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On Hadamard difference sets with weak multiplier minus one

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Abstract. A note on the intersection numbers of an Hadamard difference set (HDS) with weak multiplier -1 is given and a necessary condition for the existence of an HDS with weak multiplier -1 in the group $G = H \times \mathbb{Z}_4$, a direct product where H is any group of order u^2 with $u \geq 1$, an odd integer is obtained.

Key words: difference set, Hadamard difference set, reversible difference set, multiplier, weak multiplier, intersection number.

1. Preliminaries

A (v, k, λ) *difference set* (DS) is a k-element subset D of a group G of order v such that every element $g \neq 1$ of G has exactly λ representations $g = d_1 d_2^{-1}$ with $d_1, d_2 \in D$. The *order* of a difference set D is the integer $n = k - \lambda$ and D is called non-trivial if $n > 1$. A difference set D is called cyclic, abelian, etc., if the underlying group G has the respective property.

Many results have been obtained by studying difference sets in the context of the group ring $\mathbb{Z}[G]$ of a group G over the ring of integers $\mathbb{Z}$. For $X \subseteq G$ and $t \in \mathbb{Z}$, we denote $X^t = \{x^t \mid x \in X\}$. With this notation and viewing D as an element of $\mathbb{Z}[G]$, D satisfies the basic equation $DD^{-1} = n + \lambda G$ from which it follows that $k^2 = n + \lambda v$.

When $v = 4n$, we call D an *Hadamard Difference Set* (HDS). In this case, D has parameters of the form $(4u^2, 2u^2 - u, u^2 - u)$ for some $u \in \mathbb{Z}$ ([12], p. 38). Refer to [1], [2], or [12] for a more detailed discussion on difference sets and Hadamard difference sets.

A mapping χ from an abelian group G into the nonzero complex numbers is called a *character* on G if $\chi(ab) = \chi(a)\chi(b)$ for any $a, b \in G$. We note that χ maps every element of G into an e-th root of unity where $e = \exp(G)$, the exponent of G. We denote by G^* the character group of G and by χ_0,

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we mean the principal character. Refer to [12] for the inversion formula and for some applications of characters of abelian groups to difference sets.

Let G be an abelian group and let t be an integer such that $(t, |G|) = 1$ so that t induces an automorphism of G given by $x \to x^t$. If t has the property $D^t = Dg$ for some $g \in G$, we call t a *numerical multiplier* of D. On the other hand, if G is non-abelian, there is no reason that t induces an automorphism of G. In this case, we call t a *weak multiplier* of D if $D^t = Dg$ for some $g \in G$ (see [1]). If G is any group, without specifying whether it is abelian or non-abelian, we also call t a weak multiplier of D if t satisfies the above property. Thus we consider weak multipliers as the generalization of the usual concept of numerical multipliers. In addition, if D is a difference set in any group G with $D^{-1} = D$, we call D *reversible* (see [2]). If G is abelian, $D = D^{-1}$ if and only if $\chi(D) = \overline{\chi(D)}$ for every $\chi \in G$. Interested readers may refer to [1] for a comprehensive survey and latest results on abelian difference sets with multiplier -1 and those that are reversible.

There exist reversible difference sets in some non-abelian groups. The difference sets constructed by Miyamoto [11] and Ma [7] are non-abelian reversible difference sets. Also, the difference set with parameters $(100, 45, 20)$ constructed by Smith [13] is an example of a non-abelian reversible difference set with Hadamard parameters. We note that in Smith's construction, a Sylow 2-subgroup is not a direct factor of the group.

In Section two, we give a note on the intersection numbers of HDS's with weak multiplier -1. We then obtain a necessary condition for the existence of an Hadamard difference set in the group $G = H \times \mathbb{Z}_4$, a direct product where H is any group of order u^2 with $u \geq 1$, an odd integer in Section three.

2. On the intersection numbers of HDS's

Let H and K be groups such that $G = HK$ with H normal in G and $H \cap K = 1$. A mapping "—" given by $\overline{hk} = k$ where $h \in H$ and $k \in K$ is a homomorphism from G to K. Also, we have $\overline{g} = g$ if $g \in K$ and $Hg = H\overline{g}$ for every $g \in G$ and as a set $\overline{G} = K$. If D is a difference set in G, the integers $d_{\overline{g}} = |D \cap Hg|$ are called the *intersection numbers* of D with respect to H.

Let G be any group of order $4m^2u^2$ with $m, u \geq 1$ and $(2m, u) = 1$. Assume G contains a normal subgroup H of order u^2 such that the factor group $\overline{G} = G/H$ is abelian. By Schur-Zassenhaus Theorem, G contains a subgroup K such that $G = HK$ and $H \cap K = 1$ (see p. 221 in [5]). If G contains an HDS D then D has parameters $(4m^2u^2, 2m^2u^2 - mu, m^2u^2 - mu)$ and D satisfies

$$DD^{-1} = m^2u^2 + (m^2u^2 - mu)G. \tag{2.1}$$

Let $G = \sum_{g \in K} Hg$ so that $D = \sum_{g \in K} D_g g$ where $D_g \subseteq H$. Clearly, $d_g = |D_g|$ for every $g \in K$.

Theorem 2.1 *Let G be any group of order $4m^2u^2$ with $m, u \geq 1$, $(2m, u) = 1$. Assume $G = HK$ where H is a normal subgroup of G of order u^2, $H \cap K = 1$ and $G/H \cong K$ is abelian. If G contains an HDS D with weak multiplier -1 then:*

(i) *for every $g \in K$, $d_g = \frac{1}{2}u(u + l_g)$ where $l_g \in \{\pm 1\}$.*

(ii) *Set $A = \{g \in K \mid l_g = 1\}$ and $B = \{g \in K \mid l_g = -1\}$. Then A and B are complementary HDS's in K with weak multiplier -1 and with parameters*

$$(4t^2, 2t^2 - t, t^2 - t) \tag{2.2}$$

where $t = m$ and $-m$, respectively.

Proof. Assume $D^{-1} = Dhk$ for some $h \in H$ and $k \in K$ so that $\overline{D^{-1}} = \overline{D}k$. We have $\overline{D} = \sum_{g \in K} d_g g \in \mathbb{Z}[K]$ where $0 \leq d_g \leq u^2$ and $\bar{D}\bar{D}^{-1} = m^2u^2 + (m^2u^2 - mu)u^2K$. Then

$$\chi(\overline{D}) = \begin{cases} 2m^2u^2 - mu & \text{if } \chi = \chi_0 \text{ on } K \\ \epsilon_\chi mu & \text{if } \chi \neq \chi_0 \text{ on } K \end{cases}$$

where $\epsilon_\chi \in \{\pm\sqrt{\chi(k^{-1})}\}$ and $\sqrt{\chi(k^{-1})}$ is a $2e$-th root of unity, $e = \exp(K)$.

For a fixed $g \in K$, we have $\sum_{\chi \in K^*} \chi(\overline{D}g^{-1}) = 4m^2 d_g$ by the inversion formula. On the other hand,

$$\sum_{\chi \in K^*} \chi(\overline{D}g^{-1}) = 2m^2u^2 - mu + mu \sum_{\chi_0 \neq \chi \in K^*} \epsilon_\chi \chi(g^{-1})$$

where $\chi(g)$ is an e-th root of unity. Thus

$$4m^2 d_g = 2m^2u^2 - mu + mul'_g \tag{2.3}$$

where $l'_g = \sum_{\chi_0 \neq \chi \in K^*} \epsilon_\chi \chi(g^{-1})$, an algebraic integer. We note that

$$|l'_g| \leq 4m^2 - 1. \tag{2.4}$$

From (2.3), we get $2mu - 1 + l'_g \equiv 0 \pmod{4m}$. Since u is odd, we obtain $l'_g \equiv 1 - 2m \pmod{4m}$. Let $l'_g = 1 - 2m + 4mr_g$ for some integer r_g. Using (2.4), one can easily show that $-m < r_g \leq m$. Substituting the expression for l'_g in (2.3), we have $d_g = \frac{1}{2}u(u - 1 + 2r_g)$ and so $d_g = \frac{1}{2}u(u + l_g)$ where $l_g = -1 + 2r_g$. As $|D| = \sum_{g \in K} d_g = 2m^2u^2 - mu$, we get $\sum_{g \in K} \frac{1}{2}u(u + l_g) = 2m^2u^2 - mu$ and so $\sum_{g \in K} l_g = -2m$. From (2.1), we also have

$$\sum_{g \in K} D_g D_g^{-1} = m^2u^2 + (m^2u^2 - mu)H$$

and so $\sum_{g \in K} d_g^2 = m^2u^2 + (m^2u^2 - mu)u^2$. Substituting the expression for d_g to this last equation gives $\sum_{g \in K} l_g^2 = 4m^2$. As l_g is odd and $|K| = 4m^2$, we get $l_g^2 = 1$ which gives $l_g = \pm 1$. Hence $d_g = \frac{1}{2}u(u + l_g)$ where $l_g \in \{\pm 1\}$.

To prove the second statement, we note that

$$\overline{D} = \frac{1}{2}u(u+1)A + \frac{1}{2}u(u-1)(K - A) = \frac{1}{2}u(u-1)K + uA.$$

As $|\overline{D}| = 2m^2u^2 - mu$ and $|K| = 2m^2$, we obtain $|A| = 2m^2 - m$. Moreover, as $\chi(\overline{D}) = \epsilon_\chi mu$ where $\epsilon_\chi \in \{\pm\sqrt{\chi(k^{-1})}\}$ for every $\chi_0 \neq \chi \in K^*$, we have $\chi(A) = \epsilon_\chi m$ and so $AA^{-1} = m^2 + (m^2 - m)K$ by the inversion formula. Since $\overline{D} = \frac{1}{2}u(u-1)K + uA$ and $D^{-1} = Dhk$, we have $A^{-1} = Ak$. Thus A is an HDS in K with weak multiplier -1 and with parameters in (2.2). $\square$

Example 2.2 Let G be a group of order $4p^{2\alpha}$ with $\alpha \geq 1$ and $p \geq 3$, a prime. Let $H \in \mathrm{Syl}_p(G)$, the set of all Sylow p-subgroups of G and set $n_p = |\mathrm{Syl}_p(G)|$. By Sylow Theorem, $n_p = 1$ unless $p = 3$ in which case we have $n_3 = 1$ or 4. Thus if $p > 3$, a Sylow p-subgroup H is always normal in G and so $G = HG_2$ where G_2 is a Sylow 2-subgroup of G. Let $G_2 = \{1, x_1, x_2, x_3\}$ and let $D = D_0 + D_1x_1 + D_2x_2 + D_3x_3$ where $D_i \subset H$, $(0 \leq i \leq 3)$ be an HDS in G with weak multiplier -1. By Theorem 2.1, the intersection numbers of D with respect to H are $\{d_0, d_1, d_2, d_3\} = \big\{\frac{1}{2}p^\alpha(p^\alpha - 1), \frac{1}{2}p^\alpha(p^\alpha - 1), \frac{1}{2}p^\alpha(p^\alpha - 1), \frac{1}{2}p^\alpha(p^\alpha + 1)\big\}$ where $d_i = |D_i|$. In particular, if we set $\alpha = 1$ and $p = 5$ then the intersection numbers are $\{d_1, d_2, d_3, d_4\} = \{15, 10, 10, 10\}$. The reversible HDS constructed by Smith in the group $\langle x, y, z \mid x^5 = y^5 = z^4 = [x, y] = 1,\ zx = x^2z,\ zy = y^2z\rangle$ is an example where these intersection numbers hold true (see [1], p. 410 for a particular example of D).

Without the assumption that the HDS admits -1 as a multiplier, McFarland also gave a proof of a case of Theorem 2.1 in the group $G = G_2 \times G_p$ where p is an odd prime, $G_p = \mathbb{Z}_p \times \mathbb{Z}_p$ and G_2 is an abelian group of order 2^{2a+2} with exponent 2 if $p \equiv 1 \pmod 4$ and exponent 2 or 4 if $p \equiv 3 \pmod 4$ (see Lemma 4.2 of [8]).

Theorem 2.1 is also related to abelian reversible HDS's. We note that there exists a reversible HDS in $\mathbb{Z}_{2^{a+1}} \times \mathbb{Z}_{2^{a+1}}$ for every $a \geq 0$ by a construction of Dillon [3]. On the other hand, McFarland in [9] proved that there exists no reversible HDS in the abelian group $G_2 \times G_p$ where G_2 is an abelian group of order 2^{2a+2} and $|G_p| = p^{2\alpha}$ where $p > 3$ is a prime and α is an odd integer. The group $\mathbb{Z}_{2^a} \times \mathbb{Z}_{2^a} \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ contains a reversible HDS for every $a \geq 0$ (see [1]) and if we let $G = \mathbb{Z}_{2^{a+1}} \times \mathbb{Z}_{2^{a+1}} \times \mathbb{Z}_3 \times \mathbb{Z}_3$ then there exists a reversible HDS in G when $a = 0$ by a construction of Turyn [14]. However when $a = 1$, Xiang [15] gave a proof on the non-existence of a reversible HDS in G. The case $a > 1$ in G is still an open problem.

In the next section, we study the HDS's with multiplier -1 in the group $G = H \times \mathbb{Z}_4$, a direct product with $|H| = u^2$, $u \geq 1$, an odd integer and obtain a necessary condition for the existence of these HDS's in G.

3. On HDS's with weak multiplier -1 in a direct product

Let G be a group with $G \cong H \times \mathbb{Z}_4$, a direct product where H is any group of order u^2, $u \geq 1$, an odd integer. Let $\mathbb{Z}_4 = \langle x \rangle$ and suppose D is an HDS in G with weak multiplier -1. We can write

$$D = A_0 + A_1 x + A_2 x^2 + A_3 x^3 \tag{3.1}$$

where $A_i \subseteq H$ for $i = 0, 1, 2, 3$. Let $|A_i| = a_i$ be the intersection numbers of D with respect to H. By Theorem 2.1, the following lemma is immediate.

Lemma 3.1 *If D is an HDS with weak multiplier -1 in $G \cong H \times \mathbb{Z}_4$ where H is a group of order u^2, $u \geq 1$, an odd integer then* $\{a_0, a_1, a_2, a_3\} = \{\frac{1}{2}u(u-1),\ \frac{1}{2}u(u-1),\ \frac{1}{2}u(u-1),\ \frac{1}{2}u(u+1)\}$.

Lemma 3.2 *We can assume that $D^{-1} = D$.*

Proof. Suppose $D^{-1} = Dgx^i$ where $g \in H$ and $i \in \{0, 1, 2, 3\}$. If $i \in \{1, 3\}$ then $a_0 = a_{i+2}$ and $a_2 = a_i$ where the subscripts are integers modulo 4. This contradicts Lemma 3.1. If $D^{-1} = Dgx^2$ then $D_1 = Dx$ satisfies $D_1^{-1} = D_1 g$. Thus $D^{-1} = Dg$ for some $g \in H$.

We now assume $D^{-1} = Dg$ for some $g \in H$. Then

$$A_0^{-1} = A_0 g,\ A_1^{-1} = A_3 g,\ A_2^{-1} = A_2 g,\ A_3^{-1} = A_1 g. \tag{3.2}$$

By (3.2), we have $A_0^{-1} = A_0 g$ and by taking the inverse of both sides, we get $A_0 = g^{-1} A_0^{-1} = g^{-1} A_0 g$. Thus $gA_0 = A_0 g$. Similarly, $gA_2 = A_2 g$. Also by (3.2), $A_1^{-1} = A_3 g$ and $A_3^{-1} = A_1 g$. Then $A_1 = g^{-1} A_3^{-1} = g^{-1} A_1 g$ which gives $gA_1 = A_1 g$. Similarly, $gA_3 = A_3 g$.

Finally, we note that the order of g is odd, and so $\langle g \rangle = \langle g^2 \rangle$. Thus $g = g^{2t}$ for some integer t and we have $(Dg^t)^{-1} = Dg^t$. Thus we can assume that $D^{-1} = D$. □

By Lemma 3.2, we can now assume $D^{-1} = D$. From (3.1), we have the following:

$$A_0^{-1} = A_0,\ A_1^{-1} = A_3,\ A_2^{-1} = A_2,\ A_3^{-1} = A_1. \tag{3.3}$$

By substituting (3.1) into $DD^{-1} = u^2 + (u^2 - u)G$ and equating the coefficients of x^i on both sides for $i \in \{0,1,2,3\}$, we obtain four equations. Also, by substituting the expressions in (3.3) into these four equations we get:

$$(A_0)^2 + A_1A_3 + A_3A_1 + (A_2)^2 = u^2 + (u^2 - u)H \tag{3.4}$$

$$A_0A_1 + A_1A_0 + A_2A_3 + A_3A_2 = (u^2 - u)H \tag{3.5}$$

$$(A_1)^2 + A_0A_2 + A_2A_0 + (A_3)^2 = (u^2 - u)H \tag{3.6}$$

$$A_0A_3 + A_3A_0 + A_1A_2 + A_2A_1 = (u^2 - u)H. \tag{3.7}$$

By (3.3) and Lemma 3.1, it is immediate that only two cases can hold for the values of the a_i's, namely: $\{a_0, a_2\} = \left\{\frac{1}{2}u(u+1), \frac{1}{2}u(u-1)\right\}$ and $a_1 = a_3 = \frac{1}{2}u(u-1)$.

We now assume $u > 1$. By Feit-Thompson Theorem on groups of odd order, $H \neq H'$ where $H' = [H,H]$, the commutator subgroup of H (see [4]). Let $\overline{H} = H/H'$.

Lemma 3.3 *For every $\chi \in \overline{H}^*$, $\chi \neq \chi_0$, we have $\chi(\overline{A_1}) = \chi(\overline{A_3}) = 0$.*

Proof. Let $\chi \in \overline{H}^*$, $\chi \neq \chi_0$ and set $\alpha_i = \chi(\overline{A_i})$, $(0 \leq i \leq 3)$. From (3.4)–(3.7), we have

$$\alpha_0^2 + 2\alpha_1\alpha_3 + \alpha_2^2 = u^2 \tag{3.8}$$

$$\alpha_0\alpha_1 + \alpha_2\alpha_3 = 0 \tag{3.9}$$

$$\alpha_1^2 + 2\alpha_0\alpha_2 + \alpha_3^2 = 0 \tag{3.10}$$

$$\alpha_0\alpha_3 + \alpha_1\alpha_2 = 0. \tag{3.11}$$

By (3.9) and (3.11), $(\alpha_0 + \alpha_2)(\alpha_1 + \alpha_3) = 0$ and so $\alpha_0 = -\alpha_2$ or $\alpha_1 = -\alpha_3$. If $\alpha_0 = -\alpha_2$, by (3.8) we have $\alpha_0^2 + \alpha_1\alpha_3 = \frac{u^2}{2} \in \mathbb{Z}$ as the left hand side is an algebraic integer. Since $u \geq 1$ is odd, this is a contradiction. Therefore $\alpha_1 = -\alpha_3$. From (3.8) and (3.9), respectively, we get

$$\alpha_0^2 - 2\alpha_1^2 + \alpha_2^2 = u^2 \tag{3.12}$$

$$\alpha_1(\alpha_0 - \alpha_2) = 0. \tag{3.13}$$

By (3.13), $\alpha_1 = 0$ or $\alpha_0 = \alpha_2$. If $\alpha_0 = \alpha_2$, then by (3.12), $\alpha_0^2 - \alpha_1^2 = \frac{u^2}{2} \in \mathbb{Z}$, again a contradiction. Thus $\alpha_1 = 0$ and so $\alpha_3 = 0$. $\square$

Theorem 3.4 *Let $G \cong H \times \mathbb{Z}_4$ where H is any group of order u^2 with $u \geq 1$, an odd integer. If G contains an HDS with weak multiplier -1 then u divides $|H'|$ where $H' = [H, H]$.*

Proof. By Lemma 3.3, we have $\chi(\overline{A_1}) = 0$ for every $\chi \in \overline{H}^*$, $\chi \neq \chi_0$. Thus $A_1H' = sH$ for some integer s by the inversion formula (see [12]). As $|H| = u^2$, $|A_1| = \frac{1}{2}u(u-1)$ and $\left(u, \frac{u-1}{2}\right) = 1$, u must divide $|H'|$. $\square$

We note that if G is a cyclic group containing an HDS then $G \cong H \times \mathbb{Z}_4$ where $|H| = u^2$, $u \geq 1$, an odd integer by a result of Turyn [14]. The above theorem then gives an alternate proof of the non-existence of non-trivial cyclic HDS's with multiplier -1 which was proven also by McFarland and Ma in [10]. Moreover, if $u = p$, an odd prime, then clearly G cannot contain an HDS with weak multiplier -1.

Corollary 3.5 *If $u = pq$ where $p > q$ are primes then G does not contain an HDS with weak multiplier -1.*

Proof. Let $P \in \mathrm{Syl}_p(H)$. We have $|H| = p^2q^2$ and by Theorem 3.4, pq divides $|H'|$. Let $t = |N_H(P)|$, where $N_H(P)$ is the normalizer of P in H. Then $t \in \{p^2, p^2q, p^2q^2\}$. If $t = p^2q$ then $|\,\mathrm{Syl}_p(H)| = q \equiv 1 \bmod (p)$. As $p > q$, this case cannot occur. If $t = p^2q^2$ then $H = N_H(P)$ and so $P \lhd H$. Thus $H' \leq P$ and so pq does not divide $|H'|$. This is a contradiction.

We now assume $t = p^2$. Then $N_H(P) = P$ and so $P \leq Z(N_H(P))$. By a theorem of Burnside (see Theorem 7.4.3 in [5]), there exists a subgroup $Q \lhd H$ such that $H = QP \rhd Q$. Thus $H' \leq Q$ and again pq does not divide

$|H'|$. □

We observe that Theorem 3.4 cannot rule out the non-existence of an HDS with weak multiplier minus one in the group $G = H \times \mathbb{Z}_4$ with $|H| = u^2$, $u \geq 1$, an odd integer unlike that of the abelian case. We also note that Corollary 3.5 does not include the case $p = q$ in which case $|H| = p^4$.

We also mention here that another topic worth considering is the HDS's in all groups of order $4p^2$ with $p \geq 5$, a prime. A list of all the isomorphism classes of these groups was given by Iiams in [6]. In the same paper, Iiams proved the non-existence of non-trivial HDS's in some of these groups. The other remaining cases may still be open.

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