the tensor product

$$0 \longrightarrow X \otimes V \longrightarrow R \otimes V \longrightarrow M \otimes V \longrightarrow 0$$

splits;

(3) the kernel X has no $\mathcal{P}(V)$-projective direct summand.

A $\mathcal{P}(V)$-projective cover of any kG-module exists and is uniquely determined up to isomorphism of sequences. Dually we can define $\mathcal{P}(V)$-injective hulls of modules.

A connection between the notion of relative projectivity above and cohomology theory is given by the following fact, which is originally due to

Carlson. This will be used in Section 5. Note, however, that this is not true for $p = 2$.

Lemma 2.2 *Let p be an odd prime. Let ζ in $H^{2n}(G,k)$ be an arbitrary class. Then the extension*

$$E_\zeta : 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\zeta) \longrightarrow \Omega^{2n-1}(k) \longrightarrow 0$$

associated with ζ is a $\mathcal{P}(L_\zeta)$-projective cover of the syzygy $\Omega^{2n-1}(k)$ or equivalently a $\mathcal{P}(L_\zeta)$-injective hull of the trivial module k.

The following can be used to show divisibility in cohomology algebras.

Lemma 2.3 *Let*

$$E_\rho : 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_\rho) \xrightarrow{f} \Omega^{r-1}(k) \longrightarrow 0$$

be the extension corresponding to an element ρ in $H^r(G,k)$. Suppose that the Carlson module L_ρ is relatively $\mathcal{H}$-projective, where $\mathcal{H}$ is a set of subgroups of G. If an element ξ in $H^{n+r}(G,k)$ satisfies

$$\mathrm{res}_H f^*(\xi) = 0 \quad \textit{for every } H \textit{ in } \mathcal{H},$$

where $f^* : \mathrm{Ext}^{n+r}_{kG}(k,k) \longrightarrow \mathrm{Ext}^n_{kG}(L_\rho, k)$, then there exists an element η in $H^n(G,k)$ such that

$$\xi = \rho\eta.$$

2.2. System of parameters

Let G have p-rank r. For $i = 1, \ldots, r$ let

$$\mathcal{H}_i(G) = \{C_G(E) \mid E \text{ is an elementary abelian } p\text{-subgroup of rank } i\}.$$

Our starting point is the following facts.

Theorem 2.4 (Carlson [2]) *The cohomology algebra $H^*(G,k)$ has a homogeneous system $\{\zeta_1, \ldots, \zeta_r\}$ of parameters with the property that for every $i = 1, \ldots, r$*

$$\zeta_i \in \sum_{H \in \mathcal{H}_i(G)} \mathrm{tr}_H^G H^*(H,k).$$

Corollary 2.5 (Okuyama) *If a homogeneous system $\{\zeta_1, \ldots, \zeta_r\}$ of parameters is taken as in the theorem above, then the tensor product $L_{\zeta_1} \otimes$*

$\cdots \otimes L_{\zeta_{r-1}}$ *is* $\mathcal{H}_r(G)$*-projective.*

In particular, if $r=2$*, then* L_{ζ_1} *is* $\mathcal{H}_2(G)$*-projective and the element* ζ_1 *is regular in* $H^*(G,k)$.

The following will be used to decompose a Carlson module.

Lemma 2.6 *Let* G *be a finite group of* p*-rank two. Suppose that a set* $\{\rho,\sigma\}$ *is a homogeneous system of parameters of* $H^*(G,k)$*. Then it holds that*

$$L_{\rho\sigma} \simeq L_\rho \oplus L_\sigma.$$

Lemma 2.7 *Let* G *be a finite group. Let* ρ *in* $H^r(G,k)$ *be a regular element. Then we have the following short exact sequences:*

$$0 \longrightarrow (\Omega^{r-1}(k),k)_G \longrightarrow (\Omega^{-1}(L_\rho),k)_G \longrightarrow 0;$$

$$0 \longrightarrow \mathrm{Ext}^n_{kG}(k,k) \xrightarrow{\cdot\rho} \mathrm{Ext}^{n+r}_{kG}(k,k) \longrightarrow \mathrm{Ext}^n_{kG}(L_\rho,k) \longrightarrow 0,\ n \geq 0.$$

3. Cohomology algebra of extraspecial p-group

Let

$$P = \langle a,b \mid a^p = b^p = [a,b]^p = 1, [[a,b],a] = [[a,b],b] = 1\rangle$$

be an extraspecial p-group of order p^3 and exponent p.

Definition 3.1 Let

$$c = [a,b].$$

Then $Z(P) = \langle c\rangle$. For $j = 0,\dots,p-1$, let

$$E_j = \langle ab^j, c\rangle;\quad a_j = ab^j,\quad b_j = b.$$

Let

$$E_\infty = \langle b,c\rangle;\quad a_\infty = b,\quad b_\infty = a^{-1}.$$

We put

$$\Omega = \{0,1,\dots,p-1,\infty\};\quad \mathcal{E} = \{E_j \mid j \in \Omega\}.$$

The set $\mathcal{E}$ is the collection of all elementary abelian subgroups of rank two. We note that $C_P(E) = E$ for E in $\mathcal{E}$.

Definition 3.2 For j in Ω, regarding $H^1(E_j, \mathbf{F}_p)$ as $\mathrm{Hom}(E_j, \mathbf{F}_p)$, let

$$\lambda_1^{(j)} = a_j^*, \quad \mu_1^{(j)} = c^*$$

and let

$$\lambda_2^{(j)} = \Delta(\lambda_1^{(j)}), \quad \mu_2^{(j)} = \Delta(\mu_1^{(j)}),$$

where $\Delta : H^1(E_j, \mathbf{F}_p) \longrightarrow H^2(E_j, \mathbf{F}_p)$ is the Bockstein homomorphism. Then the element b_j acts on these elements as follows:

$$(\lambda_2^{(j)})^{b_j} = \lambda_2^{(j)}, \quad (\mu_2^{(j)})^{b_j} = -\lambda_2^{(j)} + \mu_2^{(j)}.$$

Definition 3.3 Let us fix some classes in the cohomology algebra $H^*(P, \mathbf{F}_p)$, following Leary [7]. Regarding $H^1(P, \mathbf{F}_p)$ as $\mathrm{Hom}(P, \mathbf{F}_p)$, let

$$\begin{aligned} \alpha_1 &= a^*, & \beta_1 &= b^*; \\ \alpha_2 &= \Delta(\alpha_1), & \beta_2 &= \Delta(\beta_1), \end{aligned}$$

where $\Delta : H^1(P, \mathbf{F}_p) \longrightarrow H^2(P, \mathbf{F}_p)$ is the Bockstein homomorphism. Let us, as in Leary [7], denote by $\langle \ , \ , \ \rangle$ the Massey product. Let

$$\begin{aligned} \eta_2 &= \langle \alpha_1, \alpha_1, \beta_1 \rangle, & \theta_2 &= \langle \beta_1, \beta_1, \alpha_1 \rangle; \\ \eta_3 &= \Delta(\eta_2), & \theta_3 &= \Delta(\theta_2), \end{aligned}$$

where $\Delta : H^2(P, \mathbf{F}_p) \longrightarrow H^3(P, \mathbf{F}_p)$ is the Bockstein homomorphism. We let

$$\begin{aligned} \chi_{2i-1} &= \mathrm{tr}_{E_\infty}^P(\mu_1^{(\infty)}(\mu_2^{(\infty)})^{i-1}), \quad i = 2, \ldots, p-2, \\ \chi_{2i} &= \mathrm{tr}_{E_\infty}^P((\mu_2^{(\infty)})^i), \quad i = 2, \ldots, p-2, \\ \chi_{2p-3} &= \mathrm{tr}_{E_\infty}^P(\mu_1^{(\infty)}(\mu_2^{(\infty)})^{p-2}) - \alpha_2^{p-2}\alpha_1, \\ \chi_{2p-2} &= \mathrm{tr}_{E_\infty}^P((\mu_2^{(\infty)})^{p-1}) - \alpha_2^{p-1}, \\ \chi_{2p-1} &= \mathrm{tr}_{E_\infty}^P(\mu_1^{(\infty)}(\mu_2^{(\infty)})^{p-1}) + \alpha_2^{p-2}\eta_3. \end{aligned}$$

Finally, we let

$$\nu = z \in H^{2p}(P, \mathbf{F}_p) \text{ in Leary [7].}$$

Theorem 3.1 (Leary [7] Theorem 6) *Let p be greater than 3. Then the cohomology algebra $H^*(P, \mathbf{F}_p)$ is generated by the classes α_i, β_i, $i = 1, 2$,*

η_i, θ_i, $i = 2, 3$, χ_i, $i = 7, 8, \dots, 2p-1$, *and* ν *subject to the following relations*:

$$\begin{aligned}
&\alpha_1\beta_1 = 0, \quad \alpha_2\beta_1 = \beta_2\alpha_1, \quad \alpha_1\eta_2 = \beta_1\theta_2 = 0, \quad \alpha_1\theta_2 = \beta_1\eta_2,\\
&\eta_2^2 = \theta_2^2 = \eta_2\theta_2 = 0, \quad \alpha_1\eta_3 = \alpha_2\eta_2, \quad \beta_1\theta_3 = \beta_2\theta_2,\\
&\eta_3\beta_1 = 2\alpha_2\theta_2 + \beta_2\eta_2, \quad \theta_3\alpha_1 = 2\beta_2\eta_2 + \alpha_2\theta_2,\\
&\eta_2\eta_3 = \theta_2\theta_3 = 0, \quad \theta_2\eta_3 = -\eta_2\theta_3, \quad \alpha_2\theta_3 = -\beta_2\eta_3,\\
&\alpha_2(\alpha_2\theta_2 + \beta_2\eta_2) = \beta_2(\alpha_2\theta_2 + \beta_2\eta_2) = 0,\\
&\alpha_2^p\beta_1 - \beta_2^p\alpha_1 = 0, \quad \alpha_2^p\beta_2 - \beta_2^p\alpha_2 = 0,\\
&\alpha_2^p\theta_2 + \beta_2^p\eta_2 = 0, \quad \alpha_2^p\theta_3 + \beta_2^p\eta_3 = 0,
\end{aligned}$$

$$\chi_{2i}\alpha_1 = \begin{cases} 0 \\ -\alpha_2^{p-1}\alpha_1 \end{cases} \quad \chi_{2i}\beta_1 = \begin{cases} 0 & \text{for } i < p-1 \\ -\beta_2^{p-1}\beta_1 & \text{for } i = p-1 \end{cases},$$

$$\chi_{2i}\alpha_2 = \begin{cases} 0 \\ -\alpha_2^{p} \end{cases} \quad \chi_{2i}\beta_2 = \begin{cases} 0 & \text{for } i < p-1 \\ -\beta_2^{p} & \text{for } i = p-1 \end{cases},$$

$$\chi_{2i}\eta_2 = \begin{cases} 0 \\ -\alpha_2^{p-1}\eta_2 \end{cases} \quad \chi_{2i}\theta_2 = \begin{cases} 0 & \text{for } i < p-1 \\ -\beta_2^{p-1}\theta_2 & \text{for } i = p-1 \end{cases},$$

$$\chi_{2i}\eta_3 = \begin{cases} 0 \\ -\alpha_2^{p-1}\eta_3 \end{cases} \quad \chi_{2i}\theta_3 = \begin{cases} 0 & \text{for } i < p-1 \\ -\beta_2^{p-1}\theta_3 & \text{for } i = p-1 \end{cases},$$

$$\chi_{2i}\chi_{2j} = \begin{cases} 0 & \text{for } i+j < 2p-2 \\ \alpha_2^{2p-2} + \beta_2^{2p-2} - \alpha_2^{p-1}\beta_2^{p-1} & \text{for } i = j = p-1 \end{cases},$$

$$\chi_{2i-1}\alpha_1 = \begin{cases} 0 \\ -\alpha_2^{p-1}\eta_2 \end{cases} \quad \chi_{2i-1}\beta_1 = \begin{cases} 0 & \text{for } i < p \\ \beta_2^{p-1}\theta_2 & \text{for } i = p \end{cases},$$

$$\chi_{2i-1}\alpha_2 = \begin{cases} 0 \\ -\alpha_2^{p-1}\alpha_1 \\ \alpha_2^{p-1}\eta_3 \end{cases} \quad \chi_{2i-1}\beta_2 = \begin{cases} 0 & \text{for } i < p-1 \\ -\beta_2^{p-1}\beta_1 & \text{for } i = p-1 \\ -\beta_2^{p-1}\theta_3 & \text{for } i = p \end{cases},$$

$$\chi_{2i-1}\eta_2 = 0, \quad \chi_{2i-1}\theta_2 = 0,$$

$$\chi_{2i-1}\eta_3 = \begin{cases} 0 \\ -\alpha_2^{p-1}\eta_2 \end{cases} \quad \chi_{2i-1}\theta_3 = \begin{cases} 0 & \text{for } i \neq p-1 \\ -\beta_2^{p-1}\theta_2 & \text{for } i = p-1 \end{cases},$$

$$\chi_{2i-1}\chi_{2j-1} = \begin{cases} 0 & \text{for } i < p-1 \text{ or } j < p-1 \\ \alpha_2^{2p-3}\eta_2 - \beta_2^{2p-3}\theta_2 + \alpha_2^{p-1}\beta_2^{p-2}\theta_2 & \text{for } i = p \text{ and } j = p-1 \end{cases},$$

$$\chi_{2i-1}\chi_{2j}$$
$$= \begin{cases} 0 & \text{for } i < p-1 \text{ or } j < p-1 \\ \alpha_2^{2p-3}\alpha_1 + \beta_2^{2p-3}\beta_1 - \alpha_2^{p-1}\beta_2^{p-2}\beta_1 & \text{for } i = j = p-1 \\ -\alpha_2^{2p-3}\eta_3 + \beta_2^{2p-3}\theta_3 - \alpha_2^{p-1}\beta_2^{p-2}\theta_3 & \text{for } i = p \text{ and } j = p-1 \end{cases}.$$

Here we state the actions of the outer automorphisms of P on the cohomology algebra. The outer automorphism group $\mathrm{Out}(P)$ is isomorphic with the general linear group $\mathrm{GL}(2, \mathbf{F}_p)$; a nonsingular matrix

$$\psi = \begin{bmatrix} s & t \\ u & v \end{bmatrix}$$

acts on P as an automorphism as follows:

$$a^\psi = a^s b^t, \quad b^\psi = a^u b^v.$$

The general linear group $\mathrm{GL}(2, \mathbf{F}_p)$ is generated by the following matrices:

$$\varphi = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \delta = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad d_1, d_2 \in \mathbf{F}_p^*.$$

Lemma 3.2 (Leary [7] Theorem 6) *Let p be greater than 3. The automorphisms above act on the cohomology algebra $H^*(P, \mathbf{F}_p)$ as follows:*

ζ	ζ^φ	ζ^τ	ζ^δ	
α_i	α_i	β_i	$\frac{1}{d_1}\alpha_i$	$i = 1, 2$
β_i	$-\alpha_i + \beta_i$	α_i	$\frac{1}{d_2}\beta_i$	$i = 1, 2$
η_i	η_i	θ_i	$\frac{1}{d_1^2 d_2}\eta_i$	$i = 2, 3$
θ_i	$\eta_i + \theta_i$	η_i	$\frac{1}{d_1 d_2^2}\theta_i$	$i = 2, 3,$
χ_{2i-1}	χ_{2i-1}	$(-1)^i\chi_{2i-1}$	$\frac{1}{(d_1 d_2)^i}\chi_{2i-1}$	$i = 4, \ldots, p$
χ_{2i}	χ_{2i}	$(-1)^i\chi_{2i}$	$\frac{1}{(d_1 d_2)^i}\chi_{2i}$	$i = 4, \ldots, p-1$
ν	ν	$-\nu$	$\frac{1}{d_1 d_2}\nu$	

Lemma 3.3 *Let p be greater than 3. The images of the generators in Definition 3.3 under the restrictions to subgroups E in $\mathcal{E}$ are as follows:*

ζ	$\mathrm{res}_{E_j}\zeta,\ j\in\mathbf{F}_p$	$\mathrm{res}_{E_\infty}\zeta$	
α_i	$\lambda_i^{(j)}$	0	$i=1,2$
β_i	$j\lambda_i^{(j)}$	$\lambda_i^{(\infty)}$	$i=1,2$
η_2	$-\lambda_1^{(j)}\mu_1^{(j)}$	0	
θ_2	$j\lambda_1^{(j)}\mu_1^{(j)}$	$\lambda_1^{(\infty)}\mu_1^{(\infty)}$	
η_3	$\lambda_1^{(j)}\mu_2^{(j)}-\lambda_2^{(j)}\mu_1^{(j)}$	0	
θ_3	$-j(\lambda_1^{(j)}\mu_2^{(j)}-\lambda_2^{(j)}\mu_1^{(j)})$	$-(\lambda_1^{(\infty)}\mu_2^{(\infty)}-\lambda_2^{(\infty)}\mu_1^{(\infty)})$	
χ_{2i-1}	0		$i=4,\dots,p-2$
χ_{2i}	0		$i=4,\dots,p-2$
χ_{2p-3}	$-(\lambda_2^{(j)})^{p-2}\lambda_1^{(j)}$		
χ_{2p-2}	$-(\lambda_2^{(j)})^{p-1}$		
χ_{2p-1}	$(\lambda_2^{(j)})^{p-2}(\lambda_1^{(j)}\mu_2^{(j)}-\lambda_2^{(j)}\mu_1^{(j)})$		
ν	$(\mu_2^{(j)})^p-\mu_2^{(j)}(\lambda_2^{(j)})^{p-1}$		

Henceforth we let p be a prime greater than three, unless otherwise stated.

Proposition 3.4 *The set $\{\chi_{2p-2},\nu\}$ is a system of parameters for the cohomology algebra $H^*(P,\mathbf{F}_p)$.*

Proof. For every E in $\mathcal{E}$ the tensor product $L_{\chi_{2p-2}}\otimes L_\nu$ is projective over $\mathbf{F}_pE$ by Lemma 3.3. $\square$

Lemma 3.5 (1) *The cohomology groups $H^n(P,\mathbf{F}_p)$ of degree n up to $2p-1$ have the following bases over $\mathbf{F}_p$:*

	basis	
H^1	$\alpha_1,\ \beta_1$	
H^2	$\alpha_2,\ \beta_2,\ \eta_2,\ \theta_2$	
H^{2i-1}	$\alpha_1\alpha_2^{i-1},\ \alpha_1\alpha_2^{i-2}\beta_2,\ \ldots,\ \alpha_1\alpha_2\beta_2^{i-2},\ \alpha_1\beta_2^{i-1},\ \beta_1\beta_2^{i-1},$ $\alpha_2^{i-2}\eta_3,\ \alpha_2^{i-3}\beta_2\eta_3,\ \ldots,\ \alpha_2\beta_2^{i-3}\eta_3,\ \beta_2^{i-2}\eta_3,\ \beta_2^{i-2}\theta_3,$ χ_{2i-1}	$i=2,\ldots,p$
H^{2i}	$\alpha_2^i,\ \alpha_2^{i-1}\beta_2,\ \ldots,\ \alpha_2\beta_2^{i-1},\ \beta_2^i,$ $\alpha_2^{i-1}\eta_2,\ \alpha_2^{i-2}\beta_2\eta_2,\ \ldots,\ \alpha_2\beta_2^{i-2}\eta_2,\ \beta_2^{i-1}\eta_2,\ \beta_2^{i-1}\theta_2,$ χ_{2i}	$i=2,\ldots,p-1$

(2) *The factor spaces $H^{n+2p}(P,\mathbf{F}_p)/H^n(P,\mathbf{F}_p)\nu$ have the following bases*:

	basis
$H^{2l+2p}/H^{2l}\nu$	$\alpha_2^{l+p},\ \alpha_2^{l+p-1}\beta_2,\ \ldots,\ \alpha_2^{l+1}\beta_2^{p-1},\ \beta_2^{l+p},$ $\alpha_2^{l+p-1}\eta_2,\ \alpha_2^{l+p-2}\beta_2\eta_2,\ \ldots,\ \alpha_2^{l}\beta_2^{p-1}\eta_2,\ \beta_2^{l+p-1}\theta_2$
$H^{2l+1+2p}/H^{2l+1}\nu$	$\alpha_1\alpha_2^{l+p},\ \alpha_1\alpha_2^{l+p-1}\beta_2,\ \ldots,\ \alpha_1\alpha_2^{l+1}\beta_2^{p-1},\ \beta_1\beta_2^{l+p},$ $\alpha_2^{l+p-1}\eta_3,\ \alpha_2^{l+p-2}\beta_2\eta_3,\ \ldots,\ \alpha_2^{l}\beta_2^{p-1}\eta_3,\ \beta_2^{l+p-1}\theta_3$

Let A_{2l+2p} be the complement of $H^{2l}(P,\mathbf{F}_p)\nu$ in $H^{2l+2p}(P,\mathbf{F}_p)$ spanned by the classes above; let $B_{2l+1+2p}$ be the complement of $H^{2l+1}(P,\mathbf{F}_p)\nu$ in $H^{2l+1+2p}(P,\mathbf{F}_p)$ spanned by the classes above. Then we have for $n\geq 1$ and $i=0,\ldots,p-1$ that

$$H^{2i+2np}(P,\mathbf{F}_p) = \bigoplus_{j=0}^{n-1} A_{2i+2(n-j)p}\nu^j \oplus H^{2i}(P,\mathbf{F}_p)\nu^n;$$

$$H^{2i+1+2np}(P,\mathbf{F}_p) = \bigoplus_{j=0}^{n-1} B_{2i+1+2(n-j)p}\nu^j \oplus H^{2i+1}(P,\mathbf{F}_p)\nu^n.$$

Remark 3.1 The actions of automorphisms $\varphi,\ \tau,\ \delta$ on the classes $\chi_3,\ \chi_4,$

χ_5, χ_6 are similar to those on χ_i in Lemma 3.2.

The following is the key fact for our investigation.

Lemma 3.6 *It holds that*

$$\chi_{2p-2} = \sum_{j\in\Omega} \mathrm{tr}^P_{E_j}((\mu_2^{(j)})^{p-1}).$$

Proof. For j in Ω let

$$\zeta^{(j)} = \mathrm{tr}^P_{E_j}((\mu_2^{(j)})^{p-1}) \in H^{2(p-1)}(P, \mathbf{F}_p)$$

and we let

$$\zeta = \sum_{j\in\Omega} \zeta^{(j)}.$$

Then we have

$$\mathrm{res}_{E_l}\, \zeta^{(j)} = \begin{cases} -(\lambda_2^{(j)})^{p-1} & l = j \\ 0 & l \neq j \end{cases}.$$

Indeed, for $l \neq j$

$$\begin{aligned} \mathrm{res}_{E_l}\, \zeta^{(j)} &= \mathrm{res}_{E_l}\, \mathrm{tr}^P((\mu_2^{(j)})^{p-1}) \\ &= \mathrm{tr}^{E_l}\, \mathrm{res}_{E_j\cap E_l}(\mu_2^{(j)})^{p-1} \\ &= 0; \end{aligned}$$

and for $l = j$

$$\begin{aligned} \mathrm{res}_{E_j}\, \zeta^{(j)} &= \mathrm{res}_{E_j}(\mathrm{tr}^P(\mu_2^{(j)})^{p-1}) \\ &= \sum_{l=0}^{p-1}((\mu_2^{(j)})^{p-1})^{b_j^l} \\ &= \sum_{l=0}^{p-1}(\mu_2^{(j)} - l\lambda_2^{(j)})^{p-1} \\ &= -(\lambda_2^{(j)})^{p-1}. \end{aligned}$$

Therefore, Lemma 3.3 implies for all E in $\mathcal{E}$ that

$$\mathrm{res}_E(\chi_{2p-2} - \zeta) = 0,$$

and hence $\chi_{2p-2} - \zeta$ is nilpotent. Thus by Lemma 3.5 we can write, using a homogeneous polynomial $f(X,Y)$ of total degree $p-2$,

$$\chi_{2p-2} - \zeta = f(\alpha_2, \beta_2)\eta_2 + s\beta_2^{p-2}\theta_2, \quad s \in \mathbf{F}_p.$$

Nonsingular matrices φ, τ, and δ act on $\zeta^{(j)}$s as follows:

$$(\zeta^{(j)})^{\varphi} = \begin{cases} \zeta^{(j+1)} & j \in \mathbf{F}_p \\ \zeta^{(\infty)} & j = \infty \end{cases},$$

$$(\zeta^{(j)})^{\tau} = \begin{cases} \zeta^{(j^{-1})} & j \in \mathbf{F}_p^* \\ \zeta^{(0)} & j = \infty \end{cases},$$

$$(\zeta^{(j)})^{\delta} = \begin{cases} \zeta^{(d_1 d_2 j)}, & j \in \mathbf{F}_p \\ \zeta^{(\infty)}, & j = \infty \end{cases}.$$

Applying these matrices on both sides, we obtain by Lemma 3.2 that

$$\chi_{2p-2} - \zeta = 0,$$

as desired. □

Corollary 3.7 *The element ν is regular in the cohomology algebra $H^*(P, \mathbf{F}_p)$; and the Carlson module L_ν is $\mathcal{E}$-projective; in fact we have*

$$L_\nu = \bigoplus_{j \in \Omega} L_{\mu_2^{(j)}}{}^P.$$

Proof. Lemma 3.6, Proposition 3.4, and Corollary 2.5 imply that the Carlson module L_ν is $\mathcal{E}$-projective. For every E_j in $\mathcal{E}$ we have by Lemma 3.3 that

$$\begin{aligned} \mathrm{res}_{E_j} \nu &= (\mu_2^{(j)})^p - \mu_2^{(j)} (\lambda_2^{(j)})^{p-1} \\ &= \prod_{l=0}^{p-1} \left(\mu_2^{(j)}\right)^{b_j^l}. \end{aligned}$$

Hence Lemma 2.6 leads us to the decomposition asserted. □

Though the following will not be used later, it would be worthy to be noticed.

Lemma 3.8 *We have that*

$$\nu = \mathrm{norm}_{E_\infty}^P (\mu_2^{(\infty)}) \in H^{2p}(P, \mathbf{F}_p).$$

Proof. Let $\zeta = \mathrm{norm}_{E_\infty}^P(\mu_2^{(\infty)})$. Then we have that for $j \neq \infty$

$$\begin{aligned}
\mathrm{res}_{E_j}\zeta &= \mathrm{res}_{E_j}\mathrm{norm}^P(\mu_2^{(\infty)}) \\
&= \mathrm{norm}^{E_j}(\mathrm{res}_{E_\infty\cap E_j}\mu_2^{(\infty)}) \\
&= \mathrm{norm}^{E_j}(\mathrm{res}_{\langle c\rangle}\mu_2^{(\infty)}) \\
&= (\mu_2^{(j)})^p - \mu_2^{(j)}(\lambda_2^{(j)})^{p-1};
\end{aligned}$$

and that

$$\begin{aligned}
\mathrm{res}_{E_\infty}\zeta &= \mathrm{res}_{E_\infty}\mathrm{norm}^P(\mu_2^{(\infty)}) \\
&= \prod_{l=0}^{p-1}\left(\mu_2^{(\infty)}\right)^{b_\infty^l} \\
&= (\mu_2^{(\infty)})^p - \mu_2^{(\infty)}(\lambda_2^{(\infty)})^{p-1}.
\end{aligned}$$

Thus Lemma 3.3 implies for each E in $\mathcal{E}$ that

$$\mathrm{res}_E(\nu - \zeta) = 0.$$

Therefore, by Lemma 2.3 we have that $\nu = \zeta$. □

4. Finite groups with extraspecial Sylow p-subgroups

Henceforth we let k be a field of characteristic p containing $\mathbf{F}_{p^2}$. We let G denote a finite group with P as a Sylow p-subgroup, unless otherwise stated. We shall often represent by E a subgroup E_j in $\mathcal{E}$; in this case we shall write λ_2 and μ_2 for $\lambda_2^{(j)}$ and $\mu_2^{(j)}$, respectively.

Definition 4.1 We let

$$\begin{aligned}
\rho &= \nu^{p-1} - \chi_{2p-2}{}^p \in H^{2p(p-1)}(P,k), \\
\sigma &= \nu^{p-1}\chi_{2p-2} \in H^{2(p^2-1)}(P,k).
\end{aligned}$$

Note that

$$\sigma \in \sum_{E\in\mathcal{E}} \mathrm{tr}_E^P H^{2(p^2-1)}(E,k).$$

As in Tezuka-Yagita [11], we have, using Lemma 4.2, which we also need to investigate direct sum decomposition of the Carlson module L_ρ, the following.

Theorem 4.1 *The cohomologies ρ and σ are universally stable.*

Lemma 4.2 *For E in $\mathcal{E}$ we have*
(1)

$$\mathrm{res}_E\, \rho = \prod_{\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p} (\mu_2 - \xi\lambda_2);$$

(2)

$$\mathrm{res}_E\, \sigma = -\left(\lambda_2 \prod_{j \in \mathbf{F}_p} (\mu_2 - j\lambda_2)\right)^{p-1}.$$

Proof. (1) First we note that

$$t^{(p-1)p} + t^{(p-1)^2} + t^{(p-1)(p-2)} + \cdots + t^{(p-1)} + 1 = \prod_{\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p} (t - \xi).$$

Now by Lemma 3.3 we have for E in $\mathcal{E}$ that

$$\begin{aligned} \mathrm{res}_E\, \rho &= M^p + M^{p-1}\Lambda + \cdots + M\Lambda^{p-1} + \Lambda^p \quad (M = \mu_2^{p-1},\ \Lambda = \lambda_2^{p-1}) \\ &= \prod_{\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p} (\mu_2 - \xi\lambda_2). \end{aligned}$$

(2) It follows that

$$\begin{aligned} \mathrm{res}_E\, \sigma &= -\left(\lambda_2(\mu_2^p - \mu_2\lambda_2^{p-1})\right)^{p-1} \\ &= -\left(\lambda_2 \prod_{j \in \mathbf{F}_p} (\mu_2 - j\lambda_2)\right)^{p-1}. \end{aligned}$$

□

For E_j in $\mathcal{E}$ the factor group $P/E_j = \langle \overline{b_j} \rangle$, where $\overline{b_j} = E_j b_j$, acts by conjugation on the set

$$\{L_{\mu_2 - \xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}.$$

Since

$$L_{\mu_2 - \xi\lambda_2}{}^{b_j} = L_{\mu_2 - (\xi+1)\lambda_2},$$

this action induces the action of $P/E_j = \langle \overline{b_j} \rangle$ on the set $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ such that

$\xi^{b_j} = 1+\xi$ for ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. Thus, if we write $(\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P$ for the quotient set of $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ under this action, then the set

$$\{L_{\mu_2-\xi\lambda_2} \mid \xi \in (\mathbf{F}_{p^2} \setminus \mathbf{F}_p)/P\}$$

is a complete set of representatives of the conjugation on $\{L_{\mu_2-\xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$.

Theorem 4.3 (1) *The set $\{\rho, \sigma\}$ is a system of parameters of the cohomology algebra $H^*(P,k)$.*
(2) *The element ρ is regular in $H^*(P,k)$.*
(3) *The Carlson module L_ρ is $\mathcal{E}$-projective. In fact,*

$$L_\rho = \bigoplus_{E\in\mathcal{E}} \bigoplus_{\xi\in(\mathbf{F}_{p^2}\setminus\mathbf{F}_p)/P} {L_{\mu_2-\xi\lambda_2}}^P.$$

Proof. (1) This follows from Lemma 4.2.

(2) Applying Corollary 2.5 to the system of parameters $\{\rho, \sigma\}$, we have our assertion.

(3) The Carlson module L_ρ is $\mathcal{E}$-projective because of Corollary 2.5. Then Lemma 2.6 and Lemma 4.2 imply that

$$L_{\rho|E} = \bigoplus_{\xi\in\mathbf{F}_{p^2}\setminus\mathbf{F}_p} L_{\mu_2-\xi\lambda_2}.$$

Thus we obtain

$$L_\rho = \bigoplus_{E\in\mathcal{E}} \bigoplus_{\xi\in(\mathbf{F}_{p^2}\setminus\mathbf{F}_p)/P} {L_{\mu_2-\xi\lambda_2}}^P.$$

□

Definition 4.2 By Theorem 4.1 we can take a class $\widetilde{\rho}$ in $H^{2p(p-1)}(G,k)$ such that

$$\mathrm{res}_P(\widetilde{\rho}) = \rho;$$

and a class $\widetilde{\sigma}$ in $H^{2(p^2-1)}(G,k)$ such that

$$\mathrm{res}_P(\widetilde{\sigma}) = \sigma.$$

Lemma 4.4 *The centralizer $C_G(E)$ of a subgroup E in $\mathcal{E}$ has a normal* *p-complement*:

$$C_G(E) = E \times O_{p'}(C_G(E)).$$

Proof. The subgroup E is the Sylow p-subgroup of $C_G(E)$. □

Definition 4.3 The Carlson module $L_{\widetilde{\rho}}$ is projective relative to the family $\mathcal{H}_2(G) = \{C_G(E) \mid E \in \mathcal{E}\}$ because of Corollary 2.5. Since $C_G(E)$ has a normal p-complement, the module $L_{\widetilde{\rho}}$ is $\mathcal{E}$-projective. Theorem 4.3 implies that every indecomposable direct summand has vertex some E in $\mathcal{E}$ and a source some $L_{\mu_2-\xi\lambda_2}$, $\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p$. For E in $\mathcal{E}/G$ we denote by

$$\{X_i^{(E)} \mid i \in I^{(E)}\}$$

the set of indecomposable direct summands of the Carlson module $L_{\widetilde{\rho}}$ with vertices E. Theorem 4.3 also says that if $i \neq j$, then $X_i^{(E)}$ and $X_j^{(E)}$ have different sources. We denote by $X^{(E)}$ the direct sum of $X_i^{(E)}$s: $X^{(E)} = \bigoplus_{i \in I^{(E)}} X_i^{(E)}$.

Thus we have

Theorem 4.5 *The Carlson module $L_{\widetilde{\rho}}$ decomposes as follows*:

$$L_{\widetilde{\rho}} = \bigoplus_{E \in \mathcal{E}/G} \bigoplus_{i \in I^{(E)}} X_i^{(E)},$$

where $X_i^{(E)}$ is an indecomposable kG-module with vertex E and a source $L_{\mu_2-\xi_i\lambda_2}$ and if $i \neq j$, then $X_i^{(E)}$ and $X_j^{(E)}$ have different sources.

Definition 4.4 Let $Y_i^{(E)}$ be a Green correspondent of $X_i^{(E)}$ with respect to $(G, E, N_G(E))$. The module $Y_i^{(E)}$ is a direct summand of the Carlson module $L_{\rho'}$ of the restriction $\rho' = \mathrm{res}_{N_G(E)}\, \widetilde{\rho}$ by Theorem 2.1. Let us denote by $Y^{(E)}$ the direct sum of $Y_i^{(E)}$s: $Y^{(E)} = \bigoplus_{i \in I^{(E)}} Y_i^{(E)}$.

Proposition 4.6 *It holds that*

$$(Y^{(E)})^G = X^{(E)} \oplus (\text{projective}).$$

Proof. Let

$$\mathcal{X} = \{Q \mid Q \leq E^g \cap E,\ g \in G \setminus N_G(E)\}.$$

In what follows we omit the superscript ${}^{(E)}$ of $X_i^{(E)}$ and $Y_i^{(E)}$. Because the induced module ${Y_i}^G$ decomposes

$${Y_i}^G = X_i \oplus {X_i}',$$

where ${X_i}'$ is $\mathcal{X}$-projective, it is enough to show that the $\mathcal{X}$-projective module ${X_i}'$ is projective. If Z is an indecomposable direct summand of ${X_i}'$ and $\operatorname{vtx} Z = Q \leq E^g \cap E$, then a source of Z is a direct summand of $({L_{\mu_2-\xi\lambda_2}}^g{}_{|E^g\cap E})_Q$. Now for g an element in G the intersection $E^g \cap E$ is not cyclic if and only if $E = E^g \cap E$, that is g belongs to $N_G(E)$. Hence we see for g outside $N_G(E)$ that ${L_{\mu_2-\xi\lambda_2}}^g{}_{|E^g\cap E}$ is projective; consequently, we have that $\mathcal{X}$-projective module ${X_i}'$ is projective, as required. □

Corollary 4.7 *We have that*

$$\operatorname{Ext}^*_{kG}(L_{\widetilde{\rho}}, k) \simeq \bigoplus_{E\in\mathcal{E}/G} \operatorname{Ext}^*_{kN_G(E)}(Y^{(E)}, k).$$

In particular

$$\dim H^{2p(p-1)-1}(G,k) = \sum_{E\in\mathcal{E}/G} \dim(\Omega^{-1}(Y^{(E)}), k)_{N_G(E)}$$

and

$$\begin{aligned}\dim H^{n+2p(p-1)}(G,k) &= \dim H^n(G,k) \\ &\quad + \sum_{E\in\mathcal{E}/G} \dim \operatorname{Ext}^n_{kN_G(E)}(Y^{(E)}, k).\end{aligned}$$

Proof. The isomorphism follows from Proposition 4.6 and Eckmann-Shapiro Theorem. Since the element $\widetilde{\rho}$ is regular, Lemma 2.7 gives the dimension formula. □

Thus if we could know a direct summand $Y^{(E)}$ of the Carlson module $L_{\rho'}$ of the restriction $\rho' = \operatorname{res}_{N_G(E)} \widetilde{\rho}$, then we would know $X^{(E)}$.

Lemma 4.8 *Under the notation above, for each i in $I^{(E)}$ take $L_{\mu_2-\xi_i\lambda_2}$ as a source of the indecomposable $kN_G(E)$-module $Y_i^{(E)}$. Then the set $\{L_{\mu_2-\xi_i\lambda_2} \mid i \in I^{(E)}\}$ is a complete set of representatives of the action of the factor group $N_G(E)/C_G(E)$ on the set $\{L_{\mu_2-\xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$.*

Proof. Let $N = N_G(E)$ and let

$$H_i = \{g \in N \mid L_{\mu_2 - \xi_i \lambda_2}{}^g \simeq L_{\mu_2 - \xi_i \lambda_2}\}$$

be the inertia group of the source $L_{\mu_2-\xi_i\lambda_2}$ in N. Then the indecomposable kN-module $Y_i^{(E)}$ is the induced module M_i^N of an extension M_i of $L_{\mu_2-\xi_i\lambda_2}$ to the inertia group H_i. The Carlson module $L_{\rho'}$ decomposes as follows:

$$L_{\rho'} = Y^{(E)} \oplus \left(\bigoplus Z\right),$$

where Z is indecomposable with vertex $E_j \neq E$ with source $L_{\mu_2^{(j)}-\xi_j\lambda_2^{(j)}}$. The kN-modules Zs are projective over kE. Thus we obtain

$$L_{\rho'|E} = Y^{(E)}{}_E \oplus (\text{projective}).$$

On the other hand, it holds that

$$L_{\rho'|E} = \bigoplus_{\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p} L_{\mu_2 - \xi\lambda_2} \oplus (\text{projective}).$$

Consequently we have

$$\begin{aligned} \bigoplus_{\xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p} L_{\mu_2 - \xi\lambda_2} &= Y^{(E)}{}_E \\ &= \bigoplus_{i \in I^{(E)}} M_i{}^N{}_E \\ &= \bigoplus_{i \in I^{(E)}} \bigoplus_{g \in N/H_i} L_{\mu_2 - \xi_i\lambda_2}{}^g, \end{aligned}$$

which means our assertion. □

For each i in $I^{(E)}$, the module $Y_i^{(E)}$ would be investigated in the following way. In what follows we omit the super script ${}^{(E)}$ and the subscript i; namely, we denote by Y an indecomposable direct summand of $L_{\rho'}$ with vertex E and by $L_{\mu_2-\xi\lambda_2}$ a source of Y.

(1) First we investigate the inertia group

$$H_\xi = \{g \in N_G(E) \mid L_{\mu_2 - \xi\lambda_2}{}^g \simeq L_{\mu_2 - \xi\lambda_2}\}.$$

In general the factor group $H_\xi / C_G(E)$ is cyclic of order l dividing $p^2 - 1$ (see Lemma 5.1).

(2) Let us denote by L_C the extension of $L_{\mu_2-\xi\lambda_2}$ to $C_G(E)$, which is guaranteed by Lemma 4.4. The induced module ${L_C}^{H_\xi}$ has l indecomposable direct summands:

$$ {L_C}^{H_\xi} = \bigoplus_{j=0}^{l-1} M_j. $$

The module Y is the induced module ${M_j}^{N_G(E)}$ of some M_j.

(3) Let $\rho'' = \operatorname{res}_{H_\xi} \rho'$. The Carlson module $L_{\rho''}$ has M_j above as a direct summand.

(4) The module M_j would be determined by investigation of $H^*(H_\xi, k)$.

5. Green correspondents

Let the general linear group $\mathrm{GL}(2, \mathbf{F}_p)$ act on a group $E = \langle c, a \mid c^p = a^p = 1,\ ac = ca \rangle$ by

$$ a^g = a^s c^t, \quad c^g = a^u c^v \quad \text{for} \quad g = \begin{bmatrix} s & t \\ u & v \end{bmatrix} \in \mathrm{GL}(2, \mathbf{F}_p); $$

and let

$$ N = E \rtimes \mathrm{GL}(2, \mathbf{F}_p). $$

Remark 5.1 The group N is called a "Pal group" in Tezuka-Yagita [11].

A Sylow p-subgroup of N is generated by a and a matrix

$$ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, $$

which we denote by b; we identify this p-group with our extraspecial p-group P; hence the group E is identified with E_0 in Section 3. Since the class ρ in $H^*(P, k)$ is universally stable, we can take a homogeneous class ρ' in $H^{2p(p-1)}(N, k)$ such that

$$ \operatorname{res}_P \rho' = \rho. $$

Our aim is to examine the indecomposable direct summands of the Carlson module $L_{\rho'}$ with vertex E.

Definition 5.1 Regarding $H^1(E,k)$ as $\mathrm{Hom}(E,k)$, we let

$$\lambda_1 = a^*, \quad \mu_1 = c^*;$$

and let

$$\lambda_2 = \Delta(\lambda_1), \quad \mu_2 = \Delta(\mu_1),$$

where $\Delta : H^1(E,k) \longrightarrow H^2(E,k)$ is the Bockstein map.

Definition 5.2 For an arbitrary element ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ we denote by $I(\xi)$ the inertia group in $\mathrm{GL}(2,\mathbf{F}_p)$ of the Carlson module $L_{\mu_2-\xi\lambda_2}$:

$$I(\xi) = \{g \in \mathrm{GL}(2,\mathbf{F}_p) \mid {L_{\mu_2-\xi\lambda_2}}^g \simeq L_{\mu_2-\xi\lambda_2} \ \text{ as } \ kE\text{-modules}\}.$$

Lemma 5.1 *Let $X^2 - eX + f$ be the minimal polynomial of ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. Then we have*

$$I(\xi) = \left\{ s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + u\begin{bmatrix} 0 & -f \\ 1 & e \end{bmatrix} \;\middle|\; (s,u) \in \mathbf{F}_p \times \mathbf{F}_p \setminus \{(0,0)\} \right\};$$

the group $I(\xi)$ is cyclic of order $p^2 - 1$.

Proof. A nonsingular matrix $g = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$ belongs to $I(\xi)$ if and only if $(\mu_2 - \xi\lambda_2)^g = \omega(\mu_2 - \xi\lambda_2)$ for some ω in $\mathbf{F}_{p^2} \setminus \{0\}$, which is equivalent to the condition that

$$\omega g \begin{bmatrix} -\xi \\ 1 \end{bmatrix} = \begin{bmatrix} -\xi \\ 1 \end{bmatrix} \quad \text{for some } \ \omega \in \mathbf{F}_{p^2} \setminus \{0\}.$$

Our assertion follows immediately. □

Corollary 5.2 *The general linear group* $\mathrm{GL}(2,\mathbf{F}_p)$ *acts transitively on the set* $\{L_{\mu_2-\xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$.

Proof. Let ξ be an arbitrary element in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$. Then the orbit of $L_{\mu_2-\xi\lambda_2}$ has length $|\mathrm{GL}(2,\mathbf{F}_p)|/(p^2-1) = p^2 - p$, which coincides with the number of modules in the set $\{L_{\mu_2-\xi\lambda_2} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$. □

Corollary 5.2 together with Lemma 4.8 implies that there exits a unique indecomposable direct summand of the Carlson module $L_{\rho'}$ with vertex E, which we denote by Y. We take $L_{\mu_2-\xi_0\lambda_2}$ as a source of Y, where ξ_0 in $\mathbf{F}_{p^2}$ is a primitive (p^2-1)st root of unity. If we denote by $X^2 - e_0X + f_0$ the minimal polynomial of ξ_0, then we have by Lemma 5.1 that

$$H_{\xi_0} = \left\langle \begin{bmatrix} 0 & -f_0 \\ 1 & e_0 \end{bmatrix} \right\rangle \ltimes E.$$

Let $H_{\xi_0} = H_0$ and let

$$h_0 = \begin{bmatrix} 0 & -f_0 \\ 1 & e_0 \end{bmatrix}.$$

Since E is normal in N, the module Y is the induced module of an extension $M(\xi_0)$ of $L_{\mu_2 - \xi_0\lambda_2}$ to the inertia group $H_0 : Y = M(\xi_0)^N$. We have to specify the extension $M(\xi_0)$. The induced module ${L_{\mu_2-\xi_0\lambda_2}}^{H_0}$ decomposes as a direct sum of $p^2 - 1$ extensions $M_0, \dots, M_{p^2-2}$:

$${L_{\mu_2-\xi_0\lambda_2}}^{H_0} = M_0 \oplus \cdots \oplus M_{p^2-2}.$$

The extension $M(\xi_0)$ is one of these extensions.

Let us investigate the $p^2 - 1$ extensions $M_0, \dots, M_{p^2-2}$.

Definition 5.3 We let

$$\begin{aligned} u_1 &= 1 + \sum_{i=0}^{p^2-2} \xi_0^{-i}(c^{h_0^i} - 1), \\ u_p &= 1 + \sum_{i=0}^{p^2-2} \xi_0^{-ip}(c^{h_0^i} - 1). \end{aligned}$$

The elements u_1 and u_p are units in kE; and $kE = k\langle u_1, u_p\rangle$. Moreover it holds that

$$\begin{aligned} (u_1 - 1)^{h_0} &= \xi_0(u_1 - 1); \\ (u_p - 1)^{h_0} &= \xi_0^p(u_p - 1). \end{aligned}$$

We describe the Carlson module $L_{\mu_2-\xi_0\lambda_2}$ using these units.

Lemma 5.3 *It holds that*

$$L_{\mu_2-\xi_0\lambda_2} = \langle ((u_1 - 1)^{p-1}, 0),\ (u_p - 1, u_1 - 1)\rangle.$$

Proof. First we see the values of $u_1 - 1$ and $u_p - 1$ under the homomorphisms $\lambda_1,\ \mu_1 : E \longrightarrow k$. Since

$$\lambda_1 : a^s c^t \longmapsto s, \quad \mu_1 : a^s c^t \longmapsto t$$

and

$$h_0^i = \frac{1}{\xi_0 - \xi_0^p}\begin{bmatrix} -\xi_0^{p+i} + \xi_0^{1+pi} & -\xi_0^{1+p+i} + \xi_0^{1+p+pi} \\ \xi_0^i - \xi_0^{pi} & \xi_0^{1+i} - \xi_0^{p+pi} \end{bmatrix},$$

we see that

$$\begin{aligned} \lambda_1 : u_1 - 1 \longmapsto & \frac{1}{\xi_0 - \xi_0^p}\sum_{i=0}^{p^2-2}(1 - \xi_0^{(p-1)i}) \\ &= -\frac{1}{\xi_0 - \xi_0^p}, \\ \mu_1 : u_1 - 1 \longmapsto & \frac{1}{\xi_0 - \xi_0^p}\sum_{i=0}^{p^2-2}(\xi_0 - \xi_0^{p+(p-1)i}) \\ &= -\frac{\xi_0}{\xi_0 - \xi_0^p}; \end{aligned}$$

and

$$\begin{aligned} \lambda_1 : u_p - 1 \longmapsto & \frac{1}{\xi_0 - \xi_0^p}\sum_{i=0}^{p^2-2}(\xi_0^{(1-p)i} - 1) \\ &= \frac{1}{\xi_0 - \xi_0^p}, \\ \mu_1 : u_p - 1 \longmapsto & \frac{1}{\xi_0 - \xi_0^p}\sum_{i=0}^{p^2-2}(\xi_0^{1+(1-p)i} - \xi_0^p) \\ &= \frac{\xi_0^p}{\xi_0 - \xi_0^p}. \end{aligned}$$

Therefore, we obtain

$$\mu_1 - \xi_0\lambda_1 : \begin{cases} u_1 - 1 & \longmapsto 0 \\ u_p - 1 & \longmapsto -1 \end{cases}.$$

Considering the Bockstein homomorphism, we see that

$$\mu_2 - \xi_0\lambda_2 : \begin{cases} ((u_1-1)^{p-1},0) & \longmapsto 0 \\ (u_p-1, u_1-1) & \longmapsto 0 \\ (0,(u_p-1)^{p-1}) & \longmapsto -1 \end{cases} ;$$

and hence we have

$$L_{\mu_2-\xi_0\lambda_2} = \langle ((u_1-1)^{p-1},0),\ (u_p-1,u_1-1)\rangle.$$

□

Definition 5.4 We define primitive idempotents in kH_0 by

$$e_j = \frac{1}{p^2-1}\sum_{i=0}^{p^2-2} \xi_0^{-ji}h_0^i, \quad j = 0,\dots,p^2-2.$$

It holds that

$$e_j h_0 = \xi_0^j e_j.$$

We also define one-dimensional kH_0-module k_j on which the group E acts trivially and the matrix h_0 acts as multiplication by ξ_0^j.

Definition 5.5 Let us define a kH_0-module M_0 by

$$M_0 = \langle (e_1(u_1-1)^{p-1},0),\ (e_1(u_p-1), e_p(u_1-1))\rangle,$$

which is an extension of the module $L_{\mu_2-\xi_0\lambda_2}$ to the inertia group H_0. For $j = 1,\dots,p^2-2$ we let

$$M_j = M_0 \otimes k_j.$$

By direct calculation we obtain the following.

Lemma 5.4 *It holds that*

$$\Omega^{2n}(M_0) = \left\langle \begin{array}{ll} (e_{np+1}(u_1-1)^{p-1}, & 0), \\ (e_{np+1}(u_p-1) \quad , & e_{(n+1)p}(u_1-1)) \end{array} \right\rangle$$

and that

$$\Omega^{2n+1}(M_0) = \left\langle \begin{array}{ll} (e_{(n+1)p}(u_1-1), & 0), \\ (e_{(n+1)p}(u_p-1), & e_{(n+1)p+1}(u_1-1)^{p-1}) \end{array} \right\rangle.$$

Therefore, the heads and socles of the extensions M_js are described as follows.

Lemma 5.5 *We have*

$$\begin{aligned}
\operatorname{hd}\Omega^{2n}(M_j) &= k_{(n+1)p+j} \oplus k_{(n+1)p+1+j};\\
\operatorname{soc}\Omega^{2n}(M_j) &= k_{np+1+j} \oplus k_{(n+1)p+j};\\
\operatorname{hd}\Omega^{2n+1}(M_j) &= k_{(n+1)p+1+j} \oplus k_{(n+2)p+j};\\
\operatorname{soc}\Omega^{2n+1}(M_j) &= k_{(n+1)p+j} \oplus k_{(n+1)p+1+j}.
\end{aligned}$$

In particular, each extension M_j is periodic of period $2(p^2-1)$.

Proof. The heads and socles of the extension M_0 are as follows by Lemma 5.4:

$$\begin{aligned}
\operatorname{hd}\Omega^{2n}(M_0) &= k_{(n+1)p} \oplus k_{(n+1)p+1};\\
\operatorname{soc}\Omega^{2n}(M_0) &= k_{np+1} \oplus k_{(n+1)p};\\
\operatorname{hd}\Omega^{2n+1}(M_0) &= k_{(n+1)p+1} \oplus k_{(n+2)p};\\
\operatorname{soc}\Omega^{2n+1}(M_0) &= k_{(n+1)p} \oplus k_{(n+1)p+1}.
\end{aligned}$$

□

The extension $M(\xi_0)$ we need is one of the M_js above; and at the same time it is a direct summand of the Carlson module $L_{\rho''}$ of $\rho'' = \operatorname{res}_{H_0}\rho'$.

Lemma 5.6 *We have*

$$M(\xi_0) = M_{p^2-2}.$$

Proof. First let us show that

$$M(\xi_0) = M_0 \quad \text{or} \quad M_{p^2-2}.$$

The extension

$$E_{\rho''}: 0 \longrightarrow k \longrightarrow \Omega^{-1}(L_{\rho''}) \longrightarrow \Omega^{2p(p-1)-1}(k_{H_0}) \longrightarrow 0$$

is a $\mathcal{P}(L_{\rho''})$-injective hull of the trivial module k by Lemma 2.2; therefore, the image of the inclusion $k \longrightarrow \Omega^{-1}(L_{\rho''})$ above projects non-trivially to every direct summand of $\Omega^{-1}(L_{\rho''})$. Namely, we obtain

$$(k, \Omega^{-1}(M(\xi_0)))_{H_0} \neq 0.$$

The modules among $M_j,\ j=0,\dots,p^2-2$, that satisfy the condition above are

$$M_0 \quad \text{and} \quad M_{p^2-2}.$$

By calculating a minimal projective resolution of k over kH_0, we see that

$$\begin{aligned}\operatorname{hd}\Omega^{2n}(k_{H_0}) &= \bigoplus_{i=0}^{n-1}\left(k_{(n-i)p+i}\oplus k_{(n-i)p+i+1}\right)\oplus k_n,\\ \operatorname{hd}\Omega^{2n+1}(k_{H_0}) &= \bigoplus_{i=0}^{n}\left(k_{(n-i)p+i+1}\oplus k_{(n-i+1)p+i}\right).\end{aligned}$$

Therefore, we have for $n_0=p^2-p-2$ that

$$H^{2n_0+1}(H_0,k)=0, \quad H^{2(n_0+p(p-1))+1}(H_0,k)=0;$$

hence, by the cohomology exact sequence

$$\begin{aligned}0\longrightarrow H^{2n_0+1}(H_0,k)\longrightarrow H^{2(n_0+p(p-1))+1}&(H_0,k)\\ &\longrightarrow \operatorname{Ext}^{2n_0+1}_{kH_0}(L_{\rho''},k)\longrightarrow 0,\end{aligned}$$

we have

$$\operatorname{Ext}^{2n_0+1}_{kH_0}(L_{\rho''},k)=0.$$

In particular, we obtain

$$\operatorname{Ext}^{2n_0+1}_{kH_0}(M(\xi_0),k)=0.$$

Now because

$$\operatorname{hd}\Omega^{2n_0+1}(M_0)=k_{p^2-2}\oplus k_0,$$

we conclude that

$$M(\xi_0)=M_{p^2-2}.$$

□

Consequently, we have

Proposition 5.7 *It holds that*

$$Y={M_{p^2-2}}^N$$

and that

$$\operatorname{Ext}^n_{kN}(Y,k) = \begin{cases} k & \text{when } n \equiv 2p-3, 2p-2, 2p^2-4, 2p^2-3 \pmod{2p^2-2} \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Our assertions hold from Lemma 5.5. □

6. The cohomology algebra of the general linear group $\mathrm{GL}(3, \mathbf{F}_p)$

In this section, using the facts we have proved in the preceding sections, we calculate the mod p cohomology algebra of the general linear group $\mathrm{GL}(3, \mathbf{F}_p)$.

Let $G = \mathrm{GL}(3, \mathbf{F}_p)$. The set

$$P = \left\{ \begin{bmatrix} 1 & t & u \\ 0 & 1 & v \\ 0 & 0 & 1 \end{bmatrix} \middle| \, t, u, v \in \mathbf{F}_p \right\}$$

is a Sylow p-subgroup of G, which is extraspecial of order p^3 and exponent p. Let

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let us take

$$\{E_0, E_1, E_\infty\}$$

as a complete set $\mathcal{E}/G$ of representatives of conjugacy classes of elementary abelian p-subgroups of G of rank two. Then the Carlson module $L_{\widetilde{\rho}}$ decomposes as follows:

$$L_{\widetilde{\rho}} = \bigoplus_{E \in \mathcal{E}/G} X^{(E)},$$

where $X^{(E)}$ is the sum of the indecomposable direct summands of $L_{\widetilde{\rho}}$ with vertex E (see Definition 4.3). To investigate each $X^{(E)}$ we have to know the normalizers $N_G(E)$. The following three lemmas follow from Corollary 5.2.

Lemma 6.1 *We have*

$$N_G(E_0) = \left\{ \begin{bmatrix} x & y & z \\ 0 & t & u \\ 0 & v & w \end{bmatrix} \;\middle|\; \begin{bmatrix} t & u \\ v & w \end{bmatrix} \in \mathrm{GL}(2,\mathbf{F}_p),\ x,y,z \in \mathbf{F}_p,\ x \neq 0 \right\}.$$

The factor group $N_G(E_0)/C_G(E_0)$ is isomorphic to $\mathrm{Aut}\,E_0$ $(\simeq \mathrm{GL}(2,\mathbf{F}_p))$*; this factor group acts transitively on the set* $\{L_{\mu_2^{(0)}-\xi\lambda_2^{(0)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$.

Lemma 6.2 *We have*

$$N_G(E_1) = \left\{ \begin{bmatrix} x & y & z \\ 0 & xt & u \\ 0 & 0 & xt^2 \end{bmatrix} \;\middle|\; x,y,z,t,u \in \mathbf{F}_p,\ x \neq 0,\ t \neq 0 \right\};$$

$$C_G(E_1) = \left\{ \begin{bmatrix} x & y & z \\ 0 & x & y \\ 0 & 0 & x \end{bmatrix} \;\middle|\; x,y,z \in \mathbf{F}_p,\ x \neq 0 \right\}.$$

The factor group $N_G(E_1)/C_G(E_1)$ is isomorphic to the subgroup

$$\left\{ \begin{bmatrix} t & u \\ 0 & t^2 \end{bmatrix} \;\middle|\; t,u \in \mathbf{F}_p,\ t \neq 0 \right\}$$

of the automorphism group $\mathrm{Aut}\,E_1$. *For an element ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ the inertia group H_ξ of the module $L_{\mu_2^{(1)}-\xi\lambda_2^{(1)}}$ is the centralizer $C_G(E_1)$; and hence the factor group $N_G(E_1)/C_G(E_1)$ acts transitively on the set* $\{L_{\mu_2^{(1)}-\xi\lambda_2^{(1)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$.

Lemma 6.3 *We have*

$$N_G(E_\infty) = \left\{ \begin{bmatrix} t & u & x \\ v & w & y \\ 0 & 0 & z \end{bmatrix} \;\middle|\; \begin{bmatrix} t & u \\ v & w \end{bmatrix} \in \mathrm{GL}(2,\mathbf{F}_p),\ x,y,z \in \mathbf{F}_p,\ z \neq 0 \right\}.$$

The factor group $N_G(E_\infty)/C_G(E_\infty)$ is isomorphic to $\mathrm{Aut}\,E_\infty$ $(\simeq \mathrm{GL}(2,\mathbf{F}_p))$*; this factor group acts transitively on the set* $\{L_{\mu_2^{(\infty)}-\xi\lambda_2^{(\infty)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$.

For each E_j in $\mathcal{E}/G$ the factor group $N_G(E_j)/C_G(E_j)$ acts, by Lemmas 6.1, 6.2, 6.3, transitively on the set $\{L_{\mu_2^{(j)}-\xi\lambda_2^{(j)}} \mid \xi \in \mathbf{F}_{p^2} \setminus \mathbf{F}_p\}$. Therefore, there exists only one indecomposable direct summand of $L_{\widetilde{\rho}}$ with vertex E_j by Lemma 4.8. Thus by Theorem 4.5 the Carlson module $L_{\widetilde{\rho}}$ decomposes as

$$L_{\widetilde{\rho}} = X_0 \oplus X_1 \oplus X_\infty,$$

where X_i is an indecomposable module with vertex E_i. Let Y_i be a Green correspondent of X_i with respect to $(G, E_i, N_G(E_i))$. The modules Y_0 and Y_∞ are the ones obtained in the previous section. Let us examine the module Y_1. Let $C_1 = C_G(E_1)$. The inertia group H_ξ in $N_1 = N_G(E_1)$ for an element ξ in $\mathbf{F}_{p^2} \setminus \mathbf{F}_p$ is the centralizer C_1. Hence, if we denote by L_{C_1} an extension of $L_{\mu_2^{(1)} - \xi\lambda_2^{(1)}}$ to the centralizer C_1, then we see that $Y_1 = {L_{C_1}}^{N_1}$. Therefore we have

$$\dim \operatorname{Ext}^n_{kN_1}(Y_1, k) = \dim \operatorname{Ext}^n_{kE_1}(L_{\mu_2^{(1)} - \xi\lambda_2^{(1)}}, k) = 2, \quad n \geq 0.$$

This together with Proposition 5.7 leads us to the following.

Theorem 6.4 *It holds that*

$$\dim \operatorname{Ext}^n_{kG}(L_{\widetilde{\rho}}, k) = \begin{cases} 4 & \textit{when } n \equiv 2p-3, 2p-2, 2p^2-4, 2p^2-3 \pmod{2p^2-2} \\ 2 & \textit{otherwise} \end{cases}.$$

Theorem 6.5 (1) *We have*

$$\dim H^{n+2p(p-1)}(G, k) = \dim H^n(G, k) + \begin{cases} 4 & \textit{when } n \equiv 2p-3, 2p-2, 2p^2-4, 2p^2-3 \pmod{2p^2-2} \\ 2 & \textit{otherwise} \end{cases}.$$

(2) *We have*

$$\dim H^{2p(p-1)-1}(G, k) = 4.$$

Proof. These follow from Corollary 4.7 and Theorem 6.4. □

Let

$$r = 2p(p-1), \quad s = 2(p^2-1).$$

Corollary 6.6 *Let $h_i = \dim H^i(G, k)$. Then the Poincaré series of the* *cohomology algebra* $H^*(G,k)$ *is*

$$\frac{\left(\sum_{i=0}^{r-1} h_i X^i\right)(1-X^s) + 2X^r \sum_{i=0}^{s-1} X^i + 2(X^{s-1} + X^s + X^{r+s-2} + X^{r+s-1})}{(1-X^r)(1-X^s)}.$$

Proof. Let $P(X) = \sum_{n=0}^{\infty} h_n X^n$ and $f(X) = h_0 + h_1 X + \cdots + h_{r-1}X^{r-1}$. Then we have from the dimension formula above the following equation

$$\begin{aligned} P(X) - f(X) &= X^r P(X) + 2X^r(1 + X + \cdots) \\ &\quad + 2(X^{s-1} + X^s + X^{r+s-2} + X^{r+s-1})(1 + X^s + X^{2s} + \cdots). \end{aligned}$$

Resolving this equation we have our Poincaré series. □

We have to determine the dimensions of the cohomology groups of degree up to $r-1$. To do that we use Green [6] Proposition 18 as in Tezuka-Yagita [11] and Milgram-Tezuka [8]. We can also find generators by the same method. Since the classes $\widetilde{\rho}$ in $H^r(G,k)$ and $\widetilde{\sigma}$ in $H^s(G,k)$ form a system of parameters, the cohomology algebra $H^*(G,k)$ is generated by finitely many homogeneous classes of degree up to $r+s-2$ over the polynomial subalgebra $k[\widetilde{\rho}, \widetilde{\sigma}]$. First we find the classes that are stable under the Sylow normalizer $N_G(P)$. We have observed the actions of automorphisms of the Sylow p-subgroup P in Lemma 3.2. Then among the classes obtained above we find the classes which restrict to $N_G(E)$-invariant classes in the subgroups E in $\mathcal{E}/G$. We have observed the images under the restrictions to the subgroups E in $\mathcal{E}$ in Lemma 3.3. Thus we have the following Propositions 6.7 and 6.8.

Definition 6.1 Let

$$A = \alpha_2^{p-1}, \quad B = \beta_2^{p-1}, \quad N = \nu^{p-1}.$$

We identify the classes in $H^*(G,k)$ with its restrictions to $H^*(P,k)$.

Proposition 6.7 *The cohomology groups* $H^n(G,k)$ *of degree* n *up to* $2p(p-1)-1$ *have the following bases over* k :

Bases of $H^{2n}(G,k)$, $2n \leq 2p(p-1)-2$

degree	*basis*
$2(p-1)$	$A+B+\chi_{2(p-1)}$
$2(p-1)j$, $j=2,\ldots,p-3$	$A^{j-1}B$, $\chi_{2(p-j)}\nu^{j-1}$
$2(p-1)(p-2)$	$A^{p-3}B$, $\chi_4\nu^{p-3}$, $\beta_2\eta_2\nu^{p-3}$
$2(p-1)(p-1)$	$A^{p-2}B$, $\beta_2\eta_2\nu^{p-3}A$
$2i+2(p-1)j$, $0\leq j\leq p-4,\ 1\leq i\leq p-3-j$	0
$2(p-2-j)+2(p-1)j$, $1\leq j\leq p-3$	$\alpha_2^{p-2-j}\beta_2^{p-1-j}\eta_2\nu^{j-1}$
$2(p-1-j)+2(p-1)j$, $2\leq j\leq p-3$	$\alpha_2^{p-1-j}\beta_2^{p-j}\eta_2\nu^{j-2}A$
$2i+2(p-1)j$, $1\leq i\leq p-3,\ p-i\leq j\leq p-1$	$\alpha_2^i\beta_2^i\nu^{p-1-i}A^{j-p+i}$, $\alpha_2^i\beta_2^{i+1}\eta_2\nu^{p-3-i}A^{j-p+2+i}$
$2(p-2)$	0
$2(p-2)+2(p-1)$	0
$2(p-2)+2(p-1)j$, $2\leq j\leq p-2$	$\alpha_2^{p-2}\beta_2^{p-2}\nu A^{j-2}$
$2(p-2)+2(p-1)(p-1)$	$\alpha_2^{p-2}\beta_2^{p-2}\nu A^{p-3}$, $\alpha_2^{p-2}\eta_2\nu^{p-2}$, $\beta_2^{p-2}\theta_2\nu^{p-2}$

Bases of $H^{2n+1}(G,k)$, $2n+1 \le 2p(p-1)-1$

degree	*basis*
$2i+1+2(p-1)j$, $0 \le j \le p-3$, $0 \le i \le p-3-j$	0
$2(p-2-j)+1+2(p-1)j$, $0 \le i \le p-3$	$\alpha_2^{p-2-j}\beta_2^{p-1-j}\eta_3\nu^{j-1}$
$2i+1+2(p-1)j$, $2 \le j \le p-1$, $p-1-j \le i \le p-3$	$\alpha_1\alpha_2^i\beta_2^{i+1}\nu^{p-2-i}A^{j-p+1+i}$, $\alpha_2^i\beta_2^{i+1}\eta_3\nu^{p-3-i}A^{j-p+2+i}$
$2(p-2)+1$	$\alpha_1\alpha_2^{p-2}+\beta_1\beta_2^{p-2}+\chi_{2p-3}$
$2(p-2)+1+2(p-1)j$, $1 \le j \le p-3$,	$\alpha_1\alpha_2^{p-2}A^{j-1}B$, $\chi_{2(p-1-j)-1}\nu^j$
$2(p-2)+1+2(p-1)(p-2)$	$\alpha_1\alpha_2^{p-2}A^{p-3}B$
$2(p-2)+1+2(p-1)(p-1)$	$\alpha_1\alpha_2^{p-2}A^{p-2}B$, $\alpha_2^{p-2}\eta_3\nu^{p-2}$, $\beta_2^{p-2}\theta_3\nu^{p-2}$, $\chi_{2p-1}\nu^{p-2}$

Proposition 6.8 *The factor spaces* $H^{n+2p(p-1)}(G,k)/H^n(G,k)\widetilde{\rho}$, $0 \le n \le 2(p^2-1)-2$ *have the following bases*:

Bases of $H^{2n+r}(G,k)/H^{2n}(G,k)\widetilde{\rho}$ for $0 \le 2n \le s-2$

$2n$	*basis*
0	$\beta_2\eta_2\nu^{p-3}A^2$, $A^{p-1}B$

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$2n$	*basis*
$2(p-1)$	$\beta_2\eta_2\nu^{p-3}A^3,$ $A^pB,$ $AN,$ BN
$2(p-1)j,$ $j=2,\dots,p$	$\beta_2\eta_2\nu^{p-3}A^{2+j},$ $A^{p-1+j}B$
$2i+2(p-1)j,$ $j=0,\dots,p,\ i=1,\dots,p-3$	$\alpha_2^i\beta_2^i\nu^{p-1-i}A^{i+j},$ $\alpha_2^i\beta_2^{i+1}\eta_2\nu^{p-3-i}A^{2+i+j}$
$2(p-2)+2(p-1)j,$ $j=0,\dots,p-1,$	$\alpha_2^{p-2}\beta_2^{p-2}\nu A^{p-2+j},$ $\alpha_2^{p-2}\eta_2\nu^{p-2}A^jB$

Bases of $H^{2n+1+r}(G,k)/H^{2n+1}(G,k)\widetilde{\rho}$ for $0\le 2n+1\le s-2$

$2n+1$	*basis*
$2i+1+2(p-1)j,$ $j=0,\dots,p,\ i=0,\dots,p-3$	$\alpha_1\alpha_2^i\beta_2^i\nu^{p-2-i}A^{i+1+j},$ $\alpha_2^i\beta_2^{i+1}\eta_3\nu^{p-3-i}A^{i+2+j}$
$2(p-2)+1$	$\alpha_1\alpha_2^{p-2}A^{p-1}B,$ $\alpha_2^{p-2}\eta_3\nu^{p-2}B,$ $\alpha_2^{p-2}\eta_3\nu^{p-2}A+\alpha_1\alpha_2^{p-2}\nu^{p-1},$ $-\beta_2^{p-2}\theta_3\nu^{p-2}B+\beta_1\beta_2^{p-2}\nu^{p-1}$
$2(p-2)+1+2(p-1)j,$ $j=1,\dots,p-1$	$\alpha_1\alpha_2^{p-2}A^{j-1+p}B,$ $\alpha_2^{p-2}\eta_3\nu^{p-2}A^jB$

Definition 6.2 Let us define some cohomology classes of $H^*(G,k)$ as follows:

class	definition	degree
X	$A+B+\chi_{2(p-1)}$	$2(p-1)$
X_j, $j=2,\dots,p-2$	$\chi_{2(p-j)}\nu^{j-1}$	$2(p-1)j$
Ψ	$\alpha_1\alpha_2^{p-2}+\beta_1\beta_2^{p-2}+\chi_{2(p-2)+1}$	$2(p-2)+1$
Φ_j, $j=1,\dots,p-3$	$\chi_{2(p-j-2)+1}\nu^j$	$2(p-2)+1+2(p-1)j$
Ω	$\chi_{2(p-1)+1}\nu^{p-2}$	$2(p-2)+1+2(p-1)^2$
Σ	AN	$2(p^2-1)$
T	BN	$2(p^2-1)$
Γ_j, $j=2,\dots,p-1$	$\alpha_2^{p-j}\beta_2^{p-j}\nu^{j-1}$	$2(p-j)+2(p-1)j$
Δ_j, $j=2,\dots,p-1$	$\alpha_1\alpha_2^{p-1-j}\beta_2^{p-j}\nu^{j-1}$	$2(p-1-j)+1+2(p-1)j$
E_j, $j=1,\dots,p-2$	$\alpha_2^{p-2-j}\beta_2^{p-1-j}\eta_2\nu^{j-1}$	$2(p-2-j)+2(p-1)j$
Z_j, $j=1,\dots,p-2$	$\alpha_2^{p-2-j}\beta_2^{p-1-j}\eta_3\nu^{j-1}$	$2(p-2-j)+1+2(p-1)j$
H_2	$\alpha_2^{p-2}\eta_2\nu^{p-2}$	$2(p-2)+2(p-1)^2$
Θ_2	$-\beta_2^{p-2}\theta_2\nu^{p-2}$	$2(p-2)+2(p-1)^2$
H_3	$\alpha_2^{p-2}\eta_3\nu^{p-2}$	$2(p-2)+1+2(p-1)^2$

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class	definition	degree
Θ_3	$-\beta_2^{p-2}\theta_3\nu^{p-2}$	$2(p-2)+1+2(p-1)^2$
Ξ	$\alpha_2^{p-2}\eta_3\nu^{p-2}A+\alpha_1\alpha_2^{p-2}\nu^{p-1}$	$2(p-2)+1+2(p-1)p$
Π	$-\beta_2^{p-2}\theta_3\nu^{p-2}B+\beta_1\beta_2^{p-2}\nu^{p-1}$	$2(p-2)+1+2(p-1)p$

By Propositions 6.7 and 6.8 we have the following theorem. Note that the classes $\widetilde{\rho}$, $\widetilde{\sigma}$, and the classes defined in Definition 6.2 are defined over the prime field $\mathbf{F}_p$.

Theorem 6.9 *The cohomology algebra $H^*(\mathrm{GL}(3,\mathbf{F}_p),\mathbf{F}_p)$ is generated by the classes $\widetilde{\rho}$, $\widetilde{\sigma}$, and the classes defined in Definition* 6.2.

By the definitions of our generators and the relations in Theorem 3.1 we obtain

Theorem 6.10 *The generators above satisfy the relations in the tables below, where*

$$\widetilde{\rho}' = \widetilde{\rho} - X^p;$$

classes attached with dagger marks are of odd degrees; a blank entry in the upper right triangle means that corresponding product of generators has no relations; and entries lower than main diagonal are obtained from entries in the upper right triangle:

ζ	ζX	ζX_l	$\zeta \Psi$	$\zeta \Phi_l$	$\zeta \Omega$	$\zeta \Sigma$	ζT	$\zeta \Gamma_l$	$\zeta \Delta_l$	ζE_l	ζZ_l
X		0		0	$H_3 X$	$X^2 \widetilde{\rho}'$	$X^2 \widetilde{\rho}'$				
X_j		0	0	0	0	0	0	0	0	0	0
$\Psi^\dagger$			0	0	$-H_2 X$	$\Psi X \widetilde{\rho}'$	$\Psi X \widetilde{\rho}'$	$\Delta_l X$	0	0	$E_l X$
${\Phi_j}^\dagger$				0	0	0	0	0	0	0	0
$\Omega^\dagger$					0	$H_3 \Sigma$	$-\Theta_3 T$	$Z_{l-1} X \widetilde{\rho}'$	$E_{l-1} X \widetilde{\rho}'$	0	0
Σ							$X^2 \widetilde{\rho}'^2$	$\Gamma_l X \widetilde{\rho}'$	$\Delta_l X \widetilde{\rho}'$	$E_l X \widetilde{\rho}'$	$Z_l X \widetilde{\rho}'$
T								$\Gamma_l X \widetilde{\rho}'$	$\Delta_l X \widetilde{\rho}'$	$E_l X \widetilde{\rho}'$	$Z_l X \widetilde{\rho}'$
Γ_j								See below			
${\Delta_j}^\dagger$									0	0	$\Gamma_j E_l$
E_j										0	0
${Z_j}^\dagger$											0
H_2											
Θ_2											
${H_3}^\dagger$											
${\Theta_3}^\dagger$											
$\Xi^\dagger$											
$\Pi^\dagger$											
$\widetilde{\rho}$											
$\widetilde{\sigma}$											

ζ	$\zeta \Gamma_l$	$\zeta \Delta_l$	ζE_l	ζZ_l
Γ_j	$\begin{cases} \Gamma_{j+l-1} X^2 \\ \Gamma_{j+l-p} \widetilde{\rho}' \end{cases}$	$\begin{cases} \Delta_{j+l-1} X^2 \\ \Delta_{j+l-p} \widetilde{\rho}' \end{cases}$	$\begin{cases} E_{j+l-1} X^2 \\ E_{j+l-p} \widetilde{\rho}' \end{cases}$	$\begin{cases} Z_{j+l-1} X^2, & j+l \leq p \\ Z_{j+l-p} \widetilde{\rho}', & j+l > p \end{cases}$

ζ	ζH_2	$\zeta \Theta_2$	ζH_3	$\zeta \Theta_3$	$\zeta \Xi$	$\zeta \Pi$	$\zeta \widetilde{\rho}$	$\zeta \widetilde{\sigma}$
X		H_2X		H_3X	$H_3X^2 + \Psi X\widetilde{\rho}'$	$H_3X^2 + \Psi X\widetilde{\rho}'$		$-X^2\widetilde{\rho}'$
X_j	0	0	0	0	0	0		0
$\Psi^\dagger$	0	0	H_2X	H_2X	H_2X^2	H_2X^2		$-\Psi X\widetilde{\rho}'$
${\Phi_j}^\dagger$	0	0	0	0	0	0		0
$\Omega^\dagger$	0	0	0	0	$-H_2\Sigma$	Θ_2T		$-H_3\Sigma - \Theta_3T + H_3X\widetilde{\rho}'$
Σ		$H_2X\widetilde{\rho}'$		$H_3X\widetilde{\rho}'$		$H_3X^2\widetilde{\rho}' - \Psi X\widetilde{\rho}'^2$		$-\Sigma^2$
T	$H_2X\widetilde{\rho}'$		$H_3X\widetilde{\rho}'$		$H_3X^2\widetilde{\rho}' + \Psi X\widetilde{\rho}'^2$			$-T^2$
Γ_j	$E_{j-1}X\widetilde{\rho}'$	$E_{j-1}X\widetilde{\rho}'$	$Z_{j-1}X\widetilde{\rho}'$	$Z_{j-1}X\widetilde{\rho}'$	$Z_{j-1}X^2\widetilde{\rho}' + \Delta_jX\widetilde{\rho}'$	$Z_{j-1}X^2\widetilde{\rho}' + \Delta_jX\widetilde{\rho}'$		$-\Gamma_jX\widetilde{\rho}'$
${\Delta_j}^\dagger$	0	0	$E_{j-1}X\widetilde{\rho}'$	$E_{j-1}X\widetilde{\rho}'$	$E_{j-1}X^2\widetilde{\rho}'$	$-E_{j-1}X^2\widetilde{\rho}'$		$-\Delta_jX\widetilde{\rho}'$
E_j	0	0	0	0	0	0		$-E_jX\widetilde{\rho}'$
${Z_j}^\dagger$	0	0	0	0	$-E_jX\widetilde{\rho}'$	$-E_jX\widetilde{\rho}'$		$-Z_jX\widetilde{\rho}'$
H_2	0	0	0	0	0	0		$-H_2\Sigma$
Θ_2		0	0	0	0	0		Θ_2T
${H_3}^\dagger$			0	0	$-H_2\Sigma$	$-H_2X\widetilde{\rho}'$		$-H_3\Sigma$
${\Theta_3}^\dagger$				0	$H_2X\widetilde{\rho}'$	$-\Theta_2T$		Θ_3T
$\Xi^\dagger$					0	0		$-\Xi\Sigma$
$\Pi^\dagger$						0		$-\Pi T$
$\widetilde{\rho}$								
$\widetilde{\sigma}$								$\Sigma^2 + T^2 - X^2\widetilde{\rho}'^2$

Theorem 6.11 *The generators of the cohomology algebra* $H^*(\mathrm{GL}(3, \mathbf{F}_p), \mathbf{F}_p)$ *in Theorem* 6.9 *and relations in Theorem* 6.10 *are fundamental defining relations.*

Proof. Let $A = \bigoplus_{n=0}^{\infty} A_n$, $A_0 = \mathbf{F}_p$, be a commutative graded algebra over $\mathbf{F}_p$ defined by the homogeneous elements as in Definition 6.2 and two homogeneous elements that have the same degrees as $\widetilde{\rho}$, $\widetilde{\sigma}$ with relations described in Theorem 6.10. It is enough to show for each n that the homogeneous submodule A_n has the same dimension as $H^n(G, \mathbf{F}_p)$. We use the same notation for the generators of A as $H^*(G, \mathbf{F}_p)$. Then we see from the relations that the elements of A are linear combinations of the following elements:

element of even degree	degree
$X^i\widetilde{\rho}^j$	$2(p-1)(i+pj)$
$X_l\widetilde{\rho}^j$, $l=2,\ldots,p-2$	$2(p-1)(l+pj)$
$\Sigma^u\widetilde{\rho}^j$	$2(p-1)(pj+(p+1)u)$
$T^u\widetilde{\rho}^j$	$2(p-1)(pj+(p+1)u)$
$\Gamma_l X^i\widetilde{\rho}^j$, $l=2,\ldots,p-1$	$2(p-l)+2(p-1)(l+i+pj)$
$E_l X^i\widetilde{\rho}^j$, $l=1,\ldots,p-2$	$2(p-2-l)+2(p-1)(l+i+pj)$
$H_2X^i\widetilde{\rho}^j$	$2(p-2)+2(p-1)(p-1+i+pj)$
$H_2\Sigma^u\widetilde{\rho}^j$	$2(p-2)+2(p-1)(p-1+pj+(p+1)u)$
$\Theta_2T^u\widetilde{\rho}^j$	$2(p-2)+2(p-1)(p-1+pj+(p+1)u)$

element of odd degree	degree
$\Delta_l X^i \widetilde{\rho}^j$, $l = 2, \ldots, p-1$	$2(p-1-l)+1+2(p-1)(l+i+pj)$
$Z_l X^i \widetilde{\rho}^j$, $l = 1, \ldots, p-2$	$2(p-2-l)+1+2(p-1)(l+i+pj)$
$\Psi X^i \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(i+pj)$
$H_3 X^i \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(p-1+i+pj)$
$\Phi_l \widetilde{\rho}^j$, $l = 1, \ldots, p-3$	$2(p-2)+1+2(p-1)(l+pj)$
$\Omega \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(p-1+pj)$
$\Theta_3 \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(p-1+pj)$
$H_3 \Sigma^u \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(p-1+pj+(p+1)u)$
$\Theta_3 T^u \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(p-1+pj+(p+1)u)$
$\Xi \Sigma^u \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(p+pj+(p+1)u)$
$\Pi T^u \widetilde{\rho}^j$	$2(p-2)+1+2(p-1)(p+pj+(p+1)u)$

From these tables we see that

(1) for $n \leq 2p(p-1)-1$ the homogeneous submodules A_n and $H^n(G, \mathbf{F}_p)$ have the same dimensions;

(2) the factor space $A_{n+2p(p-1)}/A_n \widetilde{\rho}$ has the same dimension as that in $H^*(G, \mathbf{F}_p)$ for each $n \geq 0$.

We conclude that the algebra A is isomorphic to the cohomology algebra $H^*(G, \mathbf{F}_p)$. □

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References

[1] Benson D.J., *Representations and cohomology II: Cohomology of groups and modules.* Cambridge studies in advanced mathematics, Vol. **31**, Cambridge University Press, Cambridge, 1991.

[2] Carlson J.F., *Depth and transfer maps in the cohomology of groups.* Math. Zeitschr. **218** (1995), 461–468.

[3] Carlson J.F., *Modules and group algebras.* Lectures in Mathematics, ETH Zürich, Birkhäuser, Basel/Boston/Berlin, 1996.

[4] Evens L., *The cohomology of groups.* Oxford Mathematics Monograph, Oxford University Press, New York, 1991.

[5] Gorenstein D., *Finite groups.* Harper & Row, Publishers, New York, 1968.

[6] Green D.J., *On the cohomology of the sporadic simple group J_4.* Math. Proc. Cambridge Philos. Soc. **113** (1993), 253–266.

[7] Leary I.J., *The mod-p cohomology rings of some p-groups.* Math. Proc. Cambridge Philos. Soc. **112** (1992), No.1, 63–75.

[8] Milgram R. and Tezuka M., *The geometry and cohomology of M_{12}:II.* Bol. Soc. Mat. Mexicana **3** (1995), 91–108.

[9] Nagao H. and Tsushima Y., *Representations of finite groups.* Academic Press, New York, London, 1989.

[10] Okuyama T. and Sasaki H., *Relative projectivity of modules and cohomology theory of finite groups.* Algebras and Representation Theory, to appear.

[11] Tezuka M. and Yagita N., *On odd prime components of cohomologies sporadic simple groups and the rings of universal stable elements.* J. Algebra **186** (1996), No.2, 483–513.

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