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Approximation of periodic functions by Vallee Poussin sums

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Abstract. Vallee-Poussin sums are introduced and their approximation properties for classes of continuous 2π-periodic functions are studied.

Key words: Vallee-Poussin sums, Double Fourier series, 2π-periodic functions.

1. Introduction

Let $I=[-\pi, \pi]$ and $C(I)$ the space of continuous 2π-periodic functions on I. Let $f\in C(I)$ and its Fourier series is given by

$$f(x)\sim\frac{a_{0}}{2}+\sum_{n=1}^{\infty}(a_{n}\cos nx+b_{n}\sin nx). \tag{1.1}$$

The partial sum of (1.1) is

$$S_{n}(f;x)=\frac{a_{0}}{2}+\sum_{k=1}^{n}(a_{k}\cos kx+b_{k}\sin kx).$$

The Fejer partial sum is given by

$$\sigma_{n}(f;x)=\frac{1}{n+1}\sum_{k=0}^{n}S_{k}(f;x)$$

and the Vallee-Poussin partial sum is defined by

$$V_{n,m}(f;x)=\frac{1}{m+1}\sum_{k=n}^{n+m}S_{k}(f;x), \quad m=0,1,2,\ldots; \ n=0,1,2,\ldots.$$

Denote by $E_{n}(f)$ the degree of best approximation of a function f by trigonometric polynomials $T_{n}(x)$ of order not exceeding n, i.e.,

$$E_{n}(f)=\inf_{T_{n}}\left\{\max_{x}|f(x)-T_{n}(x)|\right\}.$$

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In this paper, we will study the deviations of the de la Vallee poussin sums for periodic functions of two variables.

The first result in this subject was in [1]

$$|f(x) - S_n(f;x)| \leq [4 + \ln(n+1)]E_n(f)$$

and in [2], it is showed that

$$|f(x) - V_{n,m}(f;x)| \leq 2\frac{n+1}{m+1}E_n(f).$$

Let $C_{2\pi}$ be the class of real-valued functions of two variables that are continuous on $Q = [-\pi, \pi] \times [-\pi, \pi]$ and 2π-periodic in each variable separately. For $f \in C_{2\pi}$, its Fourier series is given by

$$\begin{aligned} f(x,y) \sim \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \lambda_{nm}(a_{nm}\cos nx \cos my + b_{nm}\sin nx \cos my \\ + c_{nm}\cos nx \sin my + d_{nm}\sin nx \sin my) \end{aligned}$$

where

$$\lambda_{nm} = \begin{cases} 1/4, & n = m = 0 \\ 1/2, & n \geq 1,\ m = 0; \ \text{ or } \ n = 0,\ m \geq 1\ , \\ 1, & n \geq 1,\ m \geq 1 \end{cases}$$

$$a_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\cos nu \cos mv \, du\, dv,$$

$$b_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sin nu \cos mv \, du\, dv,$$

$$c_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\cos nu \sin mv \, du\, dv,$$

$$d_{nm} = \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(u,v)\sin nu \sin mv \, du\, dv,$$

and the partial sum is given by

$$\begin{aligned} S_{n,m}(f;x,y) = \sum_{k=0}^{n}\sum_{\ell=0}^{m} \lambda_{k\ell}(a_{k\ell}\cos kx \cos \ell y + b_{k\ell}\sin kx \cos \ell y \\ + c_{k\ell}\cos kx \sin \ell y + d_{k\ell}\sin kx \sin \ell y). \end{aligned}$$

The Fejer partial sum is given by

$$\sigma_{n,m}(f;x,y) = \frac{1}{(n+1)(m+1)} \sum_{k=0}^{n} \sum_{\ell=0}^{m} S_{k\ell}(f;x,y)$$

and the Vallee-Poussin sum is

$$V_{n,p}^{m,q}(f;x,y) = \frac{1}{(p+1)(q+1)} \sum_{k=n}^{n+p} \sum_{\ell=m}^{m+q} S_{k,\ell}(f;x,y), \quad (p \geq 0,\ q \geq 0).$$

For a 2π-periodic continuous function f of two variables x and y, let $E_{n,m}(f)$ denote the degree of best approximation of f by trigonometric polynomials $T_{n,m}(x,y)$ of order $\leq n$ in x, and $\leq m$ in y, i.e.,

$$E_{n,m}(f) = \inf_{T_{n,m}} \left\{ \max_{x,y} |f(x,y) - T_{n,m}(x,y)| \right\}.$$

The efficient study for approximation by Vallee Poussin sums has been carried out for several decades. Recently, we have seen the appearance of several studies dealing with the Vallee Poussin sums. See [3, 4, 5, 6].

2. The Main Result

Before we state our main result, we need the following Lemma.

Lemma 2.1 $\int_0^\pi \frac{|\sin rt|}{t} dt = \frac{2}{\pi} \ln r + \mathcal{O}(1)$.

Proof. Let $k \leq r < k+1$. Then for $k \geq 1$ we have

$$\begin{aligned}
\int_0^\pi \frac{|\sin rt|}{t} dt &= \sum_{i=0}^{k-1} (-1)^i \int_{i\pi/r}^{(i+1)\pi/r} \frac{\sin rt}{t} dt + \mathcal{O}(1) \\
&= \sum_{i=0}^{k-1} \int_0^{\pi/r} \frac{\sin rt}{t + i\pi/r} dt + \mathcal{O}(1) \\
&= \int_0^{\pi/r} \sin rt \left\{ \sum_{i=1}^{k-1} \frac{1}{t + i\pi/r} \right\} dt + \mathcal{O}(1).
\end{aligned}$$

For $t \geq 0$ we have

$$\sum_{i=1}^{k-1} \frac{1}{t + i\pi/r} = \frac{r}{\pi} \left\{ \sum_{i=1}^{k-1} \frac{1}{i} + \mathcal{O}(1) \right\} = \frac{r}{\pi} \ln k + \mathcal{O}(1).$$

Thus,

$$\begin{aligned}\int_0^{\pi} \frac{|\sin rt|}{t} dt &= \int_0^{\pi/r} \sin rt \left\{ \frac{r}{\pi} \ln k + \mathcal{O}(1) \right\} dt \\ &= \frac{2}{\pi} \ln k + \mathcal{O}(1) \\ &= \frac{2}{\pi} \ln r + \mathcal{O}(1).\end{aligned}$$

In this paper we prove the following result: □

Theorem 2.2 *If $f(x,y) \in C_{2\pi}$, then the deviations of the de la Vallee-Poussin sums $V_{n,p}^{m,q}(f;x,y)$ from f satisfy the inequality*

$$\begin{aligned}&|f(x,y) - V_{n,p}^{m,q}(f;x,y)| \\ &\leq \frac{c}{(p+1)(q+1)} \sum_{i=n}^{n+p} \sum_{j=m}^{m+q} \left(1 + \ln \frac{i+1}{i-n+1}\right)\left(1 + \ln \frac{j+1}{j-m+1}\right) E_{i,j}(f),\end{aligned} \tag{2.1}$$

where $c > 0$ is an absolute constant.

Proof. The partial sum $S_{n,m}(f;x,y)$ can be written as

$$S_{n,m}(f;x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) D_n(u) D_m(v) du\, dv,$$

where the Direchlet sum $D_k(t)$ is given by

$$D_k(t) = \frac{1}{2} + \sum_{i=1}^{k} \cos it = \frac{\sin \frac{2k+1}{2} t}{2 \sin \frac{t}{2}}.$$

Therefore we obtain

$$S_{n,m}(f;x,y) = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) \frac{\sin \frac{2n+1}{2} u \sin \frac{2m+1}{2} v}{4 \sin \frac{1}{2} u \sin \frac{1}{2} v} du\, dv$$

and

$$\begin{aligned}&V_{n,p}^{m,q}(f;x,y) \\ &= \frac{1}{\pi^2 (p+1)(q+1)} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x+u, y+v) \sum_{k=n}^{n+p} D_k(u) \sum_{\ell=m}^{m+q} D_\ell(v) du\, dv.\end{aligned}$$

Since

$$\sum_{k=n}^{n+m} D_k(t) = \frac{\sin \frac{2n+m+1}{2}t \sin \frac{m+1}{2}t}{2\sin^2 \frac{t}{2}},$$

we have

$$V_{n,p}^{m,q}(f;x,y) = \frac{1}{\pi^2(p+1)(q+1)} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+u,y+v) \frac{\sin \frac{2n+p+1}{2}u \sin \frac{p+1}{2}u \sin \frac{2m+q+1}{2}v \sin \frac{q+1}{2}v}{4\sin^2 \frac{u}{2} \sin^2 \frac{v}{2}} du\, dv.$$

Let

$$L_{n,p}^{m,q} = \frac{(p+1)^{-1}(q+1)^{-1}}{\pi^2} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \left| \frac{\sin \frac{2n+p+1}{2}u \sin \frac{p+1}{2}u \sin \frac{2m+q+1}{2}v \sin \frac{q+1}{2}v}{4\sin^2 \frac{u}{2} \sin^2 \frac{v}{2}} \right| du\, dv$$

and

$$M_{n,m} = \frac{1}{\pi(m+1)} \int_{-\pi}^{\pi} \frac{|\sin \frac{2n+m+1}{2}t \sin \frac{m+1}{2}t|}{2\sin^2 \frac{t}{2}} dt.$$

Then for $s = (m+1)/2$, $rs = (2n+m+1)/2$ $(s \geq 1/2,\ r \geq 1)$ we have

$$\begin{aligned} M_{n,m} &= \frac{1}{2\pi s} \int_0^{\pi} \frac{|\sin rpt \sin st|}{\sin^2 \frac{t}{2}} dt \\ &= \frac{2}{\pi s} \int_0^{\pi} \frac{|\sin rpt \sin pt|}{t^2} dt + \mathcal{O}(1). \end{aligned}$$

But,

$$\int_0^{\pi} \frac{|\sin rst \sin st|}{t^2} dt = s \int_0^{p\pi} \frac{|\sin rt \sin t|}{t^2} dt$$

and

$$\left| \int_0^{p\pi} \frac{|\sin rt \sin t|}{t^2} dt - \int_0^{\pi} \frac{|\sin rt \sin t|}{t^2} dt \right| \leq \int_{\pi/2}^{\infty} \frac{dt}{t^2}.$$

Thus,

$$M_{n,m} = \frac{2}{\pi} \int_0^{\pi} \frac{|\sin rt \sin t|}{t^2} dt + \mathcal{O}(1)$$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{|\sin rt|}{t} dt + \mathcal{O}(1),$$

and from Lemma 2.1 we conclude

$$M_{n,m} = \frac{4}{\pi^2} \ln \frac{2n+m+1}{m+1} + \mathcal{O}(1) = \frac{4}{\pi^2} \ln \frac{n+m+1}{m+1} + \mathcal{O}(1).$$

Consequently, we obtain

$$L_{n,p}^{m,q} = M_{n,p} \cdot M_{m,q},$$

and it is clear that

$$|f(x,y) - V_{n,p}^{m,q}(f;x,y)| \le (1 + L_{n,p}^{m,q}) E_{n,m}(f).$$

Choose positive integers α, β such that $2^\alpha \le p+1 < 2^{\alpha+1}$ and $2^\beta \le q+1 < 2^{\beta+1}$. Then we have

$$\begin{aligned}
&f(x,y) - V_{n,p}^{m,q}(f;x,y) \\
&= \frac{1}{(p+1)(q+1)} \sum_{i=n}^{n+p} \sum_{j=m}^{m+q} [f(x,y) - S_{i,j}(f;x,y)] \\
&= \frac{1}{(p+1)(q+1)} \Bigg\{ [f(x,y) - S_{n,m}(f;x,y)] \\
&\quad + \sum_{\ell=1}^{\alpha} \sum_{i=n+2^{\ell-1}}^{n+2^{\ell}-1} [f(x,y) - S_{i,m}(f;x,y)] \\
&\quad + \sum_{i=n+2^{\alpha}}^{n+p} [f(x,y) - S_{i,m}(f;x,y)] \\
&\quad + \sum_{k=1}^{\beta} \sum_{j=m+2^{k-1}}^{m+2^{k}-1} [f(x,y) - S_{n,j}(f;x,y)] \\
&\quad + \sum_{j=m+2^{\beta}}^{m+q} [f(x,y) - S_{n,j}(f;x,y)] \\
&\quad + \sum_{\ell=1}^{\alpha} \sum_{k=1}^{\beta} \sum_{i=2^{\ell-1}+n}^{n+2^{\ell}-1} \sum_{j=m+2^{k-1}}^{m+2^{k}-1} [f(x,y) - S_{i,j}(f;x,y)]
\end{aligned}$$

$$+\sum_{k=1}^{\beta}\sum_{i=n+2^\alpha}^{n+p}\sum_{j=m+2^{k-1}}^{m+2^k-1}[f(x,y)-S_{i,j}(f;x,y)]$$

$$+\sum_{\ell=1}^{\alpha}\sum_{i=n+2^{\ell-1}}^{n+2^\ell-1}\sum_{j=m+2^\beta}^{m+q}[f(x,y)-S_{i,j}(f;x,y)]$$

$$+\sum_{i=n+2^\alpha}^{n+p}\sum_{j=m+2^\beta}^{m+q}[f(x,y)-S_{i,j}(f;x,y)]\Bigg\}$$

and

$$\sum_{i=n}^{n+p}\sum_{j=m}^{m+q}S_{i,j}(f;x,y)=(p+1)(q+1)V_{n,p}^{m,q}(f;x,y).$$

Thus, we get

$$\begin{aligned}
f-V_{n,p}^{m,q}=\frac{1}{(p+1)(q+1)}\Bigg\{&\left[f-V_{n,0}^{m,0}\right]+\sum_{\ell=1}^{\alpha}2^{\ell-1}\left[f-V_{n+2^{\ell-1},2^{\ell-1}-1}^{m,0}\right]\\
&+(p+1-2^\alpha)\left[f-V_{n+2^\alpha,p-2^\alpha}^{m,0}\right]\\
&+\sum_{k=1}^{\beta}2^{k-1}\left[f-V_{n,0}^{m+2^{k-1},2^{k-1}-1}\right]\\
&+(q+1-2^\beta)\left[f-V_{n,0}^{m+2^\beta,q-2^\beta}\right]\\
&+\sum_{\ell=1}^{\alpha}\sum_{k=1}^{\beta}2^{\ell-1}\cdot 2^{k-1}\left[f-V_{n+2^{\ell-1},2^{\ell-1}-1}^{m+2^{k-1},2^{k-1}-1}\right]\\
&+(p+1-2^\alpha)\sum_{k=1}^{\beta}2^{k-1}\left[f-V_{n+2^\alpha,p-2^\alpha}^{m+2^{k-1},2^{k-1}-1}\right]\\
&+(q+1-2^\beta)\sum_{\ell=1}^{\alpha}2^{\ell-1}\left[f-V_{n+2^{\ell-1},2^{\ell-1}-1}^{m+2^\beta,q-2^\beta}\right]\\
&+(p+1-2^\alpha)(q+1-2^\beta)\left[f-V_{n+2^\alpha,p-2^\alpha}^{m+2^\beta,q-2^\beta}\right]\Bigg\},
\end{aligned}$$

and so,

$$
\begin{aligned}
&|f(x,y) - V_{n,p}^{m,q}(f;x,y)| \\
&\leq \frac{c}{(p+1)(q+1)}\Big\{[\ln(n+1)\ln(m+1) + \ln(n+1) + \ln(m+1) + 1]E_{n,m}(f) \\
&+ \sum_{\ell=1}^{\alpha} 2^{\ell-1}\Big(\ln\frac{n+2^\ell}{2^{\ell-1}}\ln\frac{m+1}{1} + \ln\frac{n+2^\ell}{2^{\ell-1}} + \ln(m+1) + 1\Big) \\
&\qquad\qquad \times E_{n+2^{\ell-1},m}(f) \\
&+ (p+1-2^\alpha)\Big(\ln\frac{n+p+1}{p+1-2^\alpha}\ln\frac{m+1}{1} + \ln\frac{n+p+1}{p+1-2^\alpha} + \ln(m+1) + 1\Big) \\
&\qquad\qquad \times E_{n+2^\alpha,m}(f) \\
&+ \sum_{k=1}^{\beta} 2^{k-1}\Big(\ln\frac{n+1}{1}\ln\frac{m+2^k}{2^{k-1}} + \ln(n+1) + \ln\frac{m+2^k}{2^{k-1}} + 1\Big) \\
&\qquad\qquad \times E_{n,m+2^{k-1}}(f) \\
&+ (q+1-2^\beta)\Big(\ln\frac{n+1}{1}\ln\frac{m+q+1}{q+1-2^\beta} + \ln(n+1) + \ln\frac{m+q+1}{q+1-2^\beta} + 1\Big) \\
&\qquad\qquad \times E_{n,m+2^\beta}(f) \\
&+ \sum_{\ell=1}^{\alpha}\sum_{k=1}^{\beta} 2^{\ell-1}\cdot 2^{k-1}\Big(\ln\frac{n+2^\ell}{2^{\ell-1}}\ln\frac{m+2^k}{2^{k-1}} + \ln\frac{n+2^\ell}{2^{\ell-1}} + \ln\frac{m+2^k}{2^{k-1}} + 1\Big) \\
&\qquad\qquad \times E_{n+2^{\ell-1},m+2^{k-1}}(f) \\
&+ (p+1-2^\alpha)\sum_{k=1}^{\beta} 2^{k-1} \\
&\qquad \times \Big(\ln\frac{n+p+1}{p+1-2^\alpha}\ln\frac{m+2^k}{2^{k-1}} + \ln\frac{n+p+1}{p+1-2^\alpha} + \ln\frac{m+2^k}{2^{k-1}} + 1\Big) \\
&\qquad\qquad \times E_{n+2^\alpha,m+2^{k-1}}(f) \\
&+ (q+1-2^\beta)\sum_{\ell=1}^{\alpha} 2^{\ell-1} \\
&\qquad \times \Big(\ln\frac{n+2^\ell}{2^{\ell-1}}\ln\frac{m+q+1}{q+1-2^\beta} + \ln\frac{n+2^\ell}{2^{\ell-1}}\ln\frac{m+q+1}{q+1-2^\beta} + 1\Big) \\
&\qquad\qquad \times E_{n+2^{\ell-1},m+2^\beta}(f)
\end{aligned}
$$

$$+ (p+1-2^{\alpha})(q+1-2^{\beta})$$
$$\times \left(\ln \frac{n+p+1}{p+1-2^{\alpha}} \ln \frac{m+q+1}{q+1-2^{\beta}} + \ln \frac{n+p+1}{p+1-2^{\alpha}} + \ln \frac{m+q+1}{q+1-2^{\beta}} + 1\right)$$
$$\times E_{n+2^{\alpha},m+2^{\beta}}(f)$$
$$= \frac{c}{(p+1)(q+1)} \sum_{k=1}^{9} I_k, \quad say.$$

Need to find an estimate for each $I_k, k = 1, 2, \ldots, 9$. For I_2, we have

$$I_2 = \ln(m+1) \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m}(f) \ln \frac{n+2^{\ell}}{2^{\ell-1}}$$
$$+ \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m}(f) \ln \frac{n+2^{\ell}}{2^{\ell-1}}$$
$$+ \ln(m+1) \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m}(f) + \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m}(f)$$
$$= A_1 + A_2 + A_3 + A_4, \quad say.$$

On the interval $[0, \infty)$, consider the function $\phi(n) = E_{n,m}(f)$ where $\phi(y)$ is linear in every interval $[n, n+1]$, $(n = 0, 1, \ldots)$. Note that $E_{n+2^{\ell-1},m}(f) \leq \phi(y)$ for $y \leq n + 2^{\ell-1}$, and $\phi(y) \leq E_{n,m}(f)$ for $y \geq n$. Therefore, we have

$$2^{\ell-2} E_{n+2^{\ell-1},m}(f) \leq \int_{n+2^{\ell-2}}^{n+2^{\ell-1}} \phi(y)dy$$

and

$$\int_{n}^{n+1} \phi(y)dy \leq E_{n,m}(f),$$

which imply that

$$A_4 = 2 \sum_{\ell=1}^{\alpha} 2^{\ell-2} E_{n+2^{\ell-1},m}(f)$$
$$\leq 2 \sum_{\ell=1}^{\alpha} \int_{n+2^{\ell-2}}^{n+2^{\ell-1}} \phi(y)dy \leq 2 \int_{n}^{n+2^{\alpha}} \phi(y)dy$$
$$= 2 \sum_{i=n}^{n+2^{\alpha}-1} \int_{i}^{i+1} \phi(y)dy \leq 2 \sum_{i=n}^{n+2^{\alpha}-1} E_{i,m}(f) \leq 2 \sum_{i=n}^{n+p} E_{i,m}(f).$$

Similarly,

$$A_3 \le c \ln(m+1) \sum_{i=n}^{n+p} E_{i,m}.$$

Note that

$$\ln(u+v) \le \ln(1+u) + \ln(1+v) \quad (u, v > 0).$$

Setting $u = x/z$ and $v = y/z$, we obtain

$$\ln\Big(\frac{x+y}{z}\Big) \le \ln\left(1 + \frac{x}{z}\right) + \ln\left(1 + \frac{y}{z}\right),$$

and so,

$$\begin{aligned} A_2 &= \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m} \ln \frac{n+2^\ell}{2^{\ell-1}} \\ &\le \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m} \ln\left(1 + \frac{n}{2^{\ell-1}}\right) \\ &\quad + \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m} \ln\left(1 + \frac{2^\ell}{2^{\ell-1}}\right) \\ &= A_2^{(1)} + A_2^{(2)}, \quad \textit{say}. \end{aligned}$$

Now we have

$$A_2^{(2)} \le c \sum_{\ell=1}^{\alpha} 2^{\ell-1} E_{n+2^{\ell-1},m}(f) \le c \sum_{i=n}^{n+p} E_{i,m}(f)$$

and

$$\begin{aligned} A_2^{(1)} &= E_{n+1,m}(f) \ln(n+1) + 2 \sum_{\ell=2}^{\alpha} 2^{\ell-2} E_{n+2^{\ell-1},m}(f) \ln\Big(1 + \frac{n}{2^{\ell-1}}\Big) \\ &= E_{n+1,m}(f) \ln(n+1) + 2 \sum_{\ell=2}^{\alpha} \sum_{i=n+2^{\ell-2}}^{n+2^{\ell-1}-1} E_{n+2^{\ell-1},m}(f) \ln\Big(1 + \frac{n}{2^{\ell-1}}\Big) \\ &\le c \sum_{i=n}^{n+2^{\alpha-1}-1} E_{i,m}(f) \ln\Big(1 + \frac{n}{i-n+1}\Big) \end{aligned}$$

$$\leq c\sum_{i=n}^{n+p} E_{i,m}(f)\ln\Big(1+\frac{n}{i-n+1}\Big).$$

Combining the estimates for $A_2^{(1)}$ and $A_2^{(2)}$, we obtain

$$A_2 \leq c\left[\sum_{i=n}^{n+p} E_{i,m}(f) + \sum_{i=n}^{n+p} E_{i,m}(f)\ln\frac{i+1}{i-n+1}\right].$$

Similarly,

$$A_1 \leq c\ln(m+1)\left[\sum_{i=n}^{n+p} E_{i,m}(f) + \sum_{i=n}^{n+p} E_{i,m}(f)\ln\frac{i+1}{i-n+1}\right].$$

From all the above estimates, we get

$$\begin{aligned} I_2 &\leq c\sum_{i=n}^{n+p}(1+\ln(m+1))\left(1+\ln\frac{i+1}{i-n+1}\right)E_{i,m}(f)\\ &\leq c\sum_{i=n}^{n+p}\sum_{j=m}^{m+q}\left(1+\ln\frac{i+1}{i-n+1}\right)\left(1+\ln\frac{j+1}{j-m+1}\right)E_{i,j}(f). \end{aligned}$$

Similarly, we can show that

$$I_4 \leq c\sum_{i=n}^{n+p}\sum_{j=m}^{m+q}\left(1+\ln\frac{i+1}{i-n+1}\right)\left(1+\ln\frac{j+1}{j-m+1}\right)E_{i,j}(f).$$

For I_3, note that

$$\begin{aligned} I_3 &= (p+1-2^\alpha)\Big(\ln\frac{n+p+1}{p+1-2^\alpha}\ln\frac{m+1}{1}\\ &\qquad\qquad + \ln\frac{n+p+1}{p+1-2^\alpha} + \ln(m+1)+1\Big)E_{n+2^\alpha,m}\\ &= \ln(m+1)(p+1-2^\alpha)E_{n+2^\alpha,m}\ln\frac{n+p+1}{p+1-2^\alpha}\\ &\quad + (p+1-2^\alpha)E_{n+2^\alpha,m}\ln\frac{n+p+1}{p+1-2^\alpha}\\ &\quad + \ln(m+1)(p+1-2^\alpha) + \ln(m+1)(p+1-2^\alpha)E_{n+2^\alpha,m}\\ &\quad + (p+1-2^\alpha)E_{n+2^\alpha,m}\\ &= B_1+B_2+B_3+B_4, \quad \textit{say}. \end{aligned}$$

Since $p+1-2^\alpha \le 2^\alpha$, we have

$$B_4 \le 2^\alpha E_{n+2^\alpha,m}(f) \le \sum_{i=n}^{n+2^\alpha-1} E_{i,m}(f) \le \sum_{i=n}^{n+p} E_{i,m}(f).$$

Similarly,

$$B_3 \le \ln(m+1)\sum_{i=n}^{n+p} E_{i,m}(f).$$

For natural numbers α, β $(1 \le \alpha \le \beta - 1)$ we have

$$\begin{aligned}
\alpha \ln\frac{\beta-\alpha}{\alpha} &= \ln\frac{\beta-\alpha}{\alpha} + \ln\frac{\beta-\alpha}{\alpha} + \cdots + \ln\frac{\beta-\alpha}{\alpha} \\
&\le \ln\frac{\beta-\alpha}{1} + \ln\frac{\beta-\alpha}{2} + \cdots + \ln\frac{\beta-\alpha}{\alpha} \\
&= \sum_{k=\beta-\alpha+1}^{\beta} \ln\frac{\beta-\alpha}{k-\beta+\alpha}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(p+1-2^\alpha)\ln\frac{n+p+1}{p+1-2^\alpha} \\
&\quad \le \sum_{k=n+p+2}^{n+2p+2-2^\alpha} \ln\frac{n+p+1}{k-n-p-1} = \sum_{k=n}^{n+p-2^\alpha} \ln\frac{n+p+1}{k-n+1} \\
&\quad \le \sum_{k=n}^{n+p-2^\alpha} \ln\left(1+\frac{n}{k-n+1}\right) + \sum_{k=n}^{n+p-2^\alpha} \ln\left(1+\frac{p+1}{k-n+1}\right)
\end{aligned}$$

But

$$\begin{aligned}
\sum_{k=n}^{n+p-2^\alpha} \ln\left(1+\frac{p+1}{k-n+1}\right) &\le \sum_{k=1}^{p+1-2^\alpha} \ln\left(1+\frac{p+1}{k}\right) \\
&\le \sum_{k=1}^{p+1} \ln\left(1+\frac{p+1}{k}\right) \le c(p+1) \le c2^\alpha,
\end{aligned}$$

where we used

$$\ln(1+x) = \ln x + \mathcal{O}\left(\frac{1}{x}\right), \quad (x \ge 1)$$

and

$$\sum_{k=1}^{q} \ln \frac{q}{k} \leq \ln e^q = q.$$

So,

$$(p+1-2^\alpha) \ln \frac{n+p+1}{p+1-2^\alpha} \leq c \left[2^\alpha + \sum_{i=n}^{n+p-2^\alpha} \ln \left(1 + \frac{n}{i-n+1} \right) \right].$$

Since $2^\alpha \leq p+1$, we have

$$\begin{aligned} &E_{n,m}(f) + E_{n+1,m}(f) + \cdots + E_{n+p,m}(f) \\ &\qquad \geq (p+1) E_{n+2^\alpha,m}(f) \geq 2^\alpha E_{n+2^\alpha,m}(f). \end{aligned}$$

Therefore,

$$\begin{aligned} B_2 &= (p+1-2^\alpha) E_{n+2^\alpha,m}(f) \ln \frac{n+p+1}{p+1-2^\alpha} \\ &\leq c \left[2^\alpha E_{n+2^\alpha,m}(f) + \sum_{i=n}^{n+p-2^\alpha} E_{n+2^\alpha,m}(f) \ln \left(1 + \frac{n}{i-n+1} \right) \right] \\ &\leq c \left[\sum_{i=n}^{n+p} E_{i,m}(f) + \sum_{i=n}^{n+p} E_{i,m}(f) \ln \frac{i+1}{i-n+1} \right]. \end{aligned}$$

Similarly, we can show that

$$B_1 \leq c \ln(m+1) \left[\sum_{i=n}^{n+p} E_{i,m}(f) + \sum_{i=n}^{n+p} E_{i,m}(f) \ln \frac{i+1}{i-n+1} \right]$$

and obtain an estimate for I_5. Consequently, we have

$$I_3 + I_5 \leq c \sum_{i=n}^{n+p} \sum_{j=m}^{m+q} \left(1 + \ln \frac{i+1}{i-n+1} \right) \left(1 + \ln \frac{j+1}{j-m+1} \right) E_{i,j}(f).$$

Also it is easy to show that for $k = 6, 7, 8, 9$ the following inequality holds:

$$I_k \leq \sum_{i=n}^{n+p} \sum_{j=m}^{m+q} \left(1 + \ln \frac{i+1}{i-n+1} \right) \left(1 + \ln \frac{j+1}{j-m+1} \right) E_{i,j}(f).$$

Combining all the above estimates for I_k ($k = 1, 2, \ldots, 9$), we get the desired inequality (2.1). □

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