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Rational solutions of the Sasano system of type $A_1^{(1)}$

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Abstract. In this paper, we completely classify the rational solutions of the Sasano system of type $A_1^{(1)}$, which is a degeneration of the Sasano system of type $A_4^{(2)}$. These systems of differential equations are both expressed as coupled Painlevé II systems. The Sasano system of type $A_1^{(1)}$ is a higher order version of the second Painlevé equation, P_{II}, with the same affine Weyl group symmetry of type $A_1^{(1)}$ as P_{II}.

Key words: affine Weyl group, Rational solutions, the Sasano system.

Introduction

Paul Painlevé and his colleagues [18], [1] intended to find new transcendental functions defined by second order nonlinear differential equations. In general, nonlinear differential equations have moving branch points. If a solution has moving branch points, it is too complicated and is not worth considering. Therefore, they determined second order nonlinear differential equations with rational coefficients which have no moving branch points. As a result, the standard forms of these equations turned out to be given by the following six equations:

$$P_I \ : \ y'' = 6y^2 + t$$

$$P_{II} \ : \ y'' = 2y^3 + ty + \alpha$$

$$P_{III}: \ y'' = \frac{1}{y}(y')^2 - \frac{1}{t}y' + \frac{1}{t}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}$$

$$P_{IV}: \ y'' = \frac{1}{2y}(y')^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2 - \alpha)y + \frac{\beta}{y}$$

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$$P_V\ :\ y'' = \left(\frac{1}{2y} + \frac{1}{y-1}\right)(y')^2 - \frac{1}{t}y' + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \frac{\gamma}{t}y + \delta\frac{y(y+1)}{y-1}$$

$$P_{VI}:\ y'' = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)(y')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)y' + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right),$$

where $' = d/dt$ and $\alpha, \beta, \gamma, \delta$ are all complex parameters.

While generic solutions of the Painlevé equations are "new transcendental functions," there are special solutions which are expressible in terms of classical special functions. In particular, the rational solutions of P_J $(J = II, III, IV, V, VI)$ were classified by Yablonski and Vorobev [22], [21], Gromak [3], [2], Murata [10], [11], Kitaev, Law and McLeod [4], Mazzoco [9], and Yuang and Li [23].

Each of P_J $(J = II, III, IV, V, VI)$ has Bäcklund transformations, which transform solutions into another solutions of the same equation with different parameters. It was shown by Okamoto [14], [15], [16], [17] that the Bäcklund transformation groups of the Painlevé equations except for P_I are isomorphic to the extended affine Weyl groups. For $P_{II}, P_{III}, P_{IV}, P_V$, and P_{VI}, the Bäcklund transformation groups correspond to $A_1^{(1)}$, $A_1^{(1)} \bigoplus A_1^{(1)}$, $A_2^{(1)}$, $A_3^{(1)}$, and $D_4^{(1)}$, respectively.

Noumi and Yamada [13] discovered the equation of type $A_l^{(1)}$ $(l \geq 2)$, whose Bäcklund transformation group is isomorphic to the extended affine Weyl group $\tilde{W}(A_l^{(1)})$. The Noumi and Yamada systems of types $A_2^{(1)}$ and $A_3^{(1)}$ correspond to the fourth and fifth Painlevé equations, respectively. Furthermore, the rational solutions of the Noumi and Yamada systems of types $A_4^{(1)}$ and $A_5^{(1)}$ are classified in [5], [6].

Sasano [19] found the coupled Painlevé V and VI systems which have the affine Weyl group symmetries of types $D_5^{(1)}$ and $D_6^{(1)}$. Moreover, he [20] obtained equations of the affine Weyl group symmetries of types $A_4^{(2)}$ and $A_1^{(1)}$. They are called the Sasano systems of types $A_4^{(2)}$ and $A_1^{(1)}$, which are defined by

$$A_4^{(2)}(\alpha_j)_{0\le j\le 2}\begin{cases} x' = 4xy - 2\alpha_1 + 2zw, & y' = -2y^2 - 4x - 2t - w, \\ z' = z^2 - w + x + 2yz, & w' = -2zw - \alpha_0 - 2yw, \\ \alpha_0 + 2\alpha_1 + 2\alpha_2 = 1, \end{cases}$$

and

$$A_1^{(1)}(\alpha_0,\alpha_1)\begin{cases} x' = 2xy - \alpha_0 + zw, & y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, & w' = -z/2 - yw, \\ \alpha_0 + \alpha_1 = 1, \end{cases}$$

respectively. $A_1^{(1)}(\alpha_0,\alpha_1)$ is a degenerate system of $A_4^{(2)}(\alpha_j)_{0\le j\le 2}$. We note that the rational solutions of the Sasano systems of types $A_4^{(2)}$ and $D_5^{(1)}$ are classified in [7] and [8].

In this paper, we classify the rational solutions of the Sasano system of type $A_1^{(1)}$, $A_1^{(1)}(\alpha_0,\alpha_1)$. This system of differential equations is also expressed by the Hamiltonian system:

$$\frac{dx}{dt} = \frac{\partial H}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dz}{dt} = \frac{\partial H}{\partial w}, \quad \frac{dw}{dt} = -\frac{\partial H}{\partial z},$$

where the Hamiltonian H is given by

$$\begin{aligned} H &= H_{II}(x,y,t,\alpha_0) + H_3(z,w) + yzw \\ &= (xy^2 + x^2 + tx - \alpha_0 y) + (z^2/4 - w^2/4) + yzw. \end{aligned}$$

$A_1^{(1)}(\alpha_0,\alpha_1)$ has the Bäcklund transformations s_0, s_1, which are given by

$$s_0 : (x,y,z,w,t;\alpha_0,\alpha_1) \to \left(x, y - \frac{\alpha_0}{x+z^2}, z, w - \frac{2\alpha_0 z}{x+z^2}, t; -\alpha_0, \alpha_1 + 2\alpha_0\right),$$

$$\begin{aligned} s_1 &: (x,y,z,w,t;\alpha_0,\alpha_1) \\ &\to \left(x + \frac{2\alpha_1 y}{f_1} - \frac{\alpha_1^2}{f_1^2}, y - \frac{\alpha_1}{f_1}, z + \frac{2\alpha_1 w}{f_1}, w, t; \alpha_0 + 2\alpha_1, -\alpha_1\right), \end{aligned}$$

where $f_1 := x + y^2 + w^2 + t$ and $\alpha_0 + \alpha_1 = 1$. The Bäcklund transformation group $\langle s_0, s_1\rangle$ is isomorphic to the extended affine Weyl group $\tilde{W}(A_1^{(1)})$.

Our main theorem is as follows:

Theorem 0.1 *For a rational solution of $A_1^{(1)}(\alpha_0, \alpha_1)$, by some Bäcklund transformations, the parameters and solution can be transformed so that $(\alpha_0, \alpha_1) = (1/2, 1/2)$ and $(x, y, z, w) = (-t/2, 0, 0, 0)$, respectively. Furthermore, for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a rational solution if and only if $\alpha_0 - 1/2 \in \mathbb{Z}$.*

This paper is organized as follows. In Section 1, we determine the meromorphic solution at $t = \infty$ for $A_1^{(1)}(\alpha_0, \alpha_1)$. Then, we find that the residues of y at $t = \infty$ are expressed by the parameters α_0, α_1, that is,

$$b_{\infty,-1} := -\operatorname*{Res}_{t=\infty} y = 1/2 - \alpha_0.$$

In Section 2, we calculate the Laurent series of x, y, z, w at $t = c \in \mathbb{C}$. Then, we see that $\mathrm{Res}_{t=c}\, y \in \mathbb{Z}$. By the residue theorem, we obtain the necessary condition for $A_1^{(1)}(\alpha_0, \alpha_1)$ to have a rational solution, which is given by $1/2 - \alpha_0 \in \mathbb{Z}$.

In Section 3, we compute the residues of the Hamiltonian H at $t = \infty$ and $t = c \in \mathbb{C}$, which are given by

$$h_{\infty,-1} := -\operatorname*{Res}_{t=\infty} H = 1/2(\alpha_0 - 1/2)(1/2 - \alpha_0) \text{ and } \operatorname*{Res}_{t=c} H = 0, 1, 3,$$

respectively.

In Section 4, for a rational solution of $A_1^{(1)}(\alpha_0, \alpha_1)$, we transform the parameters to $(\alpha_0, \alpha_1) = (1/2, 1/2)$.

In Section 5, we determine a rational solution of $A_1^{(1)}(1/2, 1/2)$ and prove our main theorem.

1. Meromorphic Solution at $t = \infty$

In this section, for $A_1^{(1)}(\alpha_0, \alpha_1)$, we determine the meromorphic solutions at $t = \infty$. For the purpose, we set

$$\begin{cases} x = a_{\infty,n_0} t^{n_0} + a_{\infty,n_0-1} t^{n_0-1} + \cdots + a_{\infty,-k} t^{-k} + \cdots, \\ y = b_{\infty,n_1} t^{n_1} + b_{\infty,n_1-1} t^{n_1-1} + \cdots + b_{\infty,-k} t^{-k} + \cdots, \\ z = c_{\infty,n_2} t^{n_2} + c_{\infty,n_2-1} t^{n_2-1} + \cdots + c_{\infty,-k} t^{-k} + \cdots, \\ w = d_{\infty,n_3} t^{n_3} + d_{\infty,n_3-1} t^{n_3-1} + \cdots + d_{\infty,-k} t^{-k} + \cdots, \end{cases}$$

where n_0, n_1, n_2, n_3 are all integers.

The aim of this section is to show that for $A_1^{(1)}(\alpha_0, \alpha_1)$, the Laurent series of (x, y, z, w) at $t = \infty$ are uniquely determined and are given by

$$\begin{cases} x = -1/2 \cdot t + O(t^{-2}), & y = (1/2 - \alpha_0)t^{-1} + O(t^{-3}), \\ z = O(t^{-2}), & w = O(t^{-2}), \end{cases}$$

respectively. For the purpose, we treat the following five cases:

(0) x, y, z, w are all holomorphic at $t = \infty$,
(1) one of x, y, z, w has a pole at $t = \infty$,
(2) two of x, y, z, w have a pole at $t = \infty$,
(3) three of x, y, z, w have a pole at $t = \infty$,
(4) all of x, y, z, w have a pole at $t = \infty$.

1.1. The case where x, y, z, w are all holomorphic at $t = \infty$

Proposition 1.1 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that x, y, z, w are all holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. We first note that $n_0, n_1, n_2, n_3 \leq 0$. Then, comparing the coefficients of the term t in

$$y' = -y^2 - 2x - t,$$

we have $0 = -1$, which is impossible. □

1.2. The case where one of (x, y, z, w) has a pole at $t = \infty$

In this subsection, we consider the following four cases:

(1) x has a pole at $t = \infty$ and y, z, w are all holomorphic at $t = \infty$,
(2) y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$,
(3) z has a pole at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$,
(4) w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$.

1.2.1 The case where x has a pole at $t = \infty$

Proposition 1.2 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x has a pole at $t = \infty$ and y, z, w are all holomorphic at $t = \infty$. Then,*

$$\begin{cases} x = -1/2 \cdot t + O(t^{-2}), \\ y = (1/2 - \alpha_0)t^{-1} + O(t^{-3}), \\ z = O(t^{-2}), \\ w = O(t^{-2}). \end{cases}$$

Proof. Let us first note that $n_0 \geq 1$ and $n_1, n_2, n_3 \leq 0$. Considering

$$y' = -y^2 - 2x - t,$$

we find that $n_0 = 1$ and $a_{\infty,1} = -1/2$.

Comparing the constant terms in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, \\ w' = -z/2 - yw, \end{cases}$$

we obtain

$$a_{\infty,1} = 2a_{\infty,1}b_{\infty,-1} - \alpha_0 + c_{\infty,0}d_{\infty,0}, \quad a_{\infty,0} = 0, \quad d_{\infty,0} = 0, \quad c_{\infty,0} = 0,$$

respectively. Thus, it follows that $b_{\infty,-1} = 1/2 - \alpha_0$.

Furthermore, comparing the coefficients of the terms t^{-1} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, \\ w' = -z/2 - yw, \end{cases}$$

we have $b_{\infty,-2} = a_{\infty,-1} = d_{\infty,-1} = c_{\infty,-1} = 0$, respectively. $\square$

Proposition 1.3 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x has a pole at $t = \infty$ and y, z, w are all holomorphic at $t = \infty$. Then, it is unique.*

Proof. By Proposition 1.2, we set

$$\begin{cases} x = -1/2t + a_{\infty,-1}t^{-1} + \cdots & +a_{\infty,-k}t^{-k} + \cdots , \\ y = b_{\infty,-1}t^{-1} + b_{\infty,-2}t^{-2} + \cdots & +b_{\infty,-k}t^{-k} + b_{\infty,-(k+1)}t^{-(k+1)} + \cdots , \\ z = c_{\infty,-1}t^{-1} + \cdots & +c_{\infty,-k}t^{-k} + \cdots , \\ w = d_{\infty,-1}t^{-1} + \cdots & +d_{\infty,-k}t^{-k} + \cdots , \end{cases}$$

where $a_{\infty,-1}$, $b_{\infty,-1}$, $b_{\infty,-2}$, $c_{\infty,-1}$, $d_{\infty,-1}$ have been determined and

$$a_{\infty,-1} = c_{\infty,-1} = d_{\infty,-1} = 0, \quad b_{\infty,-1} = 1/2 - \alpha_0, \quad b_{\infty,-2} = 0.$$

Comparing the coefficients of the term t^{-k} $(k \geq 2)$ in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, \\ w' = -z/2 - yw, \end{cases}$$

we obtain

$$\begin{cases} b_{\infty,-(k+1)} = (k-1)a_{\infty,-(k-1)} + 2\sum a_{\infty,-l}b_{\infty,-m} + \sum c_{\infty,-l}d_{\infty,-m}, \\ 2a_{\infty,-k} = (k-1)b_{\infty,-(k-1)} - \sum b_{\infty,-l}b_{\infty,-m}, \\ 1/2 \cdot d_{\infty,-k} = (k-1)c_{\infty,-(k-1)} + \sum b_{\infty,-l}c_{\infty,-m}, \\ 1/2 \cdot c_{\infty,-k} = (k-1)d_{\infty,-(k-1)} - \sum b_{\infty,-l}d_{\infty,-m}, \end{cases}$$

where the sums extend over positive integers l, m such that $l + m = k$.

Therefore, $a_{\infty,-k}$, $b_{\infty,-(k+1)}$, $c_{\infty,-k}$, $d_{\infty,-k}$ are inductively determined, which proves the proposition. $\square$

1.2.2 The case where y has a pole at $t = \infty$

Proposition 1.4 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that y has a pole at $t = \infty$ and x, z, w are all holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $b_{\infty,n_1} \neq 0$, $n_1 \geq 1$ and $n_0, n_2, n_3 \leq 0$.

On the other hand, comparing the coefficients of the term t^{2n_1} in

$$y' = -y^2 - 2x - t,$$

we have $0 = -b_{\infty,n_1}^2$, which is impossible. □

1.2.3 The case where z has a pole at $t = \infty$

Proposition 1.5 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that z has a pole at $t = \infty$ and x, y, w are all holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $c_{\infty,n_2} \neq 0$, $n_2 \geq 1$ and $n_0, n_1, n_3 \leq 0$.

On the other hand, comparing the coefficients of the term t^{n_2} in

$$w' = -z/2 - yw,$$

we have $0 = -c_{\infty,n_2}/2$, which is contradiction. □

1.2.4 The case where w has a pole at $t = \infty$

Proposition 1.6 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that w has a pole at $t = \infty$ and x, y, z are all holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $d_{\infty,n_3} \neq 0$, $n_3 \geq 1$ and $n_0, n_1, n_2 \leq 0$.

On the other hand, comparing the coefficients of the term t^{n_3} in

$$z' = -w/2 + yz,$$

we obtain $0 = -d_{\infty,n_3}/2$, which is impossible. □

1.3. The case where two of (x, y, z, w) have a pole at $t = \infty$

We consider the following six cases:

(1) x, y both have a pole at $t = \infty$ and z, w are both holomorphic at $t = \infty$,
(2) x, z both have a pole at $t = \infty$ and y, w are both holomorphic at $t = \infty$,
(3) x, w both have a pole at $t = \infty$ and y, z are both holomorphic at $t = \infty$,
(4) y, z both have a pole at $t = \infty$ and x, w are both holomorphic at $t = \infty$,
(5) y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$,
(6) z, w both have a pole at $t = \infty$ and x, y are both holomorphic at $t = \infty$.

1.3.1 The case where $\boldsymbol{x, y}$ have a pole at $\boldsymbol{t = \infty}$

Proposition 1.7 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that x, y both have a pole at $t = \infty$ and z, w are both holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $a_{\infty,n_0} b_{\infty,n_1} \neq 0$, n_0, $n_1 \geq 1$ and n_2, $n_3 \leq 0$.

On the other hand, comparing the coefficients of the term $t^{n_0+n_1}$ in

$$x' = 2xy - \alpha_0 + zw,$$

we have $0 = 2a_{\infty,n_0} b_{\infty,n_1}$, which is contradiction. □

1.3.2 The case where $\boldsymbol{x, z}$ have a pole at $\boldsymbol{t = \infty}$

Proposition 1.8 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that x, z both have a pole at $t = \infty$ and y, w are both holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $a_{\infty,n_0} c_{\infty,n_2} \neq 0$, n_0, $n_2 \geq 1$ and n_1, $n_3 \leq 0$.

On the other hand, comparing the coefficients of the term t^{n_2} in

$$w' = -z/2 - yw,$$

we obtain $0 = -c_{\infty,n_2}/2$, which is impossible. □

1.3.3 The case where $\boldsymbol{x, w}$ have a pole at $\boldsymbol{t = \infty}$

Proposition 1.9 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that x, w both have a pole at $t = \infty$ and z, w are both holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $a_{\infty,n_0} d_{\infty,n_3} \neq 0$, n_0, $n_3 \geq 1$ and n_1, $n_2 \leq 0$.

On the other hand, comparing the coefficients of the term t^{n_3} in

$$z' = -w/2 + yz,$$

we have $0 = -d_{\infty,n_3}/2$, which is impossible. □

1.3.4 The case where $\boldsymbol{y, z}$ have a pole at $\boldsymbol{t = \infty}$

Proposition 1.10 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that y, z both have a pole at $t = \infty$ and x, w are both holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $b_{\infty,n_1} c_{\infty,n_2} \neq 0$, n_1, $n_2 \geq 1$ and n_0, $n_3 \leq 0$.

On the other hand, comparing the coefficients of the term $t^{n_1+n_2}$ in

$$z' = -w/2 + yz,$$

we have $0 = b_{\infty,n_1} c_{\infty,n_2}$, which is impossible. □

1.3.5 The case where y, w have a pole at $t = \infty$

Proposition 1.11 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that y, w both have a pole at $t = \infty$ and x, z are both holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $b_{\infty,n_1} d_{\infty,n_3} \neq 0$, n_1, $n_3 \geq 1$ and n_0, $n_2 \leq 0$.

On the other hand, comparing the coefficients of the term $t^{n_1+n_3}$ in

$$w' = -z/2 - yw,$$

we have $0 = -b_{\infty,n_1} d_{\infty,n_3}$, which is impossible. □

1.3.6 The case where z, w have a pole at $t = \infty$

Proposition 1.12 *For $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists no solution such that z, w both have a pole at $t = \infty$ and x, y are both holomorphic at $t = \infty$.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $c_{\infty,n_2} d_{\infty,n_3} \neq 0$, n_2, $n_3 \geq 1$ and n_0, $n_1 \leq 0$.

On the other hand, comparing the coefficients of the term $t^{n_2+n_3}$ in

$$x' = 2xy - \alpha_0 + zw,$$

we obtain $0 = c_{\infty,n_2} d_{\infty,n_3}$, which is contradiction. □

1.4. The case where three of (x, y, z, w) have a pole at $t = \infty$

In this subsection, we treat a solution such that three of (x, y, z, w) have a pole at $t = \infty$. For the purpose, we consider the following four cases:

(1) x, y, z all have a pole at $t = \infty$ and w is holomorphic at $t = \infty$,
(2) x, y, w all have a pole at $t = \infty$ and z is holomorphic at $t = \infty$,
(3) x, z, w all have a pole at $t = \infty$ and w is holomorphic at $t = \infty$,
(4) y, z, w all have a pole at $t = \infty$ and x is holomorphic at $t = \infty$.

In order to deal with the four cases, we prove the following lemma:

Lemma 1.13 *For* $A_1^{(1)}(\alpha_0, \alpha_1)$*, there exists no meromorphic solution at* $t = \infty$ *such that* x, y *both have a pole at* $t = \infty$*.*

Proof. It can be proved in the same way as Proposition 1.7. □

By Lemma 1.13, we have only to consider cases (3) and (4).

1.4.1 The case where x, z, w have a pole at $t = \infty$

Proposition 1.14 *For* $A_1^{(1)}(\alpha_0, \alpha_1)$*, there exists no solution such that* x, z, w *all have a pole at* $t = \infty$ *and* y *is holomorphic at* $t = \infty$*.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $a_{\infty,n_0} c_{\infty,n_2} d_{\infty,n_3} \neq 0$, n_0, n_2, $n_3 \geq 1$ and $n_1 \leq 0$.

On the other hand, considering

$$y' = -y^2 - 2x - t,$$

we have $n_0 = 1$ and $a_{\infty,1} = -1/2$. Moreover, comparing the coefficients of the term $t^{n_2+n_3}$ in

$$x' = 2xy - \alpha_0 + zw,$$

we obtain $0 = c_{\infty,n_2} d_{\infty,n_3}$, which is contradiction. □

1.4.2 The case where y, z, w have a pole at $t = \infty$

Proposition 1.15 *For* $A_1^{(1)}(\alpha_0, \alpha_1)$*, there exists no solution such that* y, z, w *all have a pole at* $t = \infty$ *and* x *is holomorphic at* $t = \infty$*.*

Proof. Suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has such a solution. Then, we see that $b_{\infty,n_1} c_{\infty,n_2} d_{\infty,n_3} \neq 0$, n_1, n_2, $n_3 \geq 1$ and $n_0 \leq 0$.

On the other hand, comparing the coefficients of the term t^{2n_1} in

$$y' = -y^2 - 2x - t,$$

we obtain $0 = -b_{\infty,n_1}^2$, which is impossible. □

1.5. The case where x, y, z, w all have a pole at $t = \infty$

Proposition 1.16 *For* $A_1^{(1)}(\alpha_0, \alpha_1)$*, there exists no solution such that* x, y, z, w *all have a pole at* $t = \infty$*.*

Proof. The proposition follows from Lemma 1.13. □

1.6. Summary

Let us summarize the results in this section.

Proposition 1.17 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a meromorphic solution at $t = \infty$. Then, x, y, z, w are uniquely expanded as*

$$\begin{cases} x = -1/2 \cdot t + O(t^{-2}), \\ y = (1/2 - \alpha_0)t^{-1} + O(t^{-3}), \\ z = O(t^{-2}), \\ w = O(t^{-2}). \end{cases}$$

2. Meromorphic Solution at $t = c \in \mathbb{C}$

In the same way as Proposition 1.17, we find which of (x, y, z, w) can have a pole at $t = c \in \mathbb{C}$.

Proposition 2.1 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a meromorphic solution at $t = c \in \mathbb{C}$. Moreover, assume that some of (x, y, z, w) have a pole at $t = c$. Then, one of the following occurs*:

(1) *y has a pole at $t = c$ and x, z, w are all holomorphic at $t = c$,*
(2) *x, y both have a pole at $t = c$ and z, w are both holomorphic at $t = c$,*
(3) *y, w both have a pole at $t = c$ and x, z are both holomorphic at $t = c$.*
(4) *x, y, z all have a pole at $t = c$ and w is holomorphic at $t = c$,*
(5) *x, y, z, w all have a pole at $t = c$.*

For the computation of a meromorphic solution at $t = c \in \mathbb{C}$, we define the Laurent series of (x, y, z, w) at $t = c \in \mathbb{C}$ by

$$\begin{cases} x = a_{c,n_0}T^{n_0} + a_{c,n_0+1}T^{n_0+1} + \cdots + a_{c,n_0+k}T^{n_0+k} + \cdots, \\ y = b_{c,n_1}T^{n_1} + b_{c,n_1+1}T^{n_1+1} + \cdots + b_{c,n_1+k}T^{n_1+k} + \cdots, \\ z = c_{c,n_2}T^{n_2} + c_{c,n_2+1}T^{n_2+1} + \cdots + c_{c,n_2+k}T^{n_2+k} + \cdots, \\ w = d_{c,n_3}T^{n_3} + d_{c,n_3+1}T^{n_3+1} + \cdots + d_{c,n_3+k}T^{n_3+k} + \cdots, \end{cases}$$

where $T := t - c$ and n_0, n_1, n_2, n_3 are all integers.

The aim of this section is to prove that for a meromorphic solution at $t = c$, the residue of y at $t = c$ is an integer, and for a rational solution of

$A_1^{(1)}(\alpha_0, \alpha_1)$, the residue of y at $t = \infty$ is expressed by

$$b_{\infty,-1} := -\operatorname*{Res}_{t=\infty} y = 1/2 - \alpha_0 \in \mathbb{Z}.$$

2.1. The case where y has a pole at $t = c \in \mathbb{C}$

Proposition 2.2 *Suppose that for* $A_1^{(1)}(\alpha_0, \alpha_1)$*, there exists a solution such that* y *has a pole at* $t = c \in \mathbb{C}$ *and* x, z, w *are all holomorphic at* $t = c$*. Then,*

$$\begin{cases} x = \alpha_0 T + \cdots, & y = T^{-1} + \cdots, \\ z = O(T), & w = O(T^2). \end{cases}$$

Proof. Let us first note that $n_1 \leq -1$, $b_{c,n_1} \neq 0$ and $n_0, n_2, n_3 \geq 0$. Considering

$$y' = -y^2 - 2x - t,$$

we have $n_1 = -1$ and $-b_{c,-1} = -b_{c,-1}^2$, which implies that $b_{c,-1} = 1$.

Comparing the coefficients of the term T^{-1} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ z' = -w/2 + yz, \\ w' = -z/2 - yw, \end{cases}$$

we obtain $2a_{c,0}b_{c,-1} = b_{c,-1}c_{c,0} = b_{c,-1}d_{c,0} = 0$, which implies that $a_{c,0} = c_{c,0} = d_{c,0} = 0$. Moreover, comparing the constant terms in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ w' = -z/2 - yw, \end{cases}$$

we have

$$a_{c,1} = 2a_{c,1}b_{c,-1} - \alpha_0, \quad d_{c,1} = -b_{c,-1}d_{c,1},$$

which shows that $a_{c,1} = \alpha_0$ and $d_{c,1} = 0$. □

2.2. The case where x, y have a pole at $t = c \in \mathbb{C}$

Lemma 2.3 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y both have a pole at $t = c \in \mathbb{C}$ and z, w are both holomorphic at $t = c$. Then, $n_0 = -2, n_1 = -1$.*

Proof. Let us first note that $n_0, n_1 \leq -1$, $a_{c,n_0} b_{c,n_1} \neq 0$, and n_2, $n_3 \geq 0$. Then, considering

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \end{cases}$$

we have $n_1 = -1$, $n_0 = -1, -2$, respectively.

We next prove that $n_0 = -2$. For the purpose, we suppose that $n_0 = -1$. Then, comparing the coefficients of the term T^{-2} in

$$x' = 2xy - \alpha_0 + zw,$$

we have $-a_{c,-1} = 2a_{c,-1} b_{c,-1}$, which implies that $b_{c,-1} = -1/2$. On the other hand, comparing the coefficients of the term T^{-2} in

$$y' = -y^2 - 2x - t,$$

we obtain $-b_{c,-1} = -b_{c,-1}^2$, which shows that $b_{c,-1} = 1$. This is impossible. $\square$

Proposition 2.4 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y both have a pole at $t = c \in \mathbb{C}$ and z, w are both holomorphic at $t = c$. Then,*

$$\begin{cases} x = -T^{-2} - c/3 - 1/2 \cdot T + \cdots, \\ y = -T^{-1} + c/3 \cdot T + (3/4 - \alpha_0/2)T^2 + \cdots, \\ z = O(T), \\ w = O(T). \end{cases}$$

Proof. By Lemma 2.3, we set

$$\begin{cases} x = a_{c,-2}T^{-2} + a_{c,-1}T^{-1} + a_{c,0} + \cdots, \\ y = b_{c,-1}T^{-1} + b_{c,0} + b_{c,1}T + b_{c,2}T^2 + \cdots, \\ z = c_{c,0} + c_{c,1}T + \cdots, \\ w = d_{c,0} + d_{c,1}T + \cdots. \end{cases}$$

Comparing the coefficients of the terms T^{-3}, T^{-2} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \end{cases}$$

we have $-2a_{c,-2} = 2a_{c,-2}b_{c,-1}$, $-b_{c,-1} = -b_{c,-1}^2 - 2a_{c,-2}$, which implies that $a_{c,-2} = -1$, $b_{c,-1} = -1$. Moreover, comparing the coefficients of the term T^{-1} in

$$\begin{cases} z' = -w/2 + yz, \\ w' = -z/2 - yw, \end{cases}$$

we have $b_{c,-1}c_{c,0} = -b_{c,-1}d_{c,0} = 0$, which shows that $c_{c,0} = d_{c,0} = 0$.

The other coefficients, $a_{c,-1}, a_{c,0}, a_{c,1}, b_{c,0}, b_{c,1}, b_{c,2}$ can be computed by considering

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t \end{cases}$$

in the same way. □

2.3. The case where y, w have a pole at $t = c \in \mathbb{C}$

Lemma 2.5 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that y, w both have a pole at $t = c \in \mathbb{C}$ and x, z are both holomorphic at $t = c$. Then, $n_1 = n_3 = -1$.*

Proof. Let us first note that n_1, $n_3 \leq -1$, $b_{c,n_1}d_{c,n_3} \neq 0$ and n_0, $n_2 \geq 0$. Then, considering

$$\begin{cases} y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, \end{cases}$$

we have $n_1 = n_3 = -1$ □

Proposition 2.6 *Suppose that for $A_1^{(1)}(\alpha_0,\alpha_1)$, there exists a solution such that y,w both have a pole at $t=c\in\mathbb{C}$ and x,z are both holomorphic at $t=c$. Then,*

$$\begin{cases} x=-d_{c,-1}^2/4+a_{c,1}T+\cdots\,, \\ y=T^{-1}+b_{c,1}T+\cdots\,, \\ z=d_{c,-1}/2+c_{c,1}T+\cdots\,, \\ w=d_{c,-1}T^{-1}+d_{c,1}T+\cdots\,, \end{cases}$$

where the coefficients satisfy $a_{c,1}-\alpha_0+c_{c,1}d_{c,-1}=0$.

Proof. By Lemma 2.5, we set

$$\begin{cases} x=a_{c,0}+a_{c,1}T+\cdots\,, \\ y=b_{c,-1}T^{-1}+b_{c,0}+b_{c,1}T+\cdots\,, \\ z=c_{c,0}+c_{c,1}T+\cdots\,, \\ w=d_{c,-1}T^{-1}+d_{c,0}+d_{c,1}T+\cdots\,. \end{cases}$$

Then, comparing the coefficients of the term T^{-2} in

$$y'=-y^2-2x-t,$$

we have $-b_{c,-1}=-b_{c,-1}^2$, which implies that $b_{c,-1}=1$. Moreover, comparing the coefficients of the term T^{-1} in

$$\begin{cases} x'=2xy-\alpha_0+zw, \\ z'=-w/2+yz, \end{cases}$$

we have $2a_{c,0}b_{c,-1}+c_{c,0}d_{c,-1}=-d_{c,-1}/2+b_{c,-1}c_{c,0}=0$, which implies that $a_{c,0}=-d_{c,-1}^2/4$, $c_{c,0}=d_{c,-1}/2$.

Comparing the coefficients of the terms T^{-1},T^0 in

$$\begin{cases} y'=-y^2-2x-t, \\ z'=-w/2+yz, \end{cases}$$

we have

$$-2b_{c,-1}b_{c,0} = 0, \quad c_{c,1} = -d_{c,0}/2 + b_{c,-1}c_{c,1} + b_{c,0}c_{c,0},$$

which implies that $b_{c,0} = d_{c,0} = 0$. Furthermore, comparing the constant term in

$$x' = 2xy - \alpha_0 + zw,$$

we obtain

$$a_{c,1} - \alpha_0 + c_{c,1}d_{c,-1} = 0. \qquad \square$$

2.4. The case where x, y, z have a pole at $t = c \in \mathbb{C}$

Lemma 2.7 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y, z all have a pole at $t = c \in \mathbb{C}$ and w is holomorphic at $t = c$. Then, $n_0 = -2$, $n_1 = n_3 = -1$.*

Proof. Let us first note that n_0, n_1, $n_2 \leq -1$, $a_{c,n_0}b_{c,n_1}c_{c,n_2} \neq 0$, and $n_3 \geq 0$. Then, considering

$$z' = -w/2 + yz,$$

we have $n_1 = -1$. Furthermore, considering

$$\begin{cases} y' = -y^2 - 2x - t, \\ w' = -z/2 - yw, \end{cases}$$

we have $n_0 = -1, -2$, $n_3 = -1$, respectively.

We next prove that $n_0 = -2$. For the purpose, suppose that $n_0 = -1$. Then, comparing the coefficients of the term T^{-2} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \end{cases}$$

we have

$$-a_{c,-1} = 2a_{c,-1}b_{c,-1}, \quad -b_{c,-1} = -b_{c,-1}^2,$$

which implies that $a_{c,-1} = 0$, $b_{c,-1} = 1$. This is impossible. $\square$

Proposition 2.8 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y, z all have a pole at $t = c \in \mathbb{C}$ and w is holomorphic at $t = c$. Then,*

$$\begin{cases} x = -T^{-2} + (c_{c,-1}^2/12 - c/3) - 1/2 \cdot T + \cdots , \\ y = -T^{-1} + (c_{c,-1}^2/6 + c/3)T + (3/4 - \alpha_0/2 + c_{c,-1}d_{c,1}/2)T^2 + \cdots , \\ z = c_{c,-1}T^{-1} + c_{c,1}T + \cdots , \\ w = c_{c,-1}/2 + d_{c,1}T + \cdots . \end{cases}$$

Proof. By Lemma 2.7, we set

$$\begin{cases} x = a_{c,-2}T^{-2} + a_{c,-1}T^{-1} + a_{c,0} + a_{c,1}T + \cdots , \\ y = b_{c,-1}T^{-1} + b_{c,0} + b_{c,1}T + b_{c,2}T^2 + \cdots , \\ z = c_{c,-1}T^{-1} + c_{c,0} + c_{c,1}T + \cdots , \\ w = d_{c,0} + d_{c,1}T + d_{c,1}T + \cdots . \end{cases}$$

Then, comparing the coefficients of the terms T^{-3}, T^{-2}, T^{-1} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ w' = -z/2 - yw, \end{cases}$$

we have

$$-2a_{c,-2} = 2a_{c,-2}b_{c,-1}, \quad -b_{c,-1} = -b_{c,-1}^2 - 2a_{c,-2},$$
$$0 = -c_{c,-1}/2 - b_{c,-1}d_{c,0},$$

which implies that $a_{c,-2} = -1$, $b_{c,-1} = -1$, $d_{c,0} = c_{c,-1}/2$.

Comparing the coefficients of the terms T^{-2}, T^{-1}, T^{-1} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, \end{cases}$$

we obtain

$$-a_{c,-1} = 2a_{c,-2}b_{c,0} + 2a_{c,-1}b_{c,-1},$$
$$0 = -2b_{c,-1}b_{c,0} - 2a_{c,-1}, \quad 0 = b_{c,-1}c_{c,0} + b_{c,0}c_{c,-1},$$

which implies that $a_{c,-1} = b_{c,0} = c_{c,0} = 0$.

Comparing the coefficients of the terms T^{-1}, T^0 in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \end{cases}$$

we have

$$0 = 2a_{c,-2}b_{c,1} + 2a_{c,0}b_{c,-1} + c_{c,-1}d_{c,0}, \quad b_{c,1} = -2b_{c,-1}b_{c,1} - 2a_{c,0} - c,$$

which shows that $a_{c,0} = c_{c,-1}^2/12 - c/3$, $b_{c,1} = c_{c,-1}^2/6 + c/3$. The other coefficients, $a_{c,1}, b_{c,2}$ can be computed by comparing the coefficients of the terms T^0, T in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \end{cases}$$

in the same way. □

2.5. The case where x, y, z, w have a pole at $t = c \in \mathbb{C}$

Lemma 2.9 *Suppose that for* $A_1^{(1)}(\alpha_0, \alpha_1)$*, there exists a solution such that* x, y, z, w *all have a pole at* $t = c \in \mathbb{C}$*. Then,* $(n_0, n_1, n_2, n_3) = (-2, -1, -1, -2)$*,* $(-2, -1, -2, -1)$*.*

Proof. Let us note that n_0, n_1, n_2, $n_3 \le -1$, and $a_{c,n_0}b_{c,n_1}c_{c,n_2}d_{c,n_3} \neq 0$. We first show that $n_1 = -1$. For the purpose, suppose that $n_1 \le -2$. Then, considering

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, \\ w' = -z/2 - yw, \end{cases}$$

we have

$$n_0 + n_1 = n_2 + n_3, \quad 2n_1 = n_0, \quad n_3 = n_1 + n_2, \quad n_2 = n_1 + n_3,$$

which implies that $n_0 = 2n_1$, $n_2 = 3n_1$, $n_3 = 2n_1$. Since $n_0 + n_1 = n_2 + n_3$, it follows that $n_1 = 0$, which is impossible. Therefore, we have $n_1 = -1$.

We show that $n_0 = -2$. Considering

$$y' = -y^2 - 2x - t,$$

we obtain $n_0 = -1, -2$. Then, we suppose that $n_0 = -1$. Thus, considering

$$x' = 2xy - \alpha_0 + zw,$$

we have $n_2 = n_3 = -1$. Comparing the coefficients of the term T^{-2} in

$$\begin{cases} y' = -y^2 - 2x - t, \\ z' = -w/2 + yz, \end{cases}$$

we obtain $-b_{c,-1} = -b_{c,-1}^2$, $-c_{c,-1} = b_{c,-1}c_{c,-1}$, which implies that $b_{c,-1} = 1$, $c_{c,-1} = 0$. This is impossible. Therefore, it follows that $n_0 = -2$.

We last prove that $(n_2, n_3) = (-1, -2), (-2, -1)$. For the purpose, considering

$$x' = 2xy - \alpha_0 + zw,$$

we have $(n_2, n_3) = (-1, -1), (-1, -2), (-2, -1)$. Let us suppose that $(n_2, n_3) = (-1, -1)$. Then, comparing the coefficients of the term T^{-3}, T^{-2}, T^{-2} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ w' = -z/2 - yw, \end{cases}$$

we obtain

$$-2a_{c,-2} = 2a_{c,-2}b_{c,-1}, \quad -b_{c,-1} = -b_{c,-1}^2 - 2a_{c,-2}, \quad -d_{c,-1} = -b_{c,-1}d_{c,-1},$$

which shows that $a_{c,-2} = -1$, $b_{c,-1} = -1$, $d_{c,-1} = 0$. This is impossible. Therefore, it follows that $(n_2, n_3) = (-1, -2), (-2, -1)$. $\square$

Proposition 2.10 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y, z, w all have a pole at $t = c \in \mathbb{C}$. Then, one of the following occurs*:

$$(1)\quad \begin{cases} x = -(t-c)^{-2} + (1/12 - c/3) - 1/2 \cdot (t-c) + \cdots, \\ y = 2(t-c)^{-1} + (-1/30 - c/15)(t-c) + 0(t-c)^2 + \cdots, \\ z = (t-c)^{-1} + (c/6 - 1/24)(t-c) + (1/4 + \alpha_0/6)(t-c)^2 + \cdots, \\ w = 6(t-c)^{-2} + (c/5 - 3/20) + 0(t-c) + \cdots, \end{cases}$$

$$(2)\quad \begin{cases} x = -(t-c)^{-2} + (1/12 - c/3) - 1/2(t-c)^2 + \cdots, \\ y = 2(t-c)^{-1} + (-1/30 - c/15)(t-c) + 0(t-c)^2 + \cdots, \\ z = -(t-c)^{-1} + (1/24 - c/6)(t-c) + (-1/4 - \alpha_0/6)(t-c)^2 + \cdots, \\ w = -6(t-c)^{-2} + (3/20 - c/5) + 0(t-c) + \cdots, \end{cases}$$

$$(3)\quad \begin{cases} x = -3(t-c)^{-2} + (1/20 - 2c/5) - 1/2(t-c) + \cdots, \\ y = -2(t-c)^{-1} + (1/30 + c/15)(t-c) + 0(t-c)^2 + \cdots, \\ z = 6\sqrt{-1}(t-c)^{-2} + (c/5 - 3/20)\sqrt{-1} + 0(t-c) + \cdots, \\ w = \sqrt{-1}(t-c)^{-1} + (c/6 - 1/24)\sqrt{-1}(t-c) \\ \qquad +(5/12 - \alpha_0/6)\sqrt{-1}(t-c)^2 + \cdots, \end{cases}$$

$$(4)\quad \begin{cases} x = -3(t-c)^{-2} + (1/20 - 2c/5) - 1/2(t-c)^2 + \cdots, \\ y = -2(t-c)^{-1} + (1/30 + c/15)(t-c) + 0(t-c)^2 + \cdots, \\ z = -6\sqrt{-1}(t-c)^{-2} + (3/20 - c/5)\sqrt{-1} + 0(t-c) + \cdots, \\ w = -\sqrt{-1}(t-c)^{-1} + (1/24 - c/6)\sqrt{-1}(t-c) \\ \qquad +(\alpha_0/6 - 5/12)\sqrt{-1}(t-c)^2 + \cdots. \end{cases}$$

Proof. We treat the case where $(n_0, n_1, n_2, n_3) = (-2, -1, -1, -2)$. If $(n_0, n_1, n_2, n_3) = (-2, -1, -2, -1)$, the coefficients can be computed in the same way.

Comparing the coefficients of the term T^{-2}, T^{-3} in

$$\begin{cases} y' = -y^2 - 2x - t, \\ w' = -z/2 - yw, \end{cases}$$

we have $-b_{c,-1} = -b_{c,-1}^2 - 2a_{c,-2}$, $-2d_{c,-2} = -b_{c,-1}d_{c,-2}$, which implies that $a_{c,-2} = -1$, $b_{c,-1} = 2$.

Furthermore, comparing the coefficients of the term T^{-3}, T^{-2} in

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ z' = -w/2 + yz, \end{cases}$$

we have

$$-2a_{c,-2} = 2a_{c,-2}b_{c,-1} + c_{c,-1}d_{c,-2}, \quad -c_{c,-1} = -d_{c,-2}/2 + b_{c,-1}c_{c,-1},$$

which implies that $(c_{c,-1}, d_{c,-2}) = (1,6), (-1,-6)$. The other coefficients can be computed in the same way. $\square$

2.6. Summary

Proposition 2.11

(1) *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a meromorphic solution at $t = \in \mathbb{C}$. Moreover, assume that some of (x, y, z, w) have a pole at $t = c$. Then, y is holomorphic at $t = c$ or y has a pole of order one at $t = c$. And $\mathrm{Res}_{t=c}\, y \in \mathbb{Z}$.*

(2) *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a rational solution. Then, it follows from Proposition 1.17 and the residue theorem that*

$$b_{\infty,-1} := 1/2 - \alpha_0 = -\operatorname*{Res}_{t=\infty} y \in \mathbb{Z}.$$

Proof. Case (1) is obvious. We treat case (2) and suppose that $A_1^{(1)}(\alpha_0, \alpha_1)$ has a rational solution. Then, it follows from Proposition 1.17 and case (1) that

$$\begin{aligned} y &= \sum_{i=1}^{m} \frac{n_i}{t - c_i} \\ &= \sum_{i=1}^{m} n_i t^{-1} \left(\sum_{k=0}^{+\infty} (c_i/t)^k \right) \\ &= (n_1 + n_2 + \cdots + n_m)t^{-1} + O(t^{-2}), \end{aligned}$$

where each of c_i $(i = 1, 2, \ldots, m)$ is a pole of y and each of n_i is an integer. If y is holomorphic in $\mathbb{C}$, the sum is considered to be zero.

Comparing the coefficients of the term t^{-1} of Laurent series of y at $t=\infty$, we have $b_{\infty,-1} = 1/2 - \alpha_0 \in \mathbb{Z}$. $\square$

3. The Laurent Series of the Hamiltonian H

In this section, using the results in Section 1 and 2, we compute the residues of H at $t=\infty, c \in \mathbb{C}$.

The aim of this section is to show that for a meromorphic solution at $t = c \in \mathbb{C}$, the residue of H at $t = c$ is an integer, and for a rational solution of $A_1^{(1)}(\alpha_0, \alpha_1)$,

$$h_{\infty,-1} := -\operatorname*{Res}_{t=\infty} H = 1/2(\alpha_0 - 1/2)(\alpha_0 + 1/2) \in \mathbb{Z}.$$

3.1. The Laurent series of H at $t = \infty$

For the computation of the Laurent series of H at $t=\infty$, using Proposition 1.17, we can set

$$H = h_{\infty,2}t^2 + h_{\infty,1}t + h_{\infty,0} + h_{\infty,-1}t^{-1} + \cdots,$$

where $h_{\infty,-1} = -\operatorname{Res}_{t=\infty} H$.

Proposition 3.1 *Suppose that for for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a meromorphic solution at $t = \infty$. Then, $-\operatorname{Res}_{t=\infty} H = 1/2(\alpha_0 - 1/2)(\alpha_0 + 1/2)$.*

3.2. The Laurent series of H at $t = c \in \mathbb{C}$

3.2.1 The case where y has a pole at $t = c \in \mathbb{C}$

Proposition 3.2 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that y has a pole at $t = c \in \mathbb{C}$ and x, z, w are all holomorphic at $t = c$. Then, H is holomorphic at $t = c$.*

3.2.2 The case where x, y have a pole at $t = c \in \mathbb{C}$

Proposition 3.3 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y both have a pole at $t = c \in \mathbb{C}$ and z, w are both holomorphic at $t = c$. Then, H has a pole of order one at $t = c$ and $\operatorname{Res}_{t=c} H = 1$.*

3.3. The case where y, w have a pole at $t = c \in \mathbb{C}$

Proposition 3.4 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that y, w both have a pole at $t = c \in \mathbb{C}$ and x, z are both holomorphic at $t = c$. Then, H is holomorphic at $t = c$.*

3.4. The case where x, y, z have a pole at $t = c \in \mathbb{C}$

Proposition 3.5 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y, z all have a pole at $t = c \in \mathbb{C}$ and w is holomorphic at $t = c$. Then, H has a pole of order one at $t = c$ and $\mathrm{Res}_{t=c} H = 1$.*

3.5. The case where x, y, z, w have a pole at $t = c \in \mathbb{C}$

Proposition 3.6 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a solution such that x, y, z, w all have a pole at $t = c \in \mathbb{C}$. Then, H has a pole of order one at $t = c$ and*

$$\operatorname*{Res}_{t=c} H = \begin{cases} 1 & \text{if case (1) or (2) occurs in Proposition 2.10,} \\ 3 & \text{if case (3) or (4) occurs in Proposition 2.10.} \end{cases}$$

3.6. Summary

Proposition 3.7

(1) *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a meromorphic solution at $t = c \in \mathbb{C}$. Then, H is holomorphic at $t = c$, or has a pole of order one at $t = c$. Furthermore, if H has a pole at $t = c$, then $\mathrm{Res}_{t=c} H = 1, 3$.*

(2) *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a rational solution. Then, it follows from Proposition 3.1 and the residue theorem that*

$$h_{\infty,-1} = 1/2(\alpha_0 - 1/2)(\alpha_0 + 1/2) = -\operatorname*{Res}_{t=\infty} H \in \mathbb{Z}.$$

Proof. Case (1) is obvious. We treat case (2). For the purpose, let us suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a rational solution. Then, it follows from Proposition 1.17 and case (1) that H is a rational function of t and given by

$$\begin{aligned} H &= h_{\infty,2}t^2 + h_{\infty,1}t + h_{\infty,0} + \sum_{k=1}^{n} \frac{\epsilon_k}{t - c_k}, \\ &= h_{\infty,2}t^2 + h_{\infty,1}t + h_{\infty,0} + \sum_{k=1}^{n} \epsilon_k t^{-1} \left(\sum_{j=0}^{+\infty} \left(\frac{c_k}{t} \right)^j \right) \\ &= h_{\infty,2}t^2 + h_{\infty,1}t + h_{\infty,0} + \left(\sum_{k=1}^{n} \epsilon_k \right) t^{-1} + \cdots, \end{aligned}$$

where each of $c_k \in \mathbb{C}$ $(k = 1, 2, \ldots, n)$ is a pole of some of x, y, z, w and each of ϵ_k is given by $\epsilon_k = 0, 1, 3$. If x, y, z, w are all holomorphic in $\mathbb{C}$, the sum is considered to be zero.

Comparing the coefficients of the Laurent series of H at $t = \infty$, we have

$$h_{\infty,-1} = \sum_{k=1}^{n} \epsilon_k \in \mathbb{Z},$$

which proves the proposition. □

4. Necessary Condition And The Reduction of The Parameters

In Proposition 2.11, we have obtained a necessary condition for $A_1^{(1)}(\alpha_0, \alpha_1)$ to have a rational solution, which is given by $1/2 - \alpha_0 \in \mathbb{Z}$. In this section, using Bäcklund transformations, we transform the parameters to $(\alpha_0, \alpha_1) = (1/2, 1/2)$.

4.1. Shift operators

Following Sasano [20], we introduce the shift operator T.

Proposition 4.1 *Let the shift operator T be defined by $T := s_1 s_0$. Then,*

$$T(\alpha_0, \alpha_1) = (\alpha_0 + 2, \alpha_1 - 2).$$

Proof. From the definitions of s_0 and s_1, it follows that

$$\begin{aligned}
T(\alpha_0, \alpha_1) &= s_1 s_0(\alpha_0, \alpha_1) \\
&= s_1(-\alpha_0, \alpha_1 + 2\alpha_0) \\
&= (-\alpha_0 + 2(\alpha_1 + 2\alpha_0), -(\alpha_1 + 2\alpha_0)) \\
&= (-\alpha_0 + 2(1 + \alpha_0), -(2 - \alpha_1)) \\
&= (\alpha_0 + 2, \alpha_1 - 2),
\end{aligned}$$

where in the fourth equality, we use the relation, $\alpha_0 + \alpha_1 = 1$. □

4.2. The properties of the Bäcklund transformations

Proposition 4.2

(0) *If $x + z^2 = 0$ for $A_1^{(1)}(\alpha_0, \alpha_1)$, then $\alpha_0 = 0$.*
(1) *If $f_1 := x + y^2 + w^2 + t = 0$ for $A_1^{(1)}(\alpha_0, \alpha_1)$, then $\alpha_1 = 0$.*

Proof. We first treat case (0). If $x + z^2 = 0$, it follows that $x' + 2zz' = 0$. Considering

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ z' = -w/2 + yz, \end{cases}$$

we find that $x' + 2zz' = 2y(x + z^2) - \alpha_0 = 0$, which implies that $\alpha_0 = 0$.

We next deal with case (1). If $x + y^2 + w^2 + t = 0$, it follows that $x' + 2yy' + 2ww' + 1 = 0$. Considering

$$\begin{cases} x' = 2xy - \alpha_0 + zw, \\ y' = -y^2 - 2x - t, \\ w' = -z/2 - yw, \end{cases}$$

we see that

$$x' + 2yy' + 2ww' + 1 = -2y(x + y^2 + w^2 + t) - \alpha_0 + 1 = 0,$$

which implies that $\alpha_1 = 0$, because $\alpha_0 + \alpha_1 = 1$. □

By this proposition, we can consider s_0 or s_1 as the identical transformation, if $x + z^2 = 0$ for $A_1^{(1)}(\alpha_0, \alpha_1)$, or if $f_1 := x + y^2 + w^2 + t = 0$ for $A_1^{(1)}(\alpha_0, \alpha_1)$, respectively.

4.3. The reduction of the parameters

In Proposition 2.11, we have obtained a necessary condition for $A_1^{(1)}(\alpha_0, \alpha_1)$ to have a rational solution, which is given by $1/2 - \alpha_0 \in \mathbb{Z}$. Then, by shift operator T, we can prove the following proposition.

Proposition 4.3 *Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a rational solution. Then, by some Bäcklund transformations, the parameters can be transformed to $(\alpha_0, \alpha_1) = (1/2, 1/2)$.*

Proof. If $A_1^{(1)}(\alpha_0, \alpha_1)$ has a rational solution, it follows from Proposition 2.11 that $1/2 - \alpha_0 \in \mathbb{Z}$. Let us consider the following two cases: (1) $1/2 - \alpha_0 \in 2\mathbb{Z}$, (2) $(1/2 - \alpha_0) - 1 \in 2\mathbb{Z}$.

We first treat case (1). Then, by T, the parameters can be transformed to $(\alpha_0, \alpha_1) = (1/2, 1/2)$.

We next deal with case (2). Then, by T, the parameters can be transformed to $(\alpha_0, \alpha_1) = (3/2, -1/2)$. In addition, by s_1, the parameters are transformed to $(\alpha_0, \alpha_1) = (1/2, 1/2)$. □

5. Classification of Rational Solutions

5.1. Rational solutions of $A_1^{(1)}(1/2, 1/2)$

Proposition 5.1 *For $A_1^{(1)}(1/2, 1/2)$, there exists a rational solution such that $(x, y, z, w) = (-t/2, 0, 0, 0)$ and it is unique.*

Proof. The proposition follows from direct calculation and Proposition 1.3. □

5.2. Proof of Main Theorem

Proof. Suppose that for $A_1^{(1)}(\alpha_0, \alpha_1)$, there exists a rational solution. Then, it follows from Proposition 2.11 that $1/2 - \alpha_0 \in \mathbb{Z}$. Furthermore, Proposition 4.3 shows that by some Bäcklund transformations, the parameters can be transformed to $(\alpha_0, \alpha_1) = (1/2, 1/2)$.

By Proposition 5.1, we see that for $A_1^{(1)}(1/2, 1/2)$, there exists a unique rational solution such that $(x, y, z, w) = (-t/2, 0, 0, 0)$, which proves our main theorem. □

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