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Another general inequality for CR-warped products in complex space forms

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Abstract. We prove that every CR-warped product $N_T \times_f N_\perp$ in a complex space form $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$ satisfies a general inequality: $||\sigma||^2 \geq 2p\{||\nabla(\ln f)||^2 + \Delta(\ln f)\} + 4hpc$, where $h = \dim_{\mathbf{C}} N_T$, $p = \dim_{\mathbf{R}} N_\perp$, and σ is the second fundamental form. We also completely classify CR-warped products in a complex space form which satisfy the equality case of this inequality.

Key words: CR-submanifold, CR-warped product, squared norm of second fundamental form, warping function, warped product, tensor product.

1. Introduction

A submanifold N of a Kähler manifold is called a CR-submanifold if there exists on N a differentiable holomorphic distribution $\mathcal{D}$ whose orthogonal complementary distribution $\mathcal{D}^\perp$ is a totally real distribution, i.e., $J\mathcal{D}_x^\perp \subset T_x^\perp N$ (cf. [1]). Throughout this paper we denote the complex rank of $\mathcal{D}$ by h and the real rank of $\mathcal{D}^\perp$ by p. The study of CR-submanifolds has been a very active field of research during the last two decades (see, for instance, [1–4, 6–9, 11, 13, 14]).

A CR-submanifold is called a CR-product if it is the direct product $N_T \times N_\perp$ of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$. It was proved in [3] that a CR-product in a complex Euclidean space is a direct product of a holomorphic submanifold and a totally real submanifold of complex linear subspaces. It was also proved in [3] that there do not exist non-proper CR-products in complex hyperbolic spaces. Moreover, CR-products in the complex projective space CP^{h+p+hp} are obtained from the Segre imbedding in a natural way.

Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F, respectively, and f be a positive differentiable function on B. The warped product $B \times_f F$ is the product manifold $B \times F$ equipped with the Riemannian metric $g = g_B + f^2 g_F$. The function f is called the *warping*

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function. A warped product is said to be *proper* if its warping function is non-constant. The warping function is the main structure of a warped product manifold. It is well-known that warped products play some important roles in differential geometry as well as in mathematical physics (cf. [12]).

It was shown in [4] that there do not exist warped products of the form: $N_\perp \times_f N_T$ in a Kähler manifold beside CR-products, where $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold. By contrast, it was also shown that there exist many CR-submanifolds which are warped products of the form $N_T \times_f N_\perp$ *by reversing the two factors* N_T *and* $N_\perp$. Such a warped product CR-submanifold is simply called a *CR-warped product*.

It was known in [4] that every CR-warped product satisfies a general inequality: $||\sigma||^2 \geq 2p||\nabla(\ln f)||^2$, where $\nabla(\ln f)$ is the gradient of $\ln f$ and σ is the second fundamental form. CR-warped products in complex space forms satisfying the equality case of this inequality have been completely classified in [4].

In this paper we prove that every CR-warped product $N_T \times_f N_\perp$ in a complex space form $\tilde{M}^m(4c)$ satisfies another general inequality:

$$||\sigma||^2 \geq 2p\{||\nabla \ln f||^2 + \Delta(\ln f)\} + 4hpc, \tag{1.1}$$

where Δ denotes the Laplacian operator of the CR-warped product.

For any three natural numbers h, p, α satisfying $\alpha \leq h$, we introduce a map $\phi_\alpha^{hp} : \mathbf{C}_*^h \times S^p \to \mathbf{C}^{\alpha p+h}$, $\mathbf{C}_*^h = \mathbf{C}^h - \{0\}$, in a way similar to Segre imbedding. We show that each ϕ_α^{hp} is a CR-warped product in the complex Euclidean space $\mathbf{C}^{\alpha p+h}$ (Theorem 3.1). We also prove that, up to rigid motions, every CR-warped product in a complex Euclidean space satisfying the equality case of inequality (1.1) is one of the ϕ_α^{hp} (Theorem 4.1). Finally, we prove that every CR-warped product satisfying the equality in a complex projective space or a complex hyperbolic space is obtained from a ϕ_α^{hp} via the Hopf fibration (Theorems 5.1 and 6.1).

2. Preliminaries

Let M be a Riemannian n-manifold with inner product $\langle\ ,\ \rangle$ and $e_1, \ldots, e_n$ be an orthonormal frame fields on M. For differentiable function φ on M, the gradient $\nabla\varphi$ and the Laplacian $\Delta\varphi$ of φ are defined respectively by

$$\langle \nabla\varphi, X\rangle = X\varphi, \tag{2.1}$$

$$\Delta\varphi = \sum_{j=1}^{n} \{e_j e_j \varphi - (\nabla_{e_j} e_j)\varphi\} \tag{2.2}$$

for vector field X tangent to M, where ∇ is the Riemannian connection on M. If M is isometrically immersed in a Riemannian manifold $\tilde{M}$. Then the formulas of Gauss and Weingarten for M in $\tilde{M}$ are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \tag{2.3}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.4}$$

for vector fields X, Y tangent to N and ξ normal to M, where $\tilde{\nabla}$ denotes the Levi-Civita connection on $\tilde{M}$, σ the second fundamental form, D the normal connection, and A the shape operator of M in $\tilde{M}$. The second fundamental form and the shape operator are related by $\langle A_\xi X, Y\rangle = \langle \sigma(X, Y), \xi\rangle$, where $\langle\ ,\ \rangle$ denotes the inner product on M as well as on $\tilde{M}$.

The *equation of Gauss* is given by

$$\begin{aligned} \tilde{R}(X, Y; Z, W) = R(X, Y; Z, W) &+ \langle \sigma(X, Z), \sigma(Y, W)\rangle \\ &- \langle \sigma(X, W), \sigma(Y, Z)\rangle, \end{aligned} \tag{2.5}$$

for X, Y, Z, W tangent to M, where R and $\tilde{R}$ denote the curvature tensors of M and $\tilde{M}$, respectively.

For the second fundamental form σ, we define its covariant derivative $\bar{\nabla}\sigma$ with respect to the connection on $TM \oplus T^\perp M$ by

$$(\bar{\nabla}_X \sigma)(Y, Z) = D_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z). \tag{2.6}$$

The *equation of Codazzi* is

$$(\tilde{R}(X, Y)Z)^\perp = (\bar{\nabla}_X \sigma)(Y, Z) - (\bar{\nabla}_Y \sigma)(X, Z), \tag{2.7}$$

where $(\tilde{R}(X, Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X, Y)Z$.

For a *CR*-submanifold M in a Kähler manifold $\tilde{M}$ with complex structure J, we denote by ν the complementary orthogonal subbundle of $J\mathcal{D}^\perp$ in the normal bundle $T^\perp M$. Hence we have the following orthogonal direct sum decomposition:

$$T^\perp M = J\mathcal{D}^\perp \oplus \nu, \quad J\mathcal{D}^\perp \perp \nu. \tag{2.8}$$

We recall the following lemma from [3] for later use.

Lemma 2.1 *Let M be a CR-submanifold in a Kähler manifold $\tilde{M}$. Then we have*

(1) $\langle \nabla_U Z, X\rangle = \langle JA_{JZ}U, X\rangle$,

(2) $A_{JZ}W = A_{JW}Z$, *and*

(3) $A_{J\xi}X = -A_\xi JX$,

for any vectors U tangent to M, X, Y in $\mathcal{D}$, Z, W in $\mathcal{D}^\perp$, and ξ in ν.

Let (x,u) be a point in a CR-warped product $N_T \times_f N_\perp$. Then, for each $X \in T_x(N_T)$, there is a unique vector in $\mathcal{D}$ at (x,u) whose projection under $\pi_T : N_T \times_f N_\perp \to N_T$ is the vector X. In this way, one may regards a vector field U on N_T as a vector field U lying in the holomorphic distribution $\mathcal{D}$ in a natural way. Similarly, one may also regard a vector field Z on $N_\perp$ as a vector field in the totally real distribution $\mathcal{D}^\perp$.

For CR-warped products in Kähler manifolds we have the following [4].

Lemma 2.2 *If $N_T \times_f N_\perp$ is a CR-warped product in a Kähler manifold $\tilde{M}$, then we have*

(1) $\langle \sigma(\mathcal{D},\mathcal{D}), J\mathcal{D}^\perp\rangle = 0$;

(2) $\nabla_X Z = \nabla_Z X = (X \ln f)Z$;

(3) $\langle \sigma(JX,Z), JW\rangle = (X\ln f)\langle Z, W\rangle$

for any vector fields X on N_T and Z, W in $N_\perp$.

Recall that the Riemann curvature tensor of a complex space form $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$ is given by

$$\begin{aligned}
&\tilde{R}(X,Y;Z,W) \\
&\quad = c\{ \langle X,W\rangle\langle Y,Z\rangle - \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle \\
&\qquad - \langle JX,Z\rangle\langle JY,W\rangle + 2\langle X,JY\rangle\langle JZ,W\rangle \}.
\end{aligned} \tag{2.9}$$

3. A class of CR-warped products in complex Euclidean space

Let $\mathbf{C}^h_* = \mathbf{C}^h - \{0\}$ and $j : S^p \to \mathbf{E}^{p+1}$ be the inclusion of the unit hypersphere S^p centered at the origin into $\mathbf{E}^{p+1}$. For a natural number $\alpha \le h$ and a vector X tangent to $\mathbf{C}^\alpha_*$ at a point $z \in \mathbf{C}^\alpha_*$, we decompose X as $X = X^{||}_z + X^\perp_z$, where $X^{||}_z$ is parallel to z and $X^\perp_z$ is perpendicular to z.

For any given three natural numbers h, p, α satisfying $\alpha \le h$, we introduce a map $\phi^{hp}_\alpha : \mathbf{C}^h_* \times S^p \to \mathbf{C}^{\alpha p + h}$ by

$$\phi(z,w) = \big(w_0 z_1, \ldots, w_p z_1, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h\big) \tag{3.1}$$

for $z = (z_1, \dots, z_h) \in \mathbf{C}_*^h$ and $w = (w_0, \dots, w_p) \in S^p \subset \mathbf{E}^{p+1}$ with $\sum_{t=0}^p w_t^2 = 1$.

Theorem 3.1 *For $1 \leq \alpha \leq h$ and $p \geq 1$, the map $\phi_\alpha^{hp} : \mathbf{C}_*^h \times S^p \to \mathbf{C}^{\alpha p+h}$ defined by (3.1) satisfies the following properties:*

(1) *$\phi_\alpha^{hp} : \mathbf{C}_*^h \times_f S^p \to \mathbf{C}^{\alpha p+h}$ is an isometric immersion with warping function: $f = \sqrt{\sum_{j=1}^\alpha z_j \bar{z}_j}$.*

(2) *ϕ_α^{hp} is a CR-warped product.*

(3) *The second fundamental form σ of ϕ_α^{hp} satisfies the equality:*

$$||\sigma||^2 = 2p\{||\nabla(\ln f)||^2 + \Delta(\ln f)\}. \tag{3.2}$$

Proof. For tangent vector fields X of $\mathbf{C}_*^h$ and Z of S^p, we obtain from (3.1) that

$$X\phi_\alpha^{hp} = (X^{(1)} \otimes j, X_{\alpha+1}, \dots, X_h), \tag{3.3}$$

$$Z\phi_\alpha^{hp} = (z^{(1)} \otimes Z, 0, \dots, 0), \tag{3.4}$$

where

$$X^{(1)} \otimes j = (w_0 X_1, \dots, w_p X_1, \dots, w_0 X_\alpha, \dots, w_p X_\alpha), \tag{3.5}$$

$$z^{(1)} \otimes Z = (Z_0 z_1, \dots, Z_p z_1, \dots, Z_0 z_\alpha, \dots, Z_p z_\alpha), \tag{3.6}$$

$$X^{(1)} = (X_1, \dots, X_\alpha), \quad X^{(2)} = (X_{\alpha+1}, \dots, X_h), \tag{3.7}$$

$$X = (X^{(1)}, X^{(2)}), \quad Z = (Z_0, \dots, Z_p), \quad z^{(1)} = (z_1, \dots, z_\alpha). \tag{3.8}$$

From (3.3) and (3.4) we know that the tangent space of $\mathbf{C}_*^h \times S^p$ at a point (z, w) is spanned by vectors given by (3.3) and (3.4). Since S^p is the unit hypersphere centered at the origin, it follows from (3.3) and (3.4) that the induced metric on $\mathbf{C}_*^h \times S^p$ via ϕ_α^{hp} is the warped product metric $g = g_0 + f^2 g_1$ with warping function $f = \sqrt{\sum_{j=1}^\alpha z_j \bar{z}_j}$, where g_0 and g_1 denote the metrics of $\mathbf{C}_*^h$ and S^p, respectively. This proves statement (1).

It follows from (3.3) that $\mathbf{C}_*^h$ is immersed as a holomorphic submanifold of $\mathbf{C}^{\alpha p+h}$. From (3.3) and (3.4) we also know that S^p is immersed as a totally real submanifold of $\mathbf{C}^{\alpha p+h}$. Hence we have statement (2).

Applying (3.1) and (3.3)–(3.8) yields

$$XY\phi_\alpha^{hp} = (\tilde{\nabla}_{X^{(1)}} Y^{(1)} \otimes j, \tilde{\nabla}_{X^{(2)}} Y^{(2)}), \tag{3.9}$$

$$ZW\phi_\alpha^{hp} = (z^{(1)} \otimes \tilde{\nabla}_Z W, 0, \dots, 0), \tag{3.10}$$

$$XZ\phi_{\alpha}^{hp} = \big(X^{(1)} \otimes Z, 0, \dots, 0\big), \tag{3.11}$$

for vector fields X, Y tangent to $\mathbf{C}_*^h$ and Z, W tangent to S^p, where $\tilde{\nabla}$ denotes the Levi-Civita connection for Euclidean space as well as for complex Euclidean space.

From (3.3)–(3.4) and (3.9)–(3.11), we find

$$\sigma(X,Y) = \sigma(Z,W) = 0, \quad \sigma(X,Z) = \big(X_{z^{(1)}}^{(1)\,\perp} \otimes Z, 0, \dots, 0\big) \tag{3.12}$$

for vector fields X, Y tangent to $\mathbf{C}_*^h$ and Z, W tangent to S^p. Therefore, the squared norm of the second fundamental form is given by

$$||\sigma||^2 = \frac{2p(2\alpha - 1)}{f^2}, \quad f^2 = \sum_{j=1}^{\alpha} z_j \bar{z}_j. \tag{3.13}$$

On the other hand, it is straightforward to verify that

$$||\nabla(\ln f)||^2 = \frac{1}{f^2}, \quad \Delta(\ln f) = \frac{2(\alpha - 1)}{f^2}. \tag{3.14}$$

By combining (3.13) and (3.14) we obtain statement (3). □

4. *CR*-warped products in complex Euclidean space

The purpose of this section is to prove the following.

Theorem 4.1 *Let $\phi : N_T \times_f N_\perp \to \mathbf{C}^m$ be a CR-warped product in complex Euclidean m-space $\mathbf{C}^m$. Then we have*

(1) *The squared norm of the second fundamental form of ϕ satisfies*

$$||\sigma||^2 \geq 2p\big\{||\nabla(\ln f)||^2 + \Delta(\ln f)\big\}. \tag{4.1}$$

(2) *If the CR-warped product satisfies the equality case of* (4.1)*, then we have*

(2.a) *N_T is an open portion of $\mathbf{C}_*^h$;*

(2.b) *$N_\perp$ is an open portion of S^p;*

(2.c) *There exists a natural number $\alpha \leq h$ and a complex coordinate system $\{z_1, \dots, z_h\}$ on $\mathbf{C}_*^h$ such that the warping function f is given by $f = \sqrt{\sum_{j=1}^{\alpha} z_j \bar{z}_j}$;*

(2.d) *Up to rigid motions of* $\mathbf{C}^m$, *the immersion* ϕ *is given by* ϕ_α^{hp} *in a natural way; namely, we have*

$$\phi(z,w) = \big(w_0 z_1, \dots, w_p z_1, \dots, w_0 z_\alpha, \dots, w_p z_\alpha, z_{\alpha+1}, \dots, z_h, 0, \dots, 0\big) \tag{4.2}$$

for $z = (z_1, \dots, z_h) \in \mathbf{C}_*^h$ *and* $w = (w_0, \dots, w_p) \in S^p \subset \mathbf{E}^{p+1}$.

Proof. Let $N_T \times_f N_\perp$ be a *CR*-warped product in a complex space form $\tilde{M}^m(4c)$ of constant holomorphic sectional curvature $4c$. Then the equation of Codazzi implies

$$\begin{aligned}&\tilde{R}(X, JX, JZ, Z)\\ &\quad = \langle D_{JX}\sigma(X,Z) - \sigma(\nabla_{JX}X, Z) - \sigma(X, \nabla_{JX}Z), JZ\rangle\\ &\qquad - \langle D_X\sigma(JX,Z) - \sigma(\nabla_X JX, Z) - \sigma(JX, \nabla_X Z), JZ\rangle\,,\end{aligned} \tag{4.3}$$

for vector fields X on N_T and Z on $N_\perp$. Since N_T is totally geodesic in $N_T \times_f N_\perp$, $\nabla_X Z$ and $\nabla_{JX} Z$ lie in $\mathcal{D}^\perp$ and $\nabla_X JX$ and $\nabla_{JX} X$ lie in $\mathcal{D}$. Hence, by applying statements (2) and (3) of Lemma 2.2, we get

$$\begin{aligned}2\langle X,X\rangle\langle Z,Z\rangle c = &-JX(\langle Z,Z\rangle JX\ln f) - \langle\sigma(X,Z), D_{JX}JZ\rangle\\ &- X(\langle Z,Z\rangle X\ln f) + \langle\sigma(JX,Z), D_X JZ\rangle\\ &+ \{(J\nabla_{JX}X)\ln f - (J\nabla_X JX)\ln f\}\langle Z,Z\rangle\\ &+ \{(X\ln f)^2 + ((JX\ln f))^2\}\langle Z,Z\rangle\,.\end{aligned} \tag{4.4}$$

Applying Lemma 2.2 we find

$$\begin{aligned}&JX(\langle Z,Z\rangle JX\ln f) + X(\langle Z,Z\rangle X\ln f)\\ &\quad = \{(JX)^2\ln f + X^2\ln f + 2(JX\ln f)^2 + 2(X\ln f)^2\}\langle Z,Z\rangle\,.\end{aligned} \tag{4.5}$$

Since $\tilde{M}^m(4c)$ is Kählerian, we have

$$J\nabla_X Z = J\sigma(X,Z) = -A_{JZ}X + D_X JZ. \tag{4.6}$$

Applying (4.6) and statements (1), (2) and (3) of Lemma 2.2, we find

$$\begin{aligned}\langle\sigma(JX,Z), D_X JZ\rangle &= \langle\sigma(JX,Z), J\nabla_X Z\rangle + \langle\sigma(JX,Z), J\sigma(X,Z)\rangle\\ &= (X\ln f)^2\langle Z,Z\rangle + \langle\sigma(JX,Z), J\sigma(X,Z)\rangle\end{aligned} \tag{4.7}$$

for vector fields X in $\mathcal{D}$ and Z in $\mathcal{D}^\perp$.

On the other hand, if we denote by $\sigma_\nu(X,Z)$ the ν-component of

$\sigma(X,Z)$, then, by applying statement (3) of Lemma 2.1, we also have

$$\begin{aligned}&\langle\sigma(JX,Z),J\sigma(X,Z)\rangle=\langle\sigma(JX,Z),J\sigma_\nu(X,Z)\rangle\\&\quad=\left\langle A_{J\sigma_\nu(X,Z)}JX,Z\right\rangle=\left\langle A_{\sigma_\nu(X,Z)}X,Z\right\rangle=||\sigma_\nu(X,Z)||^2.\end{aligned}\tag{4.8}$$

Combining (4.7) and (4.8) yields

$$\langle\sigma(JX,Z),D_XJZ\rangle=(X\ln f)^2\langle Z,Z\rangle+||\sigma_\nu(X,Z)||^2.\tag{4.9}$$

Similarly, we also have

$$\langle\sigma(X,Z),D_{JX}JZ\rangle=-(JX\ln f)^2\langle Z,Z\rangle-||\sigma_\nu(X,Z)||^2.\tag{4.10}$$

Because N_T is a holomorphic submanifold of a Kähler manifold and N_T is totally geodesic in $N_T\times_f N_\perp$, we find

$$J\nabla_{JX}X=\nabla_{JX}JX,\quad J\nabla_XJX=-\nabla_XX.\tag{4.11}$$

Combining (4.4), (4.5) and (4.9)–(4.11) we obtain

$$\begin{aligned}2\langle X,X\rangle\langle Z,Z\rangle c=\{&(\nabla_XX+\nabla_{JX}JX)\ln f-X^2\ln f\\&-(JX)^2\ln f\}\langle Z,Z\rangle+2||\sigma_\nu(X,Z)||^2.\end{aligned}\tag{4.12}$$

Assume that $\{X_1,\ldots,X_{2h}\}$ is an orthonormal frame of N_T and $\{Z_1,\ldots,Z_p\}$ an orthonormal frame on $N_\perp$. Then (4.12) implies

$$2\sum_{j=1}^{2h}\sum_{t=1}^{p}||\sigma_\nu(X_j,Z_t)||^2=4hpc-2p\,\Delta(\ln f).\tag{4.13}$$

On the other hand, statement (3) of Lemma 2.2 implies

$$\sum_{j=1}^{2h}\sum_{t=1}^{p}||\sigma_{J\mathcal{D}^\perp}(X_j,Z_t)||^2=p\,||\nabla\ln f||^2,\tag{4.14}$$

where $\sigma_{J\mathcal{D}^\perp}(X_j,Z_t)$ denotes the $J\mathcal{D}^\perp$-component of $\sigma(X_j,Z_t)$. Combining (4.13) and (4.14) gives

$$2\,||\sigma(\mathcal{D},\mathcal{D}^\perp)||^2=2p\big\{||\nabla\ln f||^2+\Delta(\ln f)+2hc\big\},\tag{4.15}$$

where $||\sigma(\mathcal{D},\mathcal{D}^\perp)||^2=\sum_{j=1}^{2h}\sum_{t=1}^{p}||\sigma(X_j,Z_t)||^2$. Equation (4.15) implies

$$||\sigma||^2\geq 2p\big\{||\nabla(\ln f)||^2+\Delta(\ln f)\big\}+4hpc.\tag{4.16}$$

In particular, if $\tilde{M}^m(4c)$ is the complex Euclidean m-space, inequality (4.16) reduces to inequality (4.1).

Now, let us assume that $\phi : N_T \times_f N_\perp \to \mathbf{C}^m$ is a CR-warped product satisfying the equality case of (4.1). Then (4.15) and the equality case of (4.1) imply

$$\sigma(\mathcal{D},\mathcal{D}) = 0, \quad \sigma(\mathcal{D}^\perp,\mathcal{D}^\perp) = 0. \tag{4.17}$$

Since N_T is totally geodesic in $N_T \times_f N_\perp$, the first equation in (4.17) and the totally geodesy of N_T in $N_T \times_f N_\perp$ imply that N_T is isometrically immersed as a totally geodesic holomorphic submanifold of $\mathbf{C}^m$. Hence, N_T is a open portion of a complex Euclidean h-space $\mathbf{C}^h$.

For vector fields X in $\mathcal{D}$ and Z, W in $\mathcal{D}^\perp$, Lemma 2.1 implies

$$\langle \nabla_W Z, X\rangle = \langle JA_{JZ}W, X\rangle = -\langle \sigma(JX, W), JZ\rangle. \tag{4.18}$$

Hence, by applying statement (2) of Lemma 2.2 and (4.18), we find

$$\langle \nabla_W Z, X\rangle = -(X \ln f)\langle Z, W\rangle. \tag{4.19}$$

On the other hand, if we denote by $\sigma^\perp$ the second fundamental form of $N_\perp$ in $M = N_T \times_f N_\perp$, we get $\langle \sigma^\perp(Z,W), X\rangle = \langle \nabla_W Z, X\rangle$. Combining this with (4.19) yields

$$\sigma^\perp(Z,W) = -\langle Z,W\rangle \nabla \ln f \tag{4.20}$$

Hence, by applying (4.20) and the second equation of (4.17), we see that $N_\perp$ is immersed as a totally umbilical submanifold of $\mathbf{C}^m$. Hence, $N_\perp$ is an open portion of an ordinary p-sphere S^p (or $\mathbf{R}$ when $p=1$).

If $p \geq 2$, we may assume that S^p is of radius one, by rescaling the warping function f if necessary. Consequently, $N_T \times_f N_\perp$ is an open portion of $\mathbf{C}^h \times_f S^p$ (or $\mathbf{C}^h \times_f \mathbf{R}$ when $p=1$). Hence, we may choose a complex Euclidean coordinate system $\{z_1, \dots, z_h\}$ on $\mathbf{C}^h$ and a coordinate system $\{u_1, \dots, u_p\}$ on S^p (or on $\mathbf{R}$ if $p=1$) so that the metric tensor on $N_T \times_f N_\perp$ is given by

$$g = \sum_{j=1}^{h} dz_j d\bar{z}_j + f^2\{du_1^2 + \cos^2 u_1 du_2^2 + \cdots + \cos^2 u_1 \cdots \cos^2 u_{p-1} du_p^2\}, \tag{4.21}$$

where $z_j = x_j + iy_j$, $i = \sqrt{-1}$.

Equation (4.21) and a straightforward computation imply that the Levi-Civita connection on $N_T \times_f N_\perp$ satisfies

$$\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial y_k} = \nabla_{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial y_k} = 0, \quad j,k = 1,\dots,h, \tag{4.22}$$

$$\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial u_t} = \frac{f_{x_j}}{f}\frac{\partial}{\partial u_t}, \quad j = 1,\dots,h; \quad t = 1,\dots,p, \tag{4.23}$$

$$\nabla_{\frac{\partial}{\partial y_j}} \frac{\partial}{\partial u_t} = \frac{f_{y_j}}{f}\frac{\partial}{\partial u_t}, \quad j = 1,\dots,h; \quad t = 1,\dots,p, \tag{4.24}$$

$$\nabla_{\frac{\partial}{\partial u_s}} \frac{\partial}{\partial u_t} = -\tan u_s \frac{\partial}{\partial u_t}, \quad 1 \le s < t \le p, \tag{4.25}$$

$$\begin{aligned}\nabla_{\frac{\partial}{\partial u_t}} \frac{\partial}{\partial u_t} &= -\prod_{s=1}^{t-1} \cos^2 u_s \sum_{k=1}^{h} \left(f f_{x_k} \frac{\partial}{\partial x_k} + f f_{y_k} \frac{\partial}{\partial y_k} \right) \\ &\quad + \sum_{q=1}^{t-1} \left(\frac{\sin 2u_q}{2} \prod_{s=q+1}^{t-1} \cos^2 u_s \right) \frac{\partial}{\partial u_q}, \quad t = 1,\dots,p.\end{aligned} \tag{4.26}$$

From equations (4.17), (4.22), (4.25) and (4.26), we know that the immersion ϕ satisfies

$$\phi_{z_j z_k} = \phi_{z_j \bar{z}_k} = \phi_{\bar{z}_j \bar{z}_k} = 0, \quad j,k = 1,\dots,h, \tag{4.27}$$

$$\phi_{u_s u_t} = -\tan u_s \phi_{u_t}, \quad 1 \le s < t \le p, \tag{4.28}$$

$$\begin{aligned}\phi_{u_t u_t} &= -\prod_{s=1}^{t-1} \cos^2 u_s \sum_{k=1}^{h} \left(f f_{x_k} \phi_{x_k} + f f_{y_k} \phi_{y_k} \right) \\ &\quad + \sum_{q=1}^{t-1} \left(\frac{\sin 2u_q}{2} \prod_{s=q+1}^{t-1} \cos^2 u_s \right) \phi_{u_q}, \quad t = 1,\dots,p,\end{aligned} \tag{4.29}$$

where $\phi_{z_j \bar{z}_k} = \partial\phi/\partial z_j \partial \bar{z}_k, \dots$, etc., and

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right). \tag{4.30}$$

Solving (4.27) gives

$$\phi(z_1,\dots,z_h,u_1,\dots,u_p)=\sum_{j=1}^{h}A_j(u_1,\dots,u_p)z_j+B(u_1,\dots,u_p) \tag{4.31}$$

for some $\mathbf{C}^m$-valued functions $A_1,\dots,A_h,B$. From (4.29) with $t=1$, we find

$$\phi_{u_1u_1}=-\frac{1}{2}\sum_{k=1}^{h}\left(\frac{\partial f^2}{\partial x_k}\phi_{x_k}+\frac{\partial f^2}{\partial y_k}\phi_{y_k}\right) \tag{4.32}$$

Substituting (4.31) into (4.32) yields

$$\sum_{j=1}^{h}\frac{\partial^2 A_j}{\partial u_1^2}z_j+\frac{\partial^2 B}{\partial u_1^2}=-\sum_{j=1}^{h}\frac{\partial f^2}{\partial \bar z_j}A_j. \tag{4.33}$$

Case (1): $\sum_{j=1}^{h}(\partial f^2/\partial \bar z_j)A_j$ is independent of $z_1,\dots,z_h$.

In this case, (4.33) implies

$$\frac{\partial^2 A_j}{\partial u_1^2}=0,\quad j=1,\dots,h, \tag{4.34}$$

$$\frac{\partial^2 B}{\partial u_1^2}=-\sum_{j=1}^{h}\frac{\partial f^2}{\partial \bar z_j}A_j. \tag{4.35}$$

Solving (4.34) gives

$$A_j(u_1,\dots,u_p)=D_j(u_2,\dots,u_p)u_1+E_j(u_2,\dots,u_p),\quad j=1,\dots,h, \tag{4.36}$$

for some vector functions $D_j(u_2,\dots,u_p)$, $E_j(u_2,\dots,u_p)$. Applying (4.31) and (4.36) yields $\left\langle \phi_{z_j},\phi_{z_j}\right\rangle=|D_j|^2u_1^2+2\left\langle D_j,E_j\right\rangle u_1+|E_j|^2$, where $\langle\ ,\ \rangle$ denotes the standard Euclidean inner product on $\mathbf{C}^h$. On the other hand, (4.21) gives $\left\langle \phi_{z_j},\phi_{z_j}\right\rangle=1$ which is independent of u_1. Thus, we obtain $D_1=\cdots=D_h=0$. Hence, (4.36) reduces to

$$A_j(u_1,\dots,u_p)=E_j(u_2,\dots,u_p),\quad j=1,\dots,h, \tag{4.37}$$

From (4.35) and (4.37), we find

$$B=-\frac{1}{2}\sum_{j=1}^{h}\frac{\partial f^2}{\partial \bar z_j}E_j(u_2,\dots,u_p)u_1^2+F(u_2,\dots,u_p)u_1+G(u_2,\dots,u_p) \tag{4.38}$$

for some vector functions F, G. Thus, we obtain from (4.31), (4.37) and (4.38) that

$$\phi = \sum_{j=1}^{h} E_j\left(z_j - \frac{1}{2}\frac{\partial f^2}{\partial \bar{z}_j}u_1^2\right) + Fu_1 + G. \tag{4.39}$$

Substituting (4.39) into (4.28) with $s = 1$ and $1 < t \leq p$ gives

$$\begin{aligned}&\frac{1}{2}\sum_{j=1}^{h}\frac{\partial f^2}{\partial \bar{z}_j}\frac{\partial E_j}{\partial u_t}u_1 - \frac{\partial F}{\partial u_t}\\ &= \tan u_1\left\{\sum_{j=1}^{h}\frac{\partial E_j}{\partial u_t}z_j - \frac{1}{2}\sum_{j=1}^{h}\frac{\partial f^2}{\partial \bar{z}_j}\frac{\partial E_j}{\partial u_t}u_1^2 + \frac{\partial F}{\partial u_t}u_1 + \frac{\partial G}{\partial u_t}\right\}.\end{aligned} \tag{4.40}$$

Since E_j, F, G and $\partial f^2/\partial \bar{z}_j$ are independent on the variable u_1, equation (4.40) implies $\partial E_j/\partial u_t = \partial F/\partial u_t = \partial G/\partial u_t = 0$ for $j = 1, \ldots, h$ and $t = 2, \ldots, p$. Thus, $E_1, \ldots, E_h, F, G$ are constant vectors in $\mathbf{C}^m$.

From (4.39) we also have

$$\phi_{u_1} = -\sum_{j=1}^{h}\frac{\partial f^2}{\partial \bar{z}_j}E_j u_1 + F. \tag{4.41}$$

On the other hand, using (4.21) we find $\langle \phi_{u_1}\phi_{u_1}\rangle = f^2$ which is a non-constant function independent of u_1. Hence, (4.41) implies

$$\sum_{j=1}^{h}(\partial f^2/\partial \bar{z}_j)E_j = 0.$$

Thus, $f^2 = |F|^2$ is constant which contradicts to properness of the *CR*-warped product.

Case (2): $\sum_{j=1}^{h}(\partial^2 f^2/\partial \bar{z}_j)A_j$ depends on $z_1, \ldots, z_h$.

In this case, by taking the derivative of (4.33) with respect to $\partial/\partial z_j$, we find

$$\frac{\partial^2 A_j}{\partial u_1^2} = -\sum_{k=1}^{h}\frac{\partial^2 f^2}{\partial z_j \partial \bar{z}_k}A_k, \quad j = 1, \ldots, h. \tag{4.42}$$

On the other hand, by applying (4.31), we find $\phi_{z_j} = A_j(u_1, \ldots, u_p)$. Thus, $A_1, \ldots, A_h$ form an orthonormal frame according to (4.21). Therefore, by using the fact that $\partial^2 A_j/\partial u_1^2$ and $A_1, \ldots, A_h$ are independent of $z_1, \ldots, z_h$, we know from (4.42) that $\partial^2 f^2/\partial z_k \partial \bar{z}_j$, $j, k = 1, \ldots, h$, are constant. Thus, we may put

$$\frac{\partial^2 f^2}{\partial z_j \partial \bar{z}_k} = \gamma_{j\bar{k}}, \quad j, k = 1, \ldots, h \tag{4.43}$$

for some constants $\gamma_{j\bar{k}}$.

Solving (4.43) yields

$$f^2(z_1, \ldots, z_h) = \sum_{j,k=1}^{h} \gamma_{j\bar{k}} z_j \bar{z}_k + H + K \tag{4.44}$$

for some functions H, K satisfying

$$\frac{\partial H}{\partial \bar{z}_j} = \frac{\partial K}{\partial z_j} = 0, \quad j = 1, \ldots, h. \tag{4.45}$$

Equation (4.43) implies that $(\gamma_{j\bar{k}})$ is a Hermitian matrix, that is $\bar{\gamma}_{j\bar{k}} = \gamma_{k\bar{j}}$. Therefore, the Spectral Theorem in matrix theory implies that there is a unitary matrix which diagonalizes $(\gamma_{j\bar{k}})$. Hence, there exists a suitable complex Euclidean coordinate system $\{z_1, \ldots, z_h\}$ on $\mathbf{C}^h$ such that (4.44) reduces to the form:

$$f^2 = \sum_{j=1}^{h} b_j z_j \bar{z}_j + H(z_1, \ldots, z_h) + K(z_1, \ldots, z_h). \tag{4.46}$$

Since f is a real-valued function, we may put

$$H = X + iY, \quad K = U - iY, \tag{4.47}$$

for some real-valued functions X, Y, U. From (4.45) and (4.47), we obtain the following Cauchy-Riemann equations:

$$\frac{\partial X}{\partial x_j} = -\frac{\partial Y}{\partial y_j}, \quad \frac{\partial Y}{\partial x_j} = \frac{\partial X}{\partial y_j}, \quad \frac{\partial U}{\partial x_j} = \frac{\partial Y}{\partial y_j}, \quad \frac{\partial Y}{\partial x_j} = -\frac{\partial U}{\partial y_j}. \tag{4.48}$$

From (4.48) we find that $H + K = X + U$ is constant, say δ. Hence, (4.46) becomes $f^2 = \sum_{j=1}^{h} b_j z_j \bar{z}_j + \delta$. We may assume $\delta = 0$ by applying a suitable translation on $\mathbf{C}^m$ if necessary. Thus, we have

$$f^2 = \sum_{j=1}^{h} a_j^2 z_j \bar{z}_j, \tag{4.49}$$

for some real numbers $a_1, \dots, a_h \geq 0$, since $f > 0$. Combining (4.33) and (4.49) yields

$$\frac{\partial^2 A_j}{\partial u_1^2} = -a_j^2 A_j, \quad j = 1, \dots, h, \tag{4.50}$$

$$\frac{\partial^2 B}{\partial u_1^2} = 0. \tag{4.51}$$

Since $f > 0$, there exists at least one a_j greater than zero. Without loss of generality, we may assume

$$a_1, \dots, a_\alpha > 0, \quad a_{\alpha+1} = \cdots = a_h = 0. \tag{4.52}$$

for some natural number $\alpha \leq h$. From (4.50), (4.51) and (4.53), we obtain

$$A_j = D_j(u_2, \dots, u_p) \cos(a_j u_1) + E_j(u_2, \dots, u_p) \sin(a_j u_1), \tag{4.53}$$

$$A_k = D_k(u_2, \dots, u_p) u_1 + E_k(u_2, \dots, u_p), \tag{4.54}$$

$$B = F(u_2, \dots, u_p) u_1 + G(u_2, \dots, u_p) \tag{4.55}$$

for $j = 1, \dots, \alpha$, and $k = \alpha + 1, \dots, h$.

Substituting (4.53), (4.54) and (4.55) into (4.31) gives

$$\begin{aligned} \phi = &\sum_{j=1}^{\alpha} \big(D_j(u_2, \dots, u_p) \cos(a_j u_1) + E_j(u_2, \dots, u_p) \sin(a_j u_1)\big) z_j \\ &+ \sum_{k=\alpha+1}^{h} \big(D_k(u_2, \dots, u_p) u_1 + E_k(u_2, \dots, u_p)\big) z_k \\ &+ F(u_2, \dots, u_p) u_1 + G(u_2, \dots, u_p). \end{aligned} \tag{4.56}$$

By differentiating (4.56) with respect to z_k, we obtain $\phi_{z_k} = D_k u_1 + E_k$ for $k = \alpha + 1, \dots, h$. Thus, $\langle \phi_{z_k}, \phi_{z_k} \rangle = |D_k|^2 u_1^2 + 2 \langle D_k, E_k \rangle + |E_k|^2$. Comparing this with (4.21) yields $D_{\alpha+1} = \cdots = D_h = 0$. Therefore, (4.56) becomes

$$\begin{aligned} \phi = &\sum_{j=1}^{\alpha} \big(D_j(u_2, \dots, u_p) \cos(a_j u_1) + E_j(u_2, \dots, u_p) \sin(a_j u_1)\big) z_j \\ &+ \sum_{k=\alpha+1}^{h} E_k(u_2, \dots, u_p) z_k + F(u_2, \dots, u_p) u_1 + G(u_2, \dots, u_p). \end{aligned} \tag{4.57}$$

From (4.28) with $s = 1$, $t > 1$ and (4.57), we find

$$\sum_{j=1}^{\alpha} a_j\left(\frac{\partial D_j}{\partial u_t}\sin(a_ju_1) - \frac{\partial E_j}{\partial u_t}\cos(a_ju_1)\right)z_j + \frac{\partial F}{\partial u_t}$$
$$= \tan u_1\left\{\sum_{j=1}^{\alpha}\left(\frac{\partial D_j}{\partial u_t}\cos(a_ju_1) + \frac{\partial E_j}{\partial u_t}\sin(a_ju_1)\right)z_j\right.$$
$$\left. + \sum_{k=\alpha+1}^{h}\frac{\partial E_k}{\partial u_t}z_k + \frac{\partial F}{\partial u_t}u_1 + \frac{\partial G}{\partial u_t}\right\} \tag{4.58}$$

which implies $\partial E_k/\partial u_t = \partial F/\partial u_t = \partial G/\partial u_t = 0$, $k = \alpha+1\ldots,h$, $t = 2,\ldots,p$. Hence, $E_{\alpha+1},\ldots,E_h, F$ and G are constant vectors. Equation (4.58) also implies

$$a_j\frac{\partial D_j}{\partial u_t}\sin(a_ju_1) - a_j\frac{\partial E_j}{\partial u_t}\cos(a_ju_1)$$
$$= \tan u_1\left\{\frac{\partial D_j}{\partial u_t}\cos(a_ju_1) + \frac{\partial E_j}{\partial u_t}\sin(a_ju_1)\right\},\quad j = 1,\ldots,\alpha, \tag{4.59}$$

which are equivalent to

$$\frac{\partial D_j}{\partial u_t}\{(a_j-1)\sin((a_j+1)u_1) - (a_j+1)\sin((a_j-1)u_1)\}$$
$$= \frac{\partial E_j}{\partial u_t}\{(a_j-1)\cos((a_j+1)u_1) + (a_j+1)\cos((a_j-1)u_1)\} \tag{4.60}$$

for $j = 1,\ldots,\alpha$. By letting $u_1 = 0$, we get $\partial E_j/\partial u_t = 0$. Thus, $E_1,\ldots,E_\alpha$ are constant vectors. Consequently, we obtain from (4.57) that

$$\phi = \sum_{j=1}^{\alpha}\big(D_j(u_2,\ldots,u_p)\cos(a_ju_1) + E_j\sin(a_ju_1)\big)z_j$$
$$+ \sum_{k=\alpha+1}^{h} E_kz_k + Fu_1 + G \tag{4.61}$$

where $E_1,\ldots,E_h, F, G$ are constant vectors. From (4.61) we obtain

$$\phi_{x_j} = D_j\cos(a_ju_1) + E_j\sin(a_ju_1),\quad j = 1,\ldots,\alpha, \tag{4.62}$$

$$\phi_{y_j} = iD_j\cos(a_ju_1) + iE_j\sin(a_ju_1),\quad j = 1,\ldots,\alpha, \tag{4.63}$$

$$\phi_{x_k} = E_k,\quad k = \alpha+1,\ldots,h, \tag{4.64}$$

$$\phi_{y_k} = iE_k, \quad k = \alpha+1, \ldots, h, \tag{4.65}$$

$$\phi_{u_1} = \sum_{j=1}^{\alpha} a_j\big(E_j \cos(a_j u_1) - D_j \sin(a_j u_1)\big) z_j + F. \tag{4.66}$$

By applying (4.21) and (4.62), we find

$$\begin{aligned} 2\delta_{j\ell} = {} & \langle D_j, D_\ell\rangle \big(\cos((a_j + a_\ell)u_1) + \cos((a_j - a_\ell)u_1)\big) \\ & + \langle E_j, E_\ell\rangle \big(\cos((a_j - a_\ell)u_1) - \cos((a_j + a_\ell)u_1)\big) \\ & + \langle D_j, E_\ell\rangle \big(\sin((a_j + a_\ell)u_1) - \sin((a_j - a_\ell)u_1)\big) \\ & + \langle D_\ell, E_j\rangle \big(\sin((a_j + a_\ell)u_1) + \sin((a_j - a_\ell)u_1)\big) \end{aligned} \tag{4.67}$$

for $j, \ell = 1, \ldots, \alpha$.

Since $\cos((a_j - a_\ell)u_1)$, $\cos((a_j + a_\ell)u_1)$ and $\sin((a_j + a_\ell)u_1)$ are independent functions, (4.67) implies $\langle D_j, E_\ell\rangle + \langle D_\ell, E_j\rangle = 0$ for $j, \ell = 1, \ldots, \alpha$. By setting $u_1 = 0$, (4.67) also yields $\langle D_j, D_\ell\rangle = \delta_{j\ell}$. Thus, by combining these with (4.67), we have $\langle E_j, E_\ell\rangle = \delta_{j\ell}$. Consequently, we obtain

$$\begin{gathered} \langle D_j, D_\ell\rangle = \langle E_j, E_\ell\rangle = \delta_{j\ell}, \quad \langle D_j, E_\ell\rangle + \langle E_j, D_\ell\rangle = 0, \\ 1 \le j,\ \ell \le \alpha. \end{gathered} \tag{4.68}$$

Similarly, by differentiating (4.67) with respect to u_1, we find

$$a_\ell \langle D_j, E_\ell\rangle + a_j \langle D_\ell, E_j\rangle = 0, \quad j, \ell = 1, \ldots, \alpha. \tag{4.69}$$

Also, from (4.21), (4.62) and (4.63), we find

$$\langle D_j, iD_\ell\rangle = \langle E_j, iE_\ell\rangle = \delta_{j\ell}, \quad \langle D_j, iE_\ell\rangle + \langle E_j, iD_\ell\rangle = 0, \tag{4.70}$$

$$a_\ell \langle D_j, iE_\ell\rangle + a_j \langle D_\ell, iE_j\rangle = 0, \quad j, \ell = 1, \ldots, \alpha. \tag{4.71}$$

From (4.21) and (4.62)–(4.65), we also have

$$\langle E_k, D_j\rangle = \langle E_k, E_j\rangle = \langle E_k, iD_j\rangle = \langle E_k, iE_j\rangle = 0 \tag{4.72}$$

for $j = 1, \ldots, \alpha$; $k = \alpha+1, \ldots, h$.

Equations (4.21), (4.49), (4.66), (4.68) and (4.70) imply

$$\begin{aligned} \sum_{j=1}^{\alpha} a_j^2 z_j \bar{z}_j = {} & \sum_{j=1}^{\alpha} a_j^2 z_j \bar{z}_j \\ & + 2\sum_{j=1}^{\alpha} a_j \left\langle (E_j \cos(a_j u_1) - D_j \sin(a_j u_1)) z_j, F\right\rangle + |F|^2. \end{aligned}$$

Thus, we obtain $F = 0$. Therefore, (4.61) reduces to

$$\phi = \sum_{j=1}^{\alpha}(D_j(u_2,\ldots,u_p)\cos(a_ju_1) + E_j\sin(a_ju_1))z_j + \sum_{k=\alpha+1}^{h} E_kz_k + G, \tag{4.73}$$

where $E_1,\ldots,E_h,G$ are constant vectors.

Using (4.60) we know that either D_j is a constant vector or $a_j = 1$. Without loss of generality, we may assume that $a_1,\ldots,a_r \neq 1$ and $a_{r+1} = \cdots = a_\alpha = 1$. Then, $D_1,\ldots,D_r$ are constant vectors; hence (4.73) reduces to

$$\begin{aligned}\phi = &\sum_{j=1}^{r}(D_j\cos(a_ju_1) + E_j\sin(a_ju_1))z_j \\ &+ \sum_{j=r+1}^{\alpha}(D_j(u_2,\ldots,u_p)\cos u_1 + E_j\sin u_1)z_j \\ &+ \sum_{k=\alpha+1}^{h} E_kz_k + G,\end{aligned} \tag{4.74}$$

where $D_1,\ldots,D_r,E_1,\ldots,E_h,G$ are constant vectors satisfying (4.68)–(4.72).

Substituting (4.49) and (4.74) into (4.29) with $t = 2$ yields

$$\begin{aligned}\sum_{j=r+1}^{\alpha}\cos u_1\frac{\partial^2 D_j}{\partial u_2^2}z_j = &-\cos^2 u_1\sum_{j=1}^{\alpha}a_j\big(D_j\cos(a_ju_1) + E_j\sin(a_ju_1)\big)z_j \\ &- \sin u_1\cos u_1\sum_{j=1}^{\alpha}a_j\big(D_j\sin(a_ju_1) - E_j\cos(a_ju_1)\big)z_j,\end{aligned} \tag{4.75}$$

where $a_{r+1} = \cdots = a_\alpha = 1$.

If $r > 1$, then (4.75) implies

$$\begin{aligned}&\cos u_1\big(D_j\cos(a_ju_1) + E_j\sin(a_ju_1)\big) \\ &+ \sin u_1\big(D_j\sin(a_ju_1) - E_j\cos(a_ju_1)\big) = 0, \quad j = 1,\ldots,r.\end{aligned} \tag{4.76}$$

Since $a_1,\ldots,a_r \neq 1$, equation (4.76) implies $D_1 = \cdots = D_r = E_1 = \cdots = E_r = 0$ which is a contradiction. Therefore, $a_1 = \cdots = a_\alpha = 1$. Hence, (4.75) implies $\partial^2 D_j/\partial u_2^2 = -D_j$ for $j = 1,\ldots,\alpha$. Solving these equations gives

$$D_j = F_j(u_3, \dots, u_p)\cos u_2 + G_j(u_3, \dots, u_p)\sin u_2.$$

Consequently, (4.73) becomes

$$\phi = \sum_{j=1}^{\alpha} \big\{F_j(u_3, \dots, u_p)\cos u_1 \cos u_2 + G_j(u_3, \dots, u_p)\cos u_1 \sin u_2 + E_j \sin u_1\big\} z_j + \sum_{k=\alpha+1}^{h} E_k z_k + G. \tag{4.77}$$

By substituting (4.77) into (4.28) with $s = 2$ and $t > 2$, we know that G_j are constant vectors. Continuing these procedures sufficiently many times, we obtain

$$\begin{aligned} \phi(z_1, \dots, z_h, u_1, \dots, u_p) &= \sum_{j=1}^{\alpha} \Big\{ c_1^j \prod_{t=1}^{p} \cos u_t + c_2^j \sin u_1 + c_3^j \sin u_2 \cos u_1 + \cdots \\ &\quad + c_{p+1}^j \sin u_p \prod_{t=1}^{p-1} \cos u_t \Big\} z_j + \sum_{k=\alpha+1}^{h} E_k z_k + G, \end{aligned} \tag{4.78}$$

where c_t^j, E_k, G are constant vectors in $\mathbf{C}^m$.

Because $N_T \times_f N_\perp$ is a CR-warped product in $\mathbf{C}^m$, we may choose the following initial conditions:

$$\begin{aligned} &\phi(1, 0, \dots, 0) = (1, 0, \dots, 0, \dots, 0), \\ &\phi_{z_1}(1, 0, \dots, 0) = (1, 0, \dots, 0, \dots, 0), \\ &\phi_{z_2}(1, 0, \dots, 0) = (0, 0, \dots, 0, \overbrace{1}^{p+2\text{-th}}, 0, \dots, 0), \\ &\qquad \cdots\cdots \\ &\phi_{z_\alpha}(1, 0, \dots, 0) = (0, \dots, 0, \overbrace{1}^{\alpha p - p + \alpha\text{-th}}, 0, \dots, 0), \\ &\phi_{z_{\alpha+1}}(1, 0, \dots, 0) = (0, \dots, 0, \overbrace{1}^{1 + \alpha p + \alpha\text{-th}}, 0, \dots, 0), \\ &\qquad \cdots\cdots \end{aligned}$$

$$\phi_{z_h}(1,0,\dots,0)=(0,\dots,0,\overbrace{1}^{\alpha p+h\text{-th}},0,\dots,0),$$

$$\phi_{u_1}(1,0,\dots,0)=(0,1,\dots,0,\overbrace{1}^{p+3\text{-th}},0,\dots,0,\overbrace{1}^{1+\alpha p-p+\alpha\text{-th}},0,\dots,0),$$

$$\dots\dots$$

$$\phi_{u_p}(1,0,\dots,0)=(0,\dots,0,\overbrace{1}^{p+1\text{-th}},0,\dots,0,\overbrace{1}^{\alpha(p+1)\text{-th}},0,\dots,0). \tag{4.79}$$

Applying (4.78) and (4.79) gives

$$\phi=\big(w_0z_1,\dots,w_pz_1,\dots,w_0z_\alpha,\dots,w_pz_\alpha,z_{\alpha+1},\dots,z_h,0,\dots,0\big), \tag{4.80}$$

where

$$w_0=\prod_{t=1}^{p}\cos u_t,\quad w_1=\sin u_1,$$

$$w_2=\sin u_2\cos u_1,\dots,w_{p+1}=\sin u_p\prod_{t=1}^{p-1}\cos u_t.$$

Since ϕ is an immersion, (4.80) implies that N_T is contained in $\mathbf{C}_*^h$. □

5. *CR*-warped products in CP^m satisfying the equality

In this section we determine *CR*-warped products in complex projectable spaces which satisfy the equality case of (4.16). In order to do so, we recall briefly a procedure via Hopf fibration to obtain the desired submanifolds of complex projective spaces.

Let $\mathbf{C}^*=\mathbf{C}-\{0\}$. Consider the $\mathbf{C}^*$-action on $\mathbf{C}_*^{m+1}$ defined by $\lambda\cdot(z_0,\dots,z_m)=(\lambda z_0,\dots,\lambda z_m)$. The set of equivalent classes obtained from this action is denoted by CP^m. Let $\pi(z)$ denote the equivalent class contains z. Then $\pi:\mathbf{C}_*^{m+1}\to CP^m$ is a surjection. It is well-known that the CP^m admits a complex structure induced from the complex structure on $\mathbf{C}^{m+1}$ and a Kähler metric g with constant holomorphic sectional curvature 4.

Assume $\psi : M\to CP^m(4)$ is an isometric immersion. Then $\breve{M}=\pi^{-1}(M)$ is a $\mathbf{C}^*$-bundle over M and the lift $\breve{\psi}:\pi^{-1}(M)\to\mathbf{C}_*^{m+1}$ of ψ is an isometric immersion satisfying $\pi\circ\breve{\psi}=\psi\circ\pi$. Conversely, if $\breve{\psi}:Q\to\mathbf{C}_*^{m+1}$ is an isometric immersion invariant under the $\mathbf{C}^*$-action, then there is a unique isometric immersion $\psi:\pi(Q)\to CP^m(4)$ satisfying $\pi\circ\breve{\psi}=\psi\circ\pi$.

There is an alternate way to view the lift $\breve{\psi} : \pi^{-1}(N) \to \mathbf{C}_*^{m+1}$ via the Hopf fibration as follows: Let S^{2m+1} denote the un it hypersphere of $\mathbf{C}^{m+1}$ centered at the origin and let $U(1) = \{\lambda \in \mathbf{C} : \lambda\bar{\lambda} = 1\}$. Then we have a $U(1)$-action on S^{2m+1} defined by $z \mapsto \lambda z$. At $z \in S^{2m+1} \subset \mathbf{C}^{m+1}$, the vector $V = iz$ is tangent to the flow of this action. The quotient space $S^{2m+1}/\sim$ obtained from this $U(1)$-action is exactly the $CP^m(4)$. Let $\varphi : S^{2m+1} \to CP^m(4)$ denote the projection via the $U(1)$-action. The projection φ is known as the *Hopf fibration*.

When $\psi : M \to CP^m(4)$ is an isometric immersion, $\hat{M} = \varphi^{-1}(M)$ is a principal circle bundle over M with totally geodesic fibers. The lift $\hat{\psi} : \hat{M} \to S^{2m+1}$ of ψ is an isometric immersion satisfying $\varphi \circ \hat{\psi} = \psi \circ \varphi$. Conversely, if $\psi : U \to S^{2m+1}$ is an isometric immersion which is invariant under $U(1)$-action, there is a unique isometric immersion $\psi_\varphi : \varphi(U) \to CP^m(4)$ satisfying $\varphi \circ \hat{\psi}_\varphi = \psi_\varphi \circ \varphi$.

For each vector X tangent to $CP^m(4)$, we denote by X^* a horizontal lift of X via the Hopf fibration φ. The horizontal lift X^* and X have the same length, since the Hopf fibration is a Riemannian submersion. Since $V = iz$ generates the vertical subspaces of the Hopf fibration, we have an orthogonal decomposition:

$$T_z S^{2m+1} = (T_{\varphi(z)} CP^m)^* \oplus \operatorname{Span}\{V\}, \tag{5.1}$$

where $(T_{\varphi(z)} CP^m)^*$ is the set consisting of all horizontal lifts of $T_{\varphi(z)} CP^m$ via φ.

For an isometric immersion $\psi : M \to CP^m(4)$, $\breve{M} = \pi^{-1}(M)$ is diffeomorphic to $\mathbf{R}^* \times \hat{M}$ where $\mathbf{R}^* = \mathbf{R} - \{0\}$ and $\hat{M} = \varphi^{-1}(M)$. The immersion $\breve{\psi} : \breve{M} \to \mathbf{C}_*^{m+1}$ is related to the immersion $\hat{\psi} : \hat{M} \to S^{2m+1}$ by

$$\breve{\psi}(t, q) = t\hat{\psi}(q), \quad t \in \mathbf{R}^*, \quad q \in \hat{M}. \tag{5.2}$$

Clearly, $\breve{M}$ is the *cone over* $\hat{M}$ with the vertex at the origin of $\mathbf{C}^{m+1}$. The metric $\breve{g}$ of $\breve{M}$ and the metric $\hat{g}$ of $\hat{M}$ are related by

$$\breve{g} = t^2 \hat{g} + dt^2. \tag{5.3}$$

The purpose of this section is to prove the following.

Theorem 5.1 *Let $\phi : N_T \times_f N_\perp \to CP^m(4)$ be a CR-warped product. Then*

(1) *The squared norm of the second fundamental form of ϕ satisfies*

$$||\sigma||^2 \geq 2p\{||\nabla(\ln f)||^2 + \Delta(\ln f)\} + 4hp. \tag{5.4}$$

(2) *The CR-warped product satisfies the equality case of* (5.4) *if and only if*

(2.i) *N_T is an open portion of complex projective h-space $CP^h(4)$;*

(2.ii) *$N_\perp$ is an open portion of unit p-sphere S^p; and*

(2.iii) *There exists a natural number $\alpha \leq h$ such that, up to rigid motions, ϕ is the composition $\pi \circ \breve{\phi}$, where*

$$\breve{\phi}(z,w) = \big(w_0 z_0, \ldots, w_p z_0, \ldots, w_0 z_\alpha, \ldots, w_p z_\alpha, z_{\alpha+1}, \ldots, z_h, 0 \ldots, 0\big) \tag{5.5}$$

for $z = (z_0, \ldots, z_h) \in \mathbf{C}_*^{h+1}$ and $w = (w_0, \ldots, w_p) \in S^p \subset \mathbf{E}^{p+1}$, and π being the projection $\pi : \mathbf{C}_*^{m+1} \to CP^m(4)$.

Proof. Inequality (5.4) is a special case of (4.16).

Let $\phi : M \to CP^m(4)$ be an isometric immersion and let $\breve{\nabla}, \hat{\nabla}$ and ∇ denote the Levi-Civita connections on $\breve{M}, \hat{M}$ and M respectively. Denote by $\hat{\sigma}$ the second fundamental form of the lift $\hat{\phi} : \hat{M} \to S^{2m+1}$ of ϕ via Hopf's fibration. Then we have

$$\hat{\nabla}_{X^*}Y^* = (\nabla_X Y)^* - \langle PX, Y\rangle V, \tag{5.6}$$

$$\hat{\nabla}_V X^* = \hat{\nabla}_{X^*}V = (PX)^*, \tag{5.7}$$

$$\hat{\nabla}_V V = 0, \tag{5.8}$$

$$\hat{\sigma}(X^*, Y^*) = (\sigma(X,Y))^*, \ \ \hat{\sigma}(X^*, V) = (FX)^*, \ \ \hat{\sigma}(V,V) = 0, \tag{5.9}$$

for vector fields X, Y tangent to M, where PX and FX are the tangential and the normal components of JX, respectively.

For a vector U tangent to $\hat{M} \subset S^{2m+1} \subset \mathbf{C}_*^{m+1}$, we extend U to a vector field, also denoted by U, in $\mathbf{C}_*^{m+1}$ by parallel translation along the rays of the cone $\breve{M}$ over $\hat{M}$. We obtain from (5.2) that

$$\breve{\sigma}(U,W)(t,q) = \frac{1}{t}\hat{\sigma}(U,W)(q), \quad t \in \mathbf{R}^*, \quad q \in \hat{M}, \tag{5.10}$$

$$\breve{\sigma}\Big(U, \frac{\partial}{\partial t}\Big) = \breve{\sigma}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = 0, \tag{5.11}$$

for U, W tangent to $\hat{M}$, where $\breve{\sigma}$ denotes the second fundamental form of the lift $\breve{\phi} : \breve{M} \to \mathbf{C}_*^{m+1}$ of ϕ via π.

Now suppose that $\phi : M = N_T \times_f N_\perp \to CP^m(4)$ is a CR-warped product in $CP^m(4)$. As before, we denote by $\mathcal{D}$ and $\mathcal{D}^\perp$ the holomorphic and the totally real distributions of $N_t \times_f N_\perp$, respectively. Let $\hat{\mathcal{D}}$ denote the distribution on $\hat{M} = \varphi^{-1}(M)$ spanned by $\mathcal{D}^* = \{X^*, X \in \mathcal{D}\}$ and $V = iz$, where X^* is a horizontal lift of X via φ. Since $\mathcal{D}$ is integrable, (5.6)–(5.8) implies that the distribution $\hat{\mathcal{D}}$ is also integrable. From (5.6)–(5.8), we also know that each leaf of $\hat{\mathcal{D}}$ is a totally geodesic submanifold of $\hat{M}$.

Let $\hat{\mathcal{D}}^\perp = \{Z^* \in T\hat{M} : Z \in \mathcal{D}^\perp\}$. Then $\hat{\mathcal{D}}^\perp$ is the orthogonal complementary distribution of $\hat{\mathcal{D}}$ in $T\hat{M}$. For vector fields Z, W in $\mathcal{D}^\perp$, (5.6) implies

$$\hat{\nabla}_{Z^*} W^* = (\nabla_Z W)^*. \tag{5.12}$$

Since $\mathcal{D}^\perp$ is integrable, (5.12) implies that $\hat{\mathcal{D}}^\perp$ is also an integrable distribution.

On the other hand, (4.19) gives

$$\langle \nabla_W Z, X\rangle = -(X \ln f)\, \langle Z, W\rangle \tag{5.13}$$

for vector field X in $\mathcal{D}$ and Z, W in $\mathcal{D}^\perp$. Thus, by (5.12), (5.13), $\langle (\nabla_Z W)^*, V\rangle = 0$, and the fact that the Hopf fibration is a Riemannian submersion, we obtain

$$\left\langle \hat{\nabla}_{Z^*} W^*, X^*\right\rangle = -(X \ln f)\, \langle Z^*, W^*\rangle\,, \quad \left\langle \hat{\nabla}_{Z^*} W^*, V\right\rangle = 0. \tag{5.14}$$

Thus, each leaf of $\hat{\mathcal{D}}^\perp$ is an extrinsic sphere in $\hat{M}$, that is, a totally umbilical submanifold with parallel mean curvature vector. Therefore, by applying a result of Hiepko [10], we know that $\hat{M}$ is also a warped product $\hat{N}_T \times_{\hat{f}} N_\perp^*$, where $\hat{N}_T$ is a leaf of $\hat{\mathcal{D}}$, $N_\perp^*$ a horizontal lift of $N_\perp$ and $\hat{f}$ the warping function. From the definitions of $\hat{\mathcal{D}}$, $\hat{N}_T$ and φ, we may choose $\hat{N}_T$ to be $\varphi^{-1}(N_T)$. Because the Hopf fibration $\varphi : S^{2m+1} \to CP^m(4)$ is a Riemannian submersion, $d\varphi$ preserves the length of vectors normal to fibres. Therefore, the warping function $\hat{f}$ of $\hat{N}_T \times_{\hat{f}} N_\perp^*$ is given by $f \circ \varphi$. Since $\breve{M}$ is the punctured cone over $\hat{M}$ with 0 as its vertex, $\breve{M}$ is nothing but $\breve{N}_T \times_{t\breve{f}} \breve{N}_\perp$, where $\breve{N}_T = \pi^{-1}(N_T)$, $\breve{f} = f \circ \pi$, and $\breve{N}_\perp$ is a horizontal lift of $N_\perp$ via π. Because $\breve{N}_\perp$ is isometric to $N_\perp$, $\breve{M}$ is thus isometric to $\breve{N}_T \times_{t\breve{f}} N_\perp$. It follows from our constructions that $\breve{N}_T = \pi^{-1}(N_T)$ is a holomorphic submanifold of $\mathbf{C}_*^{m+1}$ and $\breve{N}_\perp$ is a totally real submanifold in $\mathbf{C}_*^{m+1}$. Therefore, $\breve{M}$ is isometrically immersed in $\mathbf{C}_*^{m+1}$ as a CR-warped product.

Now, suppose that $\phi : M = N_T \times_f N_\perp \to CP^m(4)$ satisfies the equality case of (5.4). Then we obtain from (4.15) and (4.16) that

$$\sigma(\mathcal{D}, \mathcal{D}) = 0, \quad \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0. \tag{5.15}$$

Let $\breve{\mathcal{D}}$ be the distribution on $\breve{M}$ spanned by $\hat{\mathcal{D}}$ and $\partial/\partial t$ and $\breve{\mathcal{D}}^\perp$ the orthogonal distribution of $\breve{\mathcal{D}}$ in $T\breve{M}$. Then $\breve{\mathcal{D}}^\perp$ is spanned by vectors in $\mathbf{C}_*^{m+1}$ obtained from $\hat{\mathcal{D}}^\perp$ by parallel translation along rays of the cone $\breve{M}$ over $\hat{M}$. Thus, from (5.9), (5.10) and the second equation of (5.15), we obtain

$$\breve{\sigma}(\breve{\mathcal{D}}^\perp, \breve{\mathcal{D}}^\perp) = 0. \tag{5.16}$$

Also, from (5.9)–(5.11) and the first equation in (5.15), we find

$$\breve{\sigma}(\breve{\mathcal{D}}, \breve{\mathcal{D}}) = 0. \tag{5.17}$$

Therefore, by (4.15), $\pi^{-1}(M) = \breve{N}_T \times_{t\breve{f}} N_\perp$ satisfies the corresponding equality: $||\breve{\sigma}||^2 = 2p\{||\breve{\nabla}(\ln t\breve{f})||^2 + \breve{\Delta}(\ln t\breve{f})\}$ in $\mathbf{C}_*^{m+1}$. Hence, Theorem 4.1 implies that, up to rigid motions, the immersion of $\breve{M}$ is the $\breve{\phi}$ defined by (5.5) for some natural number $\alpha \leq h$. Thus, up to rigid motions, ϕ is the composition $\pi \circ \breve{\phi}$.

Conversely, it is easy to see that the immersion $\breve{\phi}$ defined by (5.5) is a CR-warped product $\mathbf{C}_*^{h+1} \times_f S^p$ in $\mathbf{C}^{m+1}$ which is invariant under the $\mathbf{C}^*$-action. Thus, the projection $\pi \circ \breve{\phi}$ of $\breve{\phi}$ under $\pi : \mathbf{C}_*^{m+1} \to CP^m(4)$ defines a submanifold M in $CP^m(4)$. It is easy to verify that M is indeed a CR-warped product $CP^h(4) \times_{\tilde{f}} S^p$ in $CP^m(4)$ for some suitable warping function $\tilde{f}$. Moreover, it follows from (5.9) that the CR-warped product M satisfies condition (5.15). Hence, by applying (4.15), we know that $M = \pi(\mathbf{C}_*^{h+1} \times_f S^p)$ satisfies the equality case of (5.4). □

6. *CR*-warped products in complex hyperbolic space

Let $\mathbf{C}_1^{m+1}$ denote a complex number space endowed with pseudo-Euclidean metric $g_0 = -dz_0 d\bar{z}_0 + \sum_{j=1}^m dz_j d\bar{z}_j$. Put $\mathbf{C}_{*1}^{m+1} = \mathbf{C}_1^{m+1} - \{0\}$. Consider the $\mathbf{C}^*$-action on $\mathbf{C}_{*1}^{m+1}$ by $\lambda \cdot (z_0, \ldots, z_m) = (\lambda z_0, \ldots, \lambda z_m)$. The set of equivalent classes obtained from this action is denoted by CH^m. The CH^m admits a natural Kähler structure (J, g) with constant holomorphic sectional curvature -4. Let $\pi : \mathbf{C}_{*1}^{m+1} \to CH^m(-4)$ denote the projection obtained from the $\mathbf{C}^*$-action.

Just like CP^m, there is an alternate way to view CH^m as follows: Let

$$H_1^{2m+1} = \{z = (z_1, z_2, \ldots, z_{m+1}) \in \mathbf{C}_1^{m+1}\colon\ \langle z, z\rangle = -1\}, \tag{6.1}$$

where $\langle\ ,\ \rangle$ is the inner product on $\mathbf{C}_1^{m+1}$ induced from the pseudo-Euclidean metric g_0. H_1^{2m+1} is known as the anti-de Sitter space-time.

We have an $U(1)$-action on H_1^{2m+1} defined by $z \mapsto \lambda z$. At each point $z \in H_1^{2m+1}$, the vector $V = iz$ is tangent to the flow of the action. The orbit lies in the negative definite plane spanned by z and iz. The quotient space $H_1^{2m+1}/\sim$ under the $U(1)$-action is exactly the complex hyperbolic space CH^m with constant holomorphic sectional curvature -4. The complex structure J on CH^m is induced from the canonical complex structure J on $\mathbf{C}_1^{m+1}$ via the Riemannian submersion:

$$\varphi\colon\ H_1^{2m+1} \to CH^m(-4), \tag{6.2}$$

which has totally geodesic fibers. The submersion (6.2) is called the hyperbolic Hopf fibration.

Assume $\psi : M \to CH^m(-4)$ is an isometric immersion. Then $\breve{M} = \pi^{-1}(M)$ is a $\mathbf{C}^*$-bundle over M and the lift $\breve{\psi} : \breve{M} \to \mathbf{C}_{*1}^{m+1}$ of ψ is an isometric immersion satisfying $\pi \circ \breve{\psi} = \psi \circ \pi$. Conversely, if $\breve{\psi} : \breve{M} \to \mathbf{C}_{*1}^{m+1}$ is an isometric immersion which is invariant under the $\mathbf{C}^*$-action, then there is an isometric immersion $\psi : \pi(\breve{M}) \to CH^m(-4)$ satisfying $\pi \circ \breve{\psi} = \psi \circ \pi$.

For an isometric immersion $\psi : M \to CH^m(-4)$, $\breve{M} = \pi^{-1}(M)$ is diffeomorphic to $\mathbf{R}^* \times \hat{M}$, where $\hat{M} = \varphi^{-1}(M)$. The immersion $\breve{\psi} : \breve{M} \to \mathbf{C}_{*1}^{m+1}$ is related to $\hat{\psi} : \hat{M} \to H_1^{2m+1}$ by

$$\breve{\psi}(t, q) = t\hat{\psi}(q), \quad t \in \mathbf{R}^*, \quad q \in \hat{M}. \tag{6.3}$$

The purpose of this section is to prove the following.

Theorem 6.1 *Let $\phi : N_T \times_f N_\perp \to CH^m(-4)$ be a CR-warped product. Then*

(1) *The squared norm of the second fundamental form of ϕ satisfies*

$$||\sigma||^2 \geq 2p\{||\nabla(\ln f)||^2 + \Delta(\ln f)\} - 4hp. \tag{6.4}$$

(2) *The CR-warped product satisfies the equality case of* (6.4) *if and only if*

(2.a) *N_T is an open portion of complex hy perbolic h-space $CH^h(-4)$;*

(2.b) *$N_\perp$ is an open portion of unit p-sphere S^p (or $\mathbf{R}$, when $p = 1$); and*

(2.c) *up to rigid motions, ϕ is the composition $\pi \circ \breve{\phi}$, where either $\breve{\phi}$ is given by*

$$\breve{\phi}(z, w) = \big(z_0, \dots, z_\beta, w_0 z_{\beta+1}, \dots, w_p z_{\beta+1}, \dots, w_0 z_h, \dots, w_p z_h, 0 \dots, 0\big) \tag{6.5}$$

for $0 < \beta \le h$, $z = (z_0, \dots, z_h) \in \mathbf{C}^{h+1}_{*1}$ and $w = (w_0, \dots, w_p) \in S^p$, or $\breve{\phi}$ is given by

$$\begin{aligned}\breve{\phi}(z, u) = \big(&z_0 \cosh u, z_0 \sinh u, z_1 \cos u, z_1 \sin u, \dots,\\ &\dots, z_\alpha \cos u, z_\alpha \sin u, z_{\alpha+1}, \dots, z_h, 0 \dots, 0\big)\end{aligned} \tag{6.6}$$

for $z = (z_0, \dots, z_h) \in \mathbf{C}^{h+1}_{*1}$ and $u \in \mathbf{R}$, and π being the projection $\pi : \mathbf{C}^{m+1}_{*1} \to CH^m(-4)$.

Proof. Inequality (6.4) is a special case of (4.16). It follows from (4.15) that a CR-warped product $\phi : M = N_T \times_f N_\perp \to CH^m(-4)$ satisfies the equality case of (6.4) if and only if the second fundamental form of ϕ satisfies

$$\sigma(\mathcal{D}, \mathcal{D}) = 0, \quad \sigma(\mathcal{D}^\perp, \mathcal{D}^\perp) = 0. \tag{6.7}$$

Suppose that ϕ is a CR-warped product in $CH^m(-4)$ satisfying (6.7). Since N_T is totally geodesic in $N_T \times_f N_\perp$, the first equation of (6.7) implies that each leaf of $\mathcal{D}$ is totally geodesic in $CH^m(-4)$. Thus, N_T is an open portion of $CH^h(-4)$; thus the preimage $\breve{N}_T = \pi^{-1}(N_T)$ is an open portion of $\mathbf{C}^{h+1}_{*1}$. Moreover, by applying an argument similar to the proof of Theorem 5.1 for CR-warped products in CP^m, we know that $\breve{M} = \pi^{-1}(M)$ is isometric to $\breve{N}_T \times_{t\breve{f}} N_\perp$ with $\breve{f} = f \circ \pi$ and the lift $\breve{\phi} : \breve{N}_T \times_{t\breve{f}} N_\perp \to \mathbf{C}^{m+1}_{*1}$ is a CR-warped product in $\mathbf{C}^{m+1}_{*1}$.

Let $\breve{\nabla}$ and $\hat{\nabla}$ denote the Levi-Civita connections on $\breve{M}$ and $\hat{M}$, respectively, and $\hat{\sigma}$ be the second fundamental form of the lift $\hat{\phi} : \hat{M} \to H^{2m+1}_1$. Then we have [5]

$$\hat{\nabla}_{X^*} Y^* = (\nabla_X Y)^* + \langle PX, Y \rangle V, \tag{6.8}$$

$$\hat{\nabla}_V X^* = \hat{\nabla}_{X^*} V = (PX)^*, \quad \hat{\nabla}_V V = 0, \tag{6.9}$$

$$\hat{\sigma}(X^*, Y^*) = (\sigma(X, Y))^*, \ \hat{\sigma}(X^*, V) = (FX)^*, \ \hat{\sigma}(V, V) = 0, \tag{6.10}$$

for vector fields X, Y tangent to M.

For a vector U tangent to $\hat{M} \subset H_1^{2m+1} \subset \mathbf{C}_{*1}^{m+1}$, we extend U to a vector field in $\mathbf{C}_{*1}^{m+1}$ by parallel translation along the rays of the cone $\breve{M}$ over $\hat{M}$. From (6.3), we find

$$\breve{\sigma}(U,W)(t,q) = \frac{1}{t}\hat{\sigma}(U,W)(q), \quad t \in \mathbf{R}^*, \quad q \in \hat{M}, \tag{6.11}$$

$$\breve{\sigma}\Big(U, \frac{\partial}{\partial t}\Big) = \breve{\sigma}\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\Big) = 0, \tag{6.12}$$

for U, W tangent to $\hat{M}$, where $\breve{\sigma}$ denotes the second fundamental form of the lift $\breve{\phi} : \breve{M} = \breve{N}_T \times_{t\breve{f}} N_\perp \to \mathbf{C}_*^{m+1}$ of ϕ via π.

By applying (6.7)–(6.12), we know that the second fundamental form $\breve{\sigma}$ of $\breve{\phi}$ satisfies

$$\breve{\sigma}(\breve{\mathcal{D}}, \breve{\mathcal{D}}) = 0, \quad \sigma(\breve{\mathcal{D}}^\perp, \breve{\mathcal{D}}^\perp) = 0, \tag{6.13}$$

where $\breve{\mathcal{D}}$ and $\breve{\mathcal{D}}^\perp$ are the holomorphic and the totally real distributions of $\breve{M}$. Since $\breve{N}_\perp$ is totally umbilical in the warped product $\breve{N}_T \times_{t\breve{f}} \breve{N}_\perp$, the second equation in (6.13) implies that $\breve{B}_\perp$ is immersed as a totally umbilical submanifold in a complex Euclidean subspace. Hence, without loss of generality, we may assume that $\breve{N}_\perp$ is an open portion of S^p (or of $\mathbf{R}$ when $p = 1$). Therefore, there is a complex coordinate system $\{z_0, \ldots, z_h\}$ on $\mathbf{C}_{*1}^{h+1}$ and a coordinate system on S^p or $\mathbf{R}$ so that the metric on $\breve{M} = \breve{N}_T \times_{t\breve{f}} N_\perp$ is given by

$$g = -dz_0 d\bar{z}_0 + \sum_{j=1}^{h} dz_j d\bar{z}_j + \lambda^2 \sum_{s=1}^{p} \Big(\prod_{t=1}^{s-1} \cos^2 u_t du_t^2 \Big), \tag{6.14}$$

where $\lambda = \lambda(z_0, \ldots, z_h)$ is the corresponding warping function.

From (6.13) and (6.14) we know that $\breve{\phi}$ satisfies the following system of partial differential equations:

$$\breve{\phi}_{z_j z_k} = \breve{\phi}_{z_j \bar{z}_k} = \breve{\phi}_{\bar{z}_j \bar{z}_k} = 0, \quad j,k = 0, \ldots, h, \tag{6.15}$$

$$\breve{\phi}_{u_s u_t} = -\tan u_s \breve{\phi}_{u_t}, \quad 1 \le s < t \le p, \tag{6.16}$$

$$\begin{aligned}\breve{\phi}_{u_t u_t} &= \lambda \prod_{s=1}^{t-1} \cos^2 u_s \Big\{ \lambda_{x_0} \breve{\phi}_{x_0} + \lambda_{y_0} \breve{\phi}_{y_0} - \sum_{k=1}^{h} \Big(\lambda_{x_k} \breve{\phi}_{x_k} + \lambda_{y_k} \breve{\phi}_{y_k} \Big) \Big\} \\ &\quad + \sum_{q=1}^{t-1} \Big(\frac{\sin 2u_q}{2} \prod_{s=q+1}^{t-1} \cos^2 u_s \Big) \breve{\phi}_{u_q}, \quad t = 1, \ldots, p.\end{aligned} \tag{6.17}$$

Solving (6.15) gives

$$\breve{\phi}(z_1,\dots,z_h,u_1,\dots,u_p)=\sum_{j=0}^{h}A_j(u_1,\dots,u_p)z_j+B(u_1,\dots,u_p) \tag{6.18}$$

for some $\mathbf{C}_1^{m+1}$-valued functions $A_0,\dots,A_h,B$. From (6.17) with $t=1$, we find

$$\breve{\phi}_{u_1u_1}=\frac{1}{2}\left(\frac{\partial\lambda^2}{\partial x_0}\breve{\phi}_{x_0}+\frac{\partial\lambda^2}{\partial y_0}\breve{\phi}_{y_0}\right)-\frac{1}{2}\sum_{k=1}^{h}\left(\frac{\partial\lambda^2}{\partial x_k}\breve{\phi}_{x_k}+\frac{\partial\lambda^2}{\partial y_k}\breve{\phi}_{y_k}\right) \tag{6.19}$$

Substituting (6.18) into (6.19) yields

$$\sum_{j=0}^{h}\frac{\partial^2A_j}{\partial u_1^2}z_j+\frac{\partial^2B}{\partial u_1^2}=\frac{1}{2}\frac{\partial\lambda^2}{\partial\bar{z}_0}A_0-\frac{1}{2}\sum_{j=1}^{h}\frac{\partial\lambda^2}{\partial\bar{z}_j}A_j. \tag{6.20}$$

Applying the same argument as for *Case* (1) in the proof of Theorem 4.1, we know that $\sum_{j=0}^{h}(\partial A_j/\partial u_1)A_j$ cannot be independent on all $z_0,\dots,z_h$. Then, by applying an argument similar to that given in the first part of *Case* (2) of the proof of Theorem 4.1, we know that the warping function λ can be chosen to be

$$\lambda=\left(\sum_{j=0}^{n}a_j^2z_j\bar{z}_j\right)^{1/2},\quad a_0,\dots,a_h\geq 0. \tag{6.21}$$

Substituting (6.21) into (6.20) gives

$$\frac{\partial^2A_0}{\partial u_1^2}=a_0^2A_0,\quad \frac{\partial^2A_j}{\partial u_1^2}=-a_j^2A_j,\quad j=1,\dots,h, \tag{6.22}$$

$$\frac{\partial^2B}{\partial u_1^2}=0. \tag{6.23}$$

Case (a): $a_0=\cdots=a_\beta=0,\ a_{\beta+1},\dots,a_h>0$ for some β satisfying $0<\beta\leq h$.

In this case, by applying an argument similar to *Case* (2) in the proof of Theorem 4.1, we may obtain

$$\breve{\phi} = \sum_{j=0}^{\beta}\Big\{c_1^j \prod_{t=1}^{p} \cos u_t + c_2^j \sin u_1 + c_3^j \sin u_2 \cos u_1 + \cdots$$
$$+ c_{p+1}^j \sin u_p \prod_{t=1}^{p-1} \cos u_t\Big\} z_j + \sum_{k=\beta+1}^{h} E_k z_k + G, \tag{6.24}$$

for some constant vectors c_t^j, E_k, G in $\mathbf{C}_{*1}^{m+1}$. Thus, after choosing some suitable initial conditions, we obtain (6.5).

Case (b): $a_0, \ldots, a_\alpha > 0$, $a_{\alpha+1} = \cdots = a_h = 0$ for some natural number $\alpha \leq h$.

In this case, after solving (6.22) and (6.23), we find

$$\begin{aligned} A_0 &= D_0(u_2, \ldots, u_p) \cosh(a_0 u_1) + E_0(u_2, \ldots, u_p) \sinh(a_0 u_1), \\ A_j &= D_j(u_2, \ldots, u_p) \cos(a_j u_1) + E_j(u_2, \ldots, u_p) \sin(a_j u_1), \\ A_k &= D_k(u_2, \ldots, u_p) u_1 + E_k(u_2, \ldots, u_p), \\ B &= F(u_2, \ldots, u_p) u_1 + G(u_2, \ldots, u_p) \end{aligned} \tag{6.25}$$

for some vector functions $D_0, \ldots, D_h, E_0, \ldots, E_h, G, G$, where $j = 1, \ldots, \alpha$, and $k = \alpha+1, \ldots, h$. Substituting (4.53), (4.54) and (4.55) into (4.31) gives

$$\breve{\phi} = (D_0(u_2, \ldots, u_p) \cosh(a_0 u_1) + E_0(u_2, \ldots, u_p) \sinh(a_0 u_1)) z_0$$
$$+ \sum_{j=1}^{\alpha} (D_j(u_2, \ldots, u_p) \cos(a_j u_1) + E_j(u_2, \ldots, u_p) \sin(a_j u_1)) z_j \tag{6.26}$$
$$+ \sum_{k=\alpha+1}^{h} (D_k(u_2, \ldots, u_p) u_1 + E_k(u_2, \ldots, u_p)) z_k \tag{6.27}$$
$$+ F(u_2, \ldots, u_p) u_1 + G(u_2, \ldots, u_p).$$

Because $\breve{\phi}$ is invariant under the $\mathbf{C}^*$-action, we have $F = G = 0$.

If $p = 1$, then $D_0, \ldots, D_h, E_0, \ldots, E_h$ are constant vectors.

If $p > 1$, then (6.26) and (6.16) with $s = 1$ and $t = 2, \ldots, p$ imply that D_0 and E_0 are constant vectors. Also, by applying arguments similar to that given in *Case* (2) of the proof of Theorem 4.1, we also know that $E_0, \ldots, E_h$ are constant vectors and $a_0 = \cdots = a_\alpha = 1$. The latter condition implies

$$\lambda^2 = \sum_{j=0}^{\alpha} z_j \bar{z}_j. \tag{6.28}$$

Thus, from (6.26), we get

$$\breve{\phi} = (D_0 \cosh u_1 + E_0 \sinh u_1) z_0 + \sum_{j=1}^{\alpha} \big(D_j(u_2, \ldots, u_p) \cos u_1 + E_j \sin u_1\big) z_j + \sum_{k=\alpha+1}^{h} E_k z_k. \tag{6.29}$$

If $p > 1$, then by substituting (4.27) and (4.28) into (6.17) with $t = 2$, we find

$$\begin{aligned} \sum_{j=1}^{\alpha} \cos u_1 \frac{\partial^2 D_j}{\partial u_2^2} z_j &= \cos^2 u_1 \Big\{ (D_0 \cosh u_1 + E_0 \sinh u_1) z_0 + \sum_{j=1}^{\alpha} \big(D_j \cos u_1 + E_j \sin u_1\big) z_j \Big\} \\ &\quad - \frac{\sin 2u_1}{2} \Big\{ (D_0 \sinh u_1 + E_0 \cosh u_1) z_0 + \sum_{j=1}^{\alpha} \big(D_j \sin u_1 - E_j \cos u_1\big) z_j \Big\}. \end{aligned} \tag{6.30}$$

By comparing the coefficients of z_0 in (6.30) we find

$$\cos u_1 (D_0 \cosh u_1 + E_0 \sinh u_1) = \sin u_1 (D_0 \sinh u_1 + E_0 \cosh u_1)$$

which is impossible. Hence, we must have $p = 1$ in *Case* (b). Thus, (6.29) becomes

$$\breve{\phi} = (D_0 \cosh u_1 + E_0 \sinh u_1) z_0 + \sum_{j=1}^{\alpha} \big(D_j \cos u_1 + E_j \sin u_1\big) z_j + \sum_{k=\alpha+1}^{h} E_k z_k. \tag{6.31}$$

for some constant vectors $D_0, \ldots, D_\alpha, E_0, \ldots, E_h$. From (6.14) and (6.31), we know that D_0 is a unit time-like vector and $D_1, \ldots, D_\alpha, E_0, \ldots, E_h$ are space-like orthonormal vectors in $\mathbf{C}_1^{m+1}$. Therefore, after choosing suitable initial conditions, we may obtain (6.6).

Conversely, it is straightforward to verify that (6.5) defines a *CR*-warped product $\mathbf{C}_{*1}^{h+1} \times_\lambda S^p$ and (6.6) defines a *CR*-warped product $\mathbf{C}_{*1}^{h+1} \times_\lambda \mathbf{R}$ in $\mathbf{C}_{*1}^{m+1}$; both cases satisfy (6.13). Since the immersions $\breve{\phi}$ defined by (6.5) and (6.6) are invariant under the $\mathbf{C}^*$-action, their projections under $\pi : \mathbf{C}_{*1}^{m+1} \to CH^m(-4)$ give rise to *CR*-warped products $CH^h(-4) \times_f S^p$

and $CH^h(-4) \times_f \mathbf{R}$ in $CH^m(-4)$. Because the second fundamental form of $CH^h(-4) \times_f S^p$ and $CH^h(-4) \times_f \mathbf{R}$ both satisfy condition (6.7) in $CH^m(-4)$, their second fundamental forms satisfy the equality case of (6.4). $\square$

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