Hokkaido Mathematical Journal Vol. 40 (2011) p. 431–447

A characterization of 42 ovoids with a certain property in $PG(3,2)$

Koichi Inoue

(Received July 2, 2010; Revised April 7, 2011)

Abstract. In this paper, we characterize 42 ovoids with a certain property in a projective space $PG(3,2)$ described in Yucas [6]. As a corollary, we construct the Steiner 4-wise balanced design $S(4, \{5,6\}, 17)$ with 252 blocks which is an extension of the point-plane design $\boldsymbol{A}$ of an affine space $AG(4,2)$. The construction leads to not only the uniqueness of such an extension, but also a (usual) extension of the 2-repeated design $2.\boldsymbol{A}$.

Key words: Steiner 4-wise balanced design, affine space, orthogonal group, exterior algebra, alternating forms, quadratic forms

1. Introduction

A t-$(v, \mathcal{K}, \lambda)$ *structure* is a pair $(\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a set of v elements (called *points*) and $\mathcal{B}$ is a *multi-set* of subsets of $\mathcal{P}$ (called *blocks*) such that the size of every block is contained in $\mathcal{K}$ and every t-subset of $\mathcal{P}$ is contained in exactly λ blocks. If $\lambda = 1$ then the structure, which does not allow repeated blocks, is called a *Steiner t-wise balanced design* and denoted by $S(t, \mathcal{K}, v)$, and furthermore denoted by $S(t, k, v)$ if $\mathcal{K} = \{k\}$.

For two t-$(v, \mathcal{K}, \lambda)$ structures $\mathcal{D}$ and $\mathcal{E}$, we define an *isomorphism* φ from $\mathcal{D}$ onto $\mathcal{E}$ to be a one-to-one mapping from the points of $\mathcal{D}$ onto the points of $\mathcal{E}$ and the blocks of $\mathcal{D}$ onto the blocks of $\mathcal{E}$ such that p is in B if and only if $\varphi(p)$ is in $\varphi(B)$ for each point p and each block B of $\mathcal{D}$, and say that $\mathcal{D}$ and $\mathcal{E}$ are *isomorphic*.

Let $\mathcal{D} := (\mathcal{P}, \mathcal{B})$ be a t-$(v, \mathcal{K}, \lambda)$ structure and $p \in \mathcal{P}$. A pair $(\mathcal{P} \setminus \{p\}, \mathcal{B}')$ where $\mathcal{B}'$ is a multi-set of $B \setminus \{p\}$ for all $B \in \mathcal{B}$ containing p is called the *derived* structure of $\mathcal{D}$ at p and denoted by $\mathcal{D}_p$. Let $\mathcal{D} := (\mathcal{P}, \mathcal{B})$ be an $S(t, k, v)$. A pair $(\mathcal{P}, \mathcal{B}')$ where $\mathcal{B}'$ is a multiple set in which each block of $\mathcal{D}$ is repeated λ times is clearly a t-(v, k, λ) structure and denoted by $\lambda.\mathcal{D}$, which is *extendable* if there exist a $(t+1)$-$(v+1, k+1, \lambda)$ structure $\mathcal{E}$ and a point p of $\mathcal{E}$ such that $\mathcal{E}_p$ is isomorphic to $\lambda.\mathcal{D}$. The structure $\mathcal{E}$ is called

---

2000 Mathematics Subject Classification : 20B25, 05B25.

a (usual) *extension* of $\lambda.\mathcal{D}$.

Let V be a 4-dimensional vector space over $\mathbb{F}_2$. The 15 non-zero vectors of V together with the 35 blocks (called *lines*) of $l \setminus \{0\}$ for 2-dimensional subspaces l form an $S(2,3,15)$. The 16 vectors of V together with 140 blocks of cosets of 2-dimensional subspaces of V form an $S(3,4,16)$, which is denoted by $\boldsymbol{A}_2(4,2)$ according to Dembowski [2], but for convenience we call it $\boldsymbol{A}$.

Kramer and Mathon [3] have showed by exhaustive computer search that there is a unique $S(4,\mathcal{K},17)$ with $|\mathcal{K}| \geq 2$. In Yucas [6], the Steiner 4-wise balanced design $S(4,\{5,6\},17)$ with 252 blocks has been constructed by extending $\boldsymbol{A}$. There are 42 blocks of the design which do not contain a new point ∞ and contain the zero vector 0. These blocks are ovoids in the projective space $PG(V)$ and also cover the triangles of $PG(V)$ once each. We will characterize the 42 ovoids in $PG(V)$ with this property. As a corollary, we will give another construction of the $S(4,\{5,6\},17)$ which is an extension of $\boldsymbol{A}$. The construction is based on a set of certain alternating forms on V associated with the alternating group A_7 of degree 7 (see Section 3) and leads to not only the uniqueness of such an extension of $\boldsymbol{A}$, but also a (usual) extension of $2.\boldsymbol{A}$. Finally, such a set of certain alternating forms could not be found in [3] and [6].

The paper is divided into four sections. In Section 2, we give some observations of the affine space $AG(V)$, and study the alternating forms and the quadratic forms on V. Since it is known that there is a bijection between the non-degenerate alternating forms on V and the non-singular vectors of an orthogonal geometry (W,Q), where W is a 6-dimensional vector space and Q is a non-degenerate quadratic form on W whose the Witt index is 3 (see e.g. Taylor [5, p. 195]), Section 3 contains some detailed observations about non-singular vectors of (W,Q) and the introduction of such a bijection. In Section 4, we characterize the 42 ovoids which cover the triangles of $PG(V)$ once each as mentioned above.

2. The Affine Space $AG(V)$

We begin with a detailed study of $AG(V)$.

A *triangle* is a 3-subset of $V\setminus\{0\}$ which is linearly independent. A *double triangle* is the set $\{p\} \cup l$ where (p,l) is a non-incident point-line pair in $PG(V)$. An *oval* is a block of $\boldsymbol{A}$ not containing the zero vector 0. There are exactly 105 ovals. An *ovoid* is a 5-subset of $V \setminus \{0\}$ in which any four vectors are linearly independent. The set of ovoids is denoted by $\mathcal{O}$. We denote by $\mathfrak{X}$ the set of unordered bases and set $\mathcal{L} = \{l \cup m \mid l, m : \text{two disjoint lines}\}$.

Elementary counting arguments show the following lemma:

Lemma 2.1 *There are exactly*:

(1) 420 *triangles,*
(2) 420 *double triangles,*
(3) 840 *unordered bases,*
(4) *one ovoid containing a given* $X \in \mathfrak{X}$*,*
(5) 168 *ovoids,*
(6) *four ovoids containing a given triangle,*
(7) *three elements of* $\mathcal{L}$ *containing a given* $X \in \mathfrak{X}$*,*
(8) 280 *elements of* $\mathcal{L}$*.*

Remark 2.2 Given a $X := \{e_1, e_2, e_3, e_4\} \in \mathfrak{X}$, the unique ovoid containing X is $X \cup \{e_1 + e_2 + e_3 + e_4\}$.

A *k-cap* is a k-subset of $V \setminus \{0\}$ in which any three vectors are linearly independent.

Lemma 2.3 *Any* 5*-subset of* $V \setminus \{0\}$ *is one of following four types*:

- *a union of two meeting lines.*
- *a form* $\{x, y, z, w, x + y\}$ *for some* $\{x, y, z, w\} \in \mathfrak{X}$*.*
- *a* 5*-cap which contains exactly one oval.*
- *an ovoid.*

Proof. Let S be a 5-subset of $V \setminus \{0\}$. If S is a 5-cap and not an ovoid, then some 4-subset $\{s_1, s_2, s_3, s_4\}$ of S is a 4-cap and linearly dependent, so $s_4 = s_1 + s_2 + s_3$. If we take $s_5 \in S \setminus \{s_1, s_2, s_3, s_4\}$ then $s_5 \notin \langle s_1, s_2, s_3 \rangle$, so ovals contained in S are just the $\{s_1, s_2, s_3, s_4\}$. If S is not a 5-cap, then we can take some line $l := \{x, y, x + y\}$ contained in S. Let $S = l \cup \{z, w\}$ and m be the line containing $\{z, w\}$. If $|l \cap m| = 1$ then $S = l \cup m$. If $|l \cap m| = 0$ then $S = \{x, y, z, w, x + y\}$, where $\{x, y, z, w\} \in \mathfrak{X}$. □

Next, to see the intersection of any two elements of $\mathcal{O} \cup \mathcal{L}$, we describe the elements of $\mathcal{O} \cup \mathcal{L}$ which correspond to the non-degenerate quadratic forms.

Let Γ be the set of non-degenerate alternating forms and $\mathcal{F}$ the set of non-degenerate quadratic forms. We define the action of $GL(V)$ on Γ by left multiplication:

$$\sigma \cdot \gamma : V \times V \ni (x, y) \longmapsto \gamma(\sigma^{-1}x, \sigma^{-1}y) \in \mathbb{F}_2$$

for all $\sigma \in GL(V)$ and $\gamma \in \Gamma$, and similarly define the action of $GL(V)$ on $\mathcal{F}$ by left multiplication:

$$\sigma \cdot f : V \ni x \longmapsto f(\sigma^{-1}x) \in \mathbb{F}_2$$

for all $\sigma \in GL(V)$ and $f \in \mathcal{F}$. For $f \in \mathcal{F}$, we set

$$Q_f = \{x \in V \mid f(x) = 0\},$$

and call the type of f *minus* or *plus* according as the Witt index of f is 1 or 2. If f is a minus type then $Q_f \setminus \{0\}$ is in $\mathcal{O}$ and $|Q_f| = 6$. If f is a plus type then the complementary set $\overline{Q_f} := V \setminus Q_f$ of Q_f is in $\mathcal{L}$ and $|\overline{Q_f}| = 6$.

Let $\gamma \in \Gamma$ and $\mathcal{F}_\gamma$ be the set of quadratic forms whose polar form is γ. Set

$$\mathcal{F}_\gamma^+ = \{f \in \mathcal{F}_\gamma \mid f \text{ is a plus type}\} \quad \text{and}$$
$$\mathcal{F}_\gamma^- = \{f \in \mathcal{F}_\gamma \mid f \text{ is a minus type}\}.$$

For the following lemma, we refer to Cameron and van Lint [1, Example 5.17] and Taylor [5, Exercise 11.17].

Lemma 2.4 $|\mathcal{F}_\gamma^+| = 10, |\mathcal{F}_\gamma^-| = 6$ *and the pair*

$$\big(V, \{Q_f \mid f \in \mathcal{F}_\gamma^-\} \cup \{\overline{Q_f} \mid f \in \mathcal{F}_\gamma^+\}\big)$$

is a symmetric 2-$(16, 6, 2)$ *design. Thus any two blocks in the design have two common points.*

Moreover, since $|\Gamma| = 28$ and $|\bigcup_{\gamma\in\Gamma} \mathcal{F}_\gamma^-| = 28 \cdot 6 = |\mathcal{O}|$, we see that the map $f \mapsto Q_f \setminus \{0\}$ is a bijection between $\bigcup_{\gamma\in\Gamma} \mathcal{F}_\gamma^-$ and $\mathcal{O}$. Similarly, the map $f \mapsto \overline{Q_f}$ is also a bijection between $\bigcup_{\gamma\in\Gamma} \mathcal{F}_\gamma^+$ and $\mathcal{L}$ since $|\bigcup_{\gamma\in\Gamma} \mathcal{F}_\gamma^+| = 28 \cdot 10 = |\mathcal{L}|$.

Lemma 2.5 *For $\gamma, \delta \in \Gamma$ such that $\gamma + \delta$ is non-degenerate, let $f \in \mathcal{F}_\gamma$ and $g \in \mathcal{F}_\delta$.*

(i) *If both f and g are minus types, then $|Q_f \cap Q_g| = 1$ or 3.*
(ii) *If both f and g are plus types, then $|\overline{Q_f} \cap \overline{Q_g}| = 1$ or 3.*
(iii) *If f is a minus type and g is a plus type, then $|Q_f \cap \overline{Q_g}| = 1$ or 3.*

Proof. We give the proof only for (i) because the other cases are similar to (i). Noting that $f+g$ is non-degenerate and $Q_f \triangle Q_g = \{x \in V \mid (f+g)(x) = 1\} = \overline{Q_{f+g}}$, we have $|Q_f \cap Q_g| = (|Q_f| + |Q_g| - |Q_f \triangle Q_g|)/2 = 1$ or 3, where $Q_f \triangle Q_g$ is the symmetric difference of Q_f and Q_g. □

Remark 2.6 For $\gamma, \delta \in \Gamma$ such that $\gamma + \delta$ is degenerate, we have similar results which are not needed in this paper.

3. The Orthogonal Geometry for $O^+(6,2)$

To define appropriate new blocks which we need to extend $\boldsymbol{A}$ to the $S(4, \{5,6\}, 17)$ with 252 blocks, we consider the geometry for an orthogonal group $O^+(6,2)$. Here the notation are consistent with [5].

Let W be an orthogonal geometry of dimension 6 over $\mathbb{F}_2$ defined by a non-degenerate quadratic form Q whose polar form is β and suppose that the Witt index is 3. Let

$$O(W) = \{f \in GL(W) \mid Q(f(w)) = Q(w) \text{ for all } w \in W\},$$

where $GL(W)$ is the group of invertible linear transformations from W to itself, and $\Omega(W)$ the derived subgroup of $O(W)$. $O(W)$ is also denoted by $O^+(6,2)$. For a non-singular vector w, the map t_w defined by

$$t_w(x) = x + \beta(x,w)w$$

for all $x \in W$ is an element of $O(W)$ and is called a *reflection.* In the graph Δ with as vertex set the non-singular vectors and join two vertices v, w whenever $\beta(v,w) = 0$, the following holds:

Lemma 3.1

(1) *There are exactly eight 7-cocliques in Δ.*
(2) *For any 7-coclique C in Δ, the sum of all vectors in C is 0 and any six*

vectors in C are linearly independent.

(3) *Any two 7-cocliques in Δ meet in a unique non-singular vector. Moreover, the size of the intersection of any three 7-cocliques in Δ is* 0.

Proof. Regarding $\mathbb{F}_2$ as $\mathbb{Z}/2\mathbb{Z}$, we define the subspace

$$U = \left\{x \in \mathbb{F}_2^8 \mid 2|\mathrm{wt}(x)\right\}$$

of $\mathbb{F}_2^8$, where $\mathrm{wt}(x)$ denotes the number of ones in x, and the quadratic form $f : U \to \mathbb{F}_2$ by

$$f(x) = \frac{\mathrm{wt}(x)}{2} \pmod 2$$

for all $x \in U$. Then the polar form of f is equal to

$$\begin{aligned} f(x+y) - f(x) - f(y) &\equiv |\mathrm{supp}(x) \cap \mathrm{supp}(y)| \pmod 2 \\ &= \sum_{i=1}^{8} x_i y_i, \end{aligned}$$

for all $x := (x_1, \ldots, x_8)$ and $y := (y_1, \ldots, y_8) \in V$, where $\mathrm{supp}(x) := \{i \in \{1, \ldots, 8\} \mid i\text{-th entry in } x = 1\}$. Since U contains the all-1 vector $\mathbf{1}$ and $\{x \in \mathrm{rad}U \mid f(x) = 0\} = \langle \mathbf{1} \rangle$, so f induces the non-degenerate quadratic form $\overline{f}$ from $\overline{U} := U/\langle \mathbf{1} \rangle$ to $\mathbb{F}_2$ by $\overline{f}(\overline{x}) := f(x)$ for all $\overline{x} := x + \langle \mathbf{1} \rangle \in \overline{U}$. Moreover $\overline{f}$ is a plus type, that is, the Witt index of $\overline{f}$ is 3. Therefore two orthogonal geometries $(\overline{U}, \overline{f})$ and (W, Q) are isomorphic. For non-singular vector $\overline{x}$, since we may have $\mathrm{wt}(x) = 2$ and identify x with $\mathrm{supp}(x)$, we write ij in the place of $\{i, j\}$, where $1 \le i \ne j \le 8$. Then it is straightforward to see that there are exactly eight 7-cocliques in Δ as follows:

$$\{18, 28, 38, 48, 58, 68, 78\}, \ldots, \{21, 31, 41, 51, 61, 71, 81\}.$$

This easily yields (2) and (3). $\square$

Suppose that G is a group acting on a set Ω and $X \subseteq \Omega$. Then we set $G_{\{X\}} = \{g \in G \mid gX = X\}$.

Lemma 3.2 *Let $C = \{w_1, \ldots, w_7\}$ be a 7-coclique in Δ.*

(1) $O(W)_{\{C\}} = \langle t_{w_i + w_j} \mid 1 \le i < j \le 7 \rangle$ *and it is isomorphic to* S_7.

(2) $\Omega(W)_{\{C\}} = \langle t_{w_i+w_j} t_{w_k+w_l} \mid \{i,j\},\{k,l\}$: *two disjoint* 2*-subsets of* $\{1,\ldots,7\}\rangle$ *and it is isomorphic to* A_7*.*

Proof. $O(W)_{\{C\}}$ acts faithfully on C and is identified with the subgroup of $S(C) = S_7$. If $\{v_1,\ldots,v_7\}$ is another 7-coclique in Δ then there exists $f \in O(W)$ such that $f(w_i) = v_i$ for all $i \in \{1,\ldots,7\}$. This implies $O(W)_{\{C\}} = S(C)$ since $|O(W)_{\{C\}}| = |O(W)|/8 = |S_7|$. The derived subgroup of $O(W)_{\{C\}}$ is $A(C) = A_7$ and clearly contained in $\Omega(W)_{\{C\}}$. Any transposition $(w_i\ w_j)$ of $S(C)$ is identified with a reflection $t_{w_i+w_j}$, but $t_{w_i+w_j} \notin \Omega(W)$ and so $|\Omega(W)_{\{C\}}| \leq |A_7|$. This implies that $\Omega(W)_{\{C\}} = A(C)$. Thus the result follows. □

The lemmas above show the following:

Proposition 3.3 $\Omega(W)$ *acts transitively on the set of eight* 7*-cocliques in* Δ*.*

We next apply the above observations to the special orthogonal geometry for $O^+(6,2)$. The *exterior algebra* of V is introduced in [5]. Let e_1, e_2, e_3, e_4 be a basis for V and $\tilde{e} := e_1 \wedge e_2 \wedge e_3 \wedge e_4$. For $\xi := \sum_{1\leq i<j\leq 4} p_{ij} e_i \wedge e_j \in \Lambda_2 V$, where $\Lambda_2 V$ is the *second exterior power* of V, we put

$$Q(\xi) = p_{12}p_{34} + p_{13}p_{24} + p_{14}p_{23}.$$

Then Q is a non-degenerate quadratic form of the Witt index 3 on $\Lambda_2 V$. We let β denote the polar form of Q. We note that Q does not depend on the basis chosen for V and it is uniquely determined. By [5, Theorems 12.17, 12.20], the map

$$GL(V) \ni f \mapsto \Lambda_2 f \in \Omega(\Lambda_2 V)$$

is an isomorphism, and furthermore by [5, p. 195] there is a bijection φ from the set of all non-singular bivectors of $\Lambda_2 V$ to Γ defined by

$$\varphi(\xi)(x \wedge y) = \beta(x \wedge y, \xi)$$

for any non-singular bivector ξ of $\Lambda_2 V$ and all $x, y \in V$.

Let ξ, η be the non-singular bivectors corresponding to distinct $\gamma, \delta \in \Gamma$, respectively. Since $\gamma + \delta = \beta(-, \xi + \eta)\alpha_2$, it is seen that $\gamma + \delta$ is non-degenerate if and only if $\beta(\xi,\eta)=1$. Take eight 7-cocliques $C_1,\dots,C_8$ in the graph Δ and put $C_1=\{\xi_1,\dots,\xi_7\}$. Moreover, take $\gamma_i\in\Gamma$ corresponding to each ξ_i and put $\mathcal{C}_1=\{\gamma_1,\dots,\gamma_7\}$. For each $i\in\{2,\dots,8\}$, we similarly let $\mathcal{C}_i$ denote the image of C_i under the correspondence. We define the following four sets:

$$\mathcal{O}_1=\bigcup_{i=1}^{7}\{Q_f\setminus\{0\}\mid f\in\mathcal{F}^-_{\gamma_i}\},\quad \overline{\mathcal{O}_1}=\mathcal{O}\setminus\mathcal{O}_1,$$

$$\mathcal{L}_1=\bigcup_{i=1}^{7}\{\overline{Q_f}\mid f\in\mathcal{F}^+_{\gamma_i}\},\qquad \overline{\mathcal{L}_1}=\mathcal{L}\setminus\mathcal{L}_1.$$

From Lemma 2.5 we have $|\mathcal{O}_1|=7\cdot 6=42$, $|\mathcal{L}_1|=7\cdot 10=70$. In this way, for each of $\mathcal{C}_2,\dots,\mathcal{C}_8$, we give the other seven sets of ovoids which we denote $\mathcal{O}_2,\dots,\mathcal{O}_8$.

We let $\binom{\Omega}{k}$ denote the set of all k-subsets of a set Ω.

Lemma 3.4

(1) *Each triangle is contained in a unique ovoid of* $\mathcal{O}_1$.
(2) *Each double triangle is contained in a unique element of* $\mathcal{L}_1$.

Proof. We have

$$\{T\mid T\text{ is a triangle}\}\supseteq\bigcup_{O\in\mathcal{O}_1}\binom{O}{3}$$

and so equality holds by Lemmas 2.4 and 2.5, thus (1) follows. The proof of (2) is similar to that of (1). $\square$

Proposition 3.5 *$GL(V)$ acts transitively on $\{\mathcal{O}_1,\dots,\mathcal{O}_8\}$.*

Proof. It is enough to show that, for $i\in\{1,\dots,8\}$, there exists $\tau\in GL(V)$ such that $\tau\mathcal{O}_1=\mathcal{O}_i$. From Proposition 3.3 there exists $\sigma\in\Omega(\Lambda_2V)$ such that $\sigma C_1=C_i$. Therefore, take $\tau\in GL(V)$ such that $\Lambda_2\tau=\sigma$, put $\eta_j=\sigma(\xi_j)$ for each $j\in\{1,\dots,7\}$ and take $\delta_j\in\mathcal{C}_i$ corresponding to each η_j. Then

$$\begin{aligned}(\tau\cdot\gamma_j)(x,y) &= \gamma_j(\tau^{-1}(x),\tau^{-1}(y)) = \beta(\tau^{-1}(x)\wedge\tau^{-1}(y),\xi_j)\\ &= \beta((\Lambda_2\tau^{-1})(x\wedge y),\xi_j) = \beta(\sigma^{-1}(x\wedge y),\xi_j)\\ &= \beta(x\wedge y,\sigma(\xi_j)) = \beta(x\wedge y,\eta_j) = \delta_j(x,y)\end{aligned}$$

for all $x,y\in V$ and so $\tau\cdot\gamma_j=\delta_j$. Hence $\tau\mathcal{C}_1=\mathcal{C}_i$.

For $O\in\mathcal{O}_1$, there exist $f\in\mathcal{F}^-_{\gamma_j}$ and $j\in\{1,\dots,7\}$ such that $O\cup\{0\}=Q_f$. Then $\tau(O\cup\{0\})\subseteq Q_{\tau\cdot f}$ and the polar form of $\tau\cdot f$ is $\tau\cdot\gamma_j=\delta_j$. For distinct $x,y\in Q_{\tau\cdot f}\setminus\{0\}$, taking $x',y'\in V$ such that $x=\tau(x')$, $y=\tau(y')$, we have $f(x')=f(y')=0$ and $1=\gamma_j(x',y')=(\tau\cdot\gamma_j)(x,y)=\delta_j(x,y)$. Thus no two vectors of $Q_{\tau\cdot f}\setminus\{0\}$ are orthogonal with respect to δ_j and so we must have $\tau\cdot f\in\mathcal{F}^-_{\delta_j}$. This proves that $\tau\mathcal{O}_1\subseteq\mathcal{O}_i$, so equality holds, as required. $\square$

4. Extending $\boldsymbol{A}$ to an $S(4,\{5,6\},17)$ with 252 blocks

In this section, we first characterize $\mathcal{O}_1$ which covers the triangles of $PG(V)$ once each. As a corollary, $\boldsymbol{A}$ is uniquely extended to an $S(4,\{5,6\},17)$ with 252 blocks.

Lemma 4.1 *For distinct $i,j\in\{1,\dots,8\}$, $\mathcal{O}_i\cap\mathcal{O}_j$ contains exactly six ovoids of which any two ovoids meet in a unique point. Moreover, for all distinct $i,j,k\in\{1,\dots,8\}$, $|\mathcal{O}_i\cap\mathcal{O}_j\cap\mathcal{O}_k|=0$.*

Proof. Lemma 3.1(3) shows that two 7-cocliques C_i,C_j in Δ corresponding to $\mathcal{C}_i,\mathcal{C}_j$, respectively, meet in a unique non-singular vector, and so $|\mathcal{C}_i\cap\mathcal{C}_j|=1$. Therefore from Lemma 2.4 the first half of the lemma follows. Moreover, from Lemma 3.1(3) again, the latter half of the lemma holds. $\square$

By Proposition 3.5, each $\mathcal{O}_i$ is characterized in the following:

Theorem 4.2 *There are exactly eight members $\mathcal{S}$ of $\binom{\mathcal{O}}{42}$ satisfying the following condition:*

$$\textit{each triangle is contained in a unique ovoid of } \mathcal{S}. \qquad (*)$$

Proof. For distinct $p,q\in V\setminus\{0\}$, we define $\mathcal{O}(p,q)$ to be the set of ovoids containing $\{p,q\}$. Define the set of (T,O) where T is a triangle containing $\{p,q\}$, $O\in\mathcal{O}(p,q)$ and $T\subset O$, and counting the set in two ways, we have $|\mathcal{O}(p,q)|=16$. So the pair $(V\setminus\{0\},\mathcal{O})$ is a 2-$(15,5,16)$ design. Fix distinct

$p, q \in V \setminus \{0\}$. Applying the method of intersection triangles to this design (see [1, p. 21]), we have $\sharp\{B \in \mathcal{O} \mid B \cap O = \{p, q\}\} = 6$ for each $O \in \mathcal{O}(p, q)$. We define the set $\mathcal{N}_i$ of i-subsets of $\mathcal{O}$ in which the intersection of any two is equal to $\{p, q\}$ for each $i \in \{2, 4\}$. Then a counting argument shows that $|\mathcal{N}_2| = 16 \cdot 6/2 = 48$. For $\{O_1, O_2\} \in \mathcal{N}_2$, we define a_{ij} to be the number of ovoids O of $\mathcal{O}(p, q)$ satisfying $|O \cap O_1| = i$ and $|O \cap O_2| = j$ for $i, j \in \{2, 3, 5\}$. Then we have the following four equations:

$$
\begin{aligned}
&16 = |\mathcal{O}(p, q)| = a_{22} + a_{23} + a_{32} + a_{33} + 1 + 1, \\
&\sharp\big\{(r, O) \in (O_1 \cup O_2 \setminus \{p, q\}) \times \mathcal{O}(p, q) \mid r \in O\big\} \\
&\quad = 6 \cdot 4 = a_{23} + a_{32} + 2a_{33} + 3 + 3, \\
&\sharp\big\{(r, O) \in (O_1 \setminus \{p, q\}) \times \mathcal{O}(p, q) \mid r \in O\big\} \\
&\quad = 3 \cdot 4 = a_{32} + a_{33} + 3, \\
&\sharp\big\{(r, O) \in (O_2 \setminus \{p, q\}) \times \mathcal{O}(p, q) \mid r \in O\big\} \\
&\quad = 3 \cdot 4 = a_{23} + a_{33} + 3.
\end{aligned}
$$

Put $e_1 = p, e_2 = q$. We write O_1 as $\{e_1, e_2, e_3, e_4, e_1 + e_2 + e_3 + e_4\}$ for some $\{e_1, e_2, e_3, e_4\} \in \mathfrak{X}$ and temporarily write $1, 2, 3, 4, 12, 123, \ldots$ for $e_1, e_2, e_3, e_4, e_1 + e_2, e_1 + e_2 + e_3, \ldots$, respectively. In particular, we can enumerate the 6 ovoids of $\{B \in \mathcal{O}(1, 2) \mid B \cap O_1 = \{1, 2\}\}$ as follows:

$$
\begin{aligned}
&B_1 := \{1, 2\} \cup \{13, 124, 134\}, \quad && B_2 := \{1, 2\} \cup \{13, 24, 34\}, \\
&B_3 := \{1, 2\} \cup \{23, 14, 34\}, && B_4 := \{1, 2\} \cup \{23, 124, 234\}, \\
&B_5 := \{1, 2\} \cup \{123, 14, 134\}, && B_6 := \{1, 2\} \cup \{123, 24, 234\}.
\end{aligned}
$$

By suitably interchanging e_1, e_2, e_3 and e_4, we may assume that $O_2 = B_1$ or B_2.

(i) If $O_2 = B_1$, then we can enumerate all the ovoids of $\{O \in \mathcal{O}(1, 2) \mid |O_1 \cap O| = |O_2 \cap O| = 3\}$ as follows:

$$
\begin{aligned}
&\{1, 2\} \cup \{3, 124, 34\}, \quad && \{1, 2\} \cup \{3, 134, 24\}, \\
&\{1, 2\} \cup \{4, 13, 234\}, && \{1, 2\} \cup \{4, 134, 23\}, \\
&\{1, 2\} \cup \{1234, 13, 14\}, && \{1, 2\} \cup \{1234, 124, 123\}.
\end{aligned}
$$

(ii) If $O_2 = B_2$, then we can enumerate all the ovoids of $\{O \in \mathcal{O}(1,2) \mid |O_1 \cap O| = |O_2 \cap O| = 3\}$ as follows:

$$\begin{array}{ll} \{1,2\} \cup \{3,24,134\}, & \{1,2\} \cup \{3,34,124\}, \\ \{1,2\} \cup \{4,13,234\}, & \{1,2\} \cup \{4,34,123\}, \\ \{1,2\} \cup \{1234,13,14\}, & \{1,2\} \cup \{1234,24,23\}. \end{array}$$

Hence $a_{33} = 6, a_{23} = a_{32} = 3$ and $a_{22} = 2$. Therefore we can determine the unique element of $\mathcal{N}_4$ containing $\{O_1, O_2\}$, which is first row or second row of the following array (by adding $\{O_1, O_2\}$) according as $O_2 = B_1$ or B_2:

$$\boxed{\begin{array}{ll} \{1,2\} \cup \{23,14,34\} & \{1,2\} \cup \{123,24,234\} \\ \{1,2\} \cup \{23,124,234\} & \{1,2\} \cup \{123,14,134\} \end{array}}$$

Thus a counting argument shows that $|\mathcal{N}_4| = 48 \cdot 1/\binom{4}{2} = 8$.

First, it is immediate from Lemma 3.4(1) that each $\mathcal{O}_i$ satisfy the condition $(*)$. To prove the converse, let $\mathcal{S}$ be a set of 42 ovoids satisfying $(*)$. An elementary counting argument shows that, for distinct $r, s \in V \setminus \{0\}$, $\{r, s\}$ is in exactly four ovoids of $\mathcal{S}$. This implies that

$$\mathcal{N}_4 = \{\mathcal{O}_1 \cap \mathcal{O}(r,s), \ldots, \mathcal{O}_8 \cap \mathcal{O}(r,s)\}$$

for all $\{r, s\} \in \binom{V \setminus \{0\}}{2}$. Therefore there exists $i \in \{1, \ldots, 8\}$ such that $\mathcal{S} \cap \mathcal{O}(p,q) = \mathcal{O}_i \cap \mathcal{O}(p,q)$. To show that i is independent of $\{p, q\}$, for $\{r, s\} \in \binom{V \setminus \{0\}}{2}$, we take $j \in \{1, \ldots, 8\}$ such that $\mathcal{S} \cap \mathcal{O}(r,s) = \mathcal{O}_j \cap \mathcal{O}(r,s)$, and it is enough to show that $i = j$. Suppose that $i \neq j$. Then we will lead a contradiction. Since there exist two triangles T_1, T_2 such that $\{p,q\} \subset T_1, \{r,s\} \subset T_2, |T_1 \cap T_2| = 2$ and $T_1 \cap T_2 \notin \{\{p,q\},\{r,s\}\}$, we take $k \in \{1, \ldots, 8\}$ such that $\mathcal{S} \cap \mathcal{O}(T_1 \cap T_2) = \mathcal{O}_k \cap \mathcal{O}(T_1 \cap T_2)$. Let B_1 and B_2 be the ovoids in $\mathcal{S}$ containing T_1 and T_2, respectively. We note the following lemmas:

Lemma 4.3 *Let i, j be distinct elements of $\{1, \ldots, 8\}$. If $O \in \mathcal{O}_i \cap \mathcal{O}_j$ and $\{x, y\} \in \binom{O}{2}$, then*

(1) *There are exactly three ovoids of $\mathcal{O}_i$ which meet O in $\{x, y\}$.*
(2) *$\{B \in \mathcal{O} \mid B \cap O = \{x, y\}\}$ is the disjoint union of $\{B \in \mathcal{O}_i \mid B \cap O = \{x, y\}\}$ and $\{B \in \mathcal{O}_j \mid B \cap O = \{x, y\}\}$.*

Lemma 4.4 *All the elements of* $\{\mathcal{O}_i \cap \mathcal{O}_j \mid \{i,j\}$ *is a* 2*-subset of* $\{1,\ldots,8\}\}$ *partition* $\mathcal{O}$*.*

Indeed, Lemma 4.3 (1) follows from the fact that the pair $(V \setminus \{0\}, \mathcal{O}_i)$ is a 2-$(15,5,4)$ design. By Lemma 4.1, Lemma 4.3(2) and Lemma 4.4 follow.

We turn to the proof of Theorem 4.2. Suppose first that $k \in \{i,j\}$. By interchanging i and j, we may assume that $k = i$.

(i) Suppose that $B_1 = B_2$. There exists $t \in V \setminus B_1$ such that $T_3 := \{p,q,t\}$ is a triangle. Taking $B_3 \in \mathcal{S}$ containing T_3, by Lemma 4.3(2), we have $l \in \{1,\ldots,8\} \setminus \{i\}$ so that $B_3 \in \mathcal{O}_i \cap \mathcal{O}_l$, and $l \neq j$ by Lemma 4.1. By interchanging p and q, we may assume that $p \in T_1 \cap T_2$. Let

$$n_i := \sharp\{B \in (\mathcal{S} \cap \mathcal{O}(T_1 \cap T_2)) \setminus \{B_1\} \mid |B \cap B_3| = i\}, \quad 0 \leq i \leq 5.$$

Since $B_3 \notin \mathcal{S} \cap \mathcal{O}(T_1 \cap T_2)$, we obtain $i \leq 3$. Since $p \in B_3$, we have $n_0 = 0$, and the condition $(*)$ implies $n_3 = 0$. Therefore it follows that $n_1 \leq 1$ and $n_2 \geq 2$. Let B and C be two elements of $(\mathcal{S} \cap \mathcal{O}(T_1 \cap T_2)) \setminus \{B_1\}$ which meet B_3 in exactly two points. From Lemma 4.3(1) we have $B, C \in \mathcal{O}_i \cup \mathcal{O}_l$ and so we may assume that $B \in \mathcal{O}_i$ and $C \in \mathcal{O}_l$. Thus it follows that $C \in \mathcal{O}_i \cap \mathcal{O}_l$, which contradicts Lemma 4.1.

(ii) Suppose that $B_1 \neq B_2$. Taking $l \in \{1,\ldots,8\} \setminus \{i\}$ so that $B_1 \in \mathcal{O}_i \cup \mathcal{O}_l$, we have $l \neq j$. By interchanging r and s, we may assume that $r \in T_1 \cap T_2$. Then the argument similar to (i) shows that there are two elements of $(\mathcal{S} \cap \mathcal{O}(r,s)) \setminus \{B_2\}$ which meet B_1 in exactly two points, but one of these elements lies in $\mathcal{O}_i \cap \mathcal{O}_j$, which contradicts Lemma 4.1.

Suppose finally that $k \notin \{i,j\}$. Lemma 4.1 shows that $B_1 \neq B_2$. By interchanging r and s, we may assume that $r \in T_1 \cap T_2$. Similarly there are two elements of $(\mathcal{S} \cap \mathcal{O}(r,s)) \setminus \{B_2\}$ which meet B_1 in exactly two points, but one of these elements lies in $\mathcal{O}_k \cap \mathcal{O}_j$, which contradicts Lemma 4.1.

Therefore it follows that $i = j$ and $\mathcal{S} \cap \mathcal{O}(p,q) \subset \mathcal{O}_i$ for all the 2-subsets $\{p,q\}$ of $V \setminus \{0\}$, which implies $\mathcal{S} \subseteq \mathcal{O}_i$, so equality holds. This completes the proof. □

We can now obtain the main result of [6]. We define the pair $\boldsymbol{D} := (\mathcal{P}, \mathcal{B})$ as follows:

$\mathcal{P} = V \cup \{\infty_1\}$ (where ∞_1 is a new point not in V),

$\mathcal{B} = \{B \cup \{\infty_1\} \mid$ B is a block of $\boldsymbol{A}\} \cup \{B \cup \{0\} \mid B \in \mathcal{O}_1\} \cup \mathcal{L}_1$.

There are 140 blocks of size 5 and $112(=42+70)$ blocks of size 6. Since $\binom{17}{4}=140\cdot\binom{5}{4}+112\cdot\binom{6}{4}$, it is enough to show that each $X\in\binom{\mathcal{P}}{4}$ is in at least one block. We have

$$\mathfrak{X}\supseteq\bigcup_{O\in\mathcal{O}_1}\binom{O}{4}\cup\bigcup_{L\in\mathcal{L}_1}\{X\in\mathfrak{X}\mid X\subset L\},$$

and the size of the right side is $840(=42\cdot 5+70\cdot 9)$ from Lemmas 2.4 and 2.5, so equality holds. Therefore it is easily seen from Lemma 3.4 that $\boldsymbol{D}$ is an $S(4,\{5,6\},17)$ with 252 blocks.

Lemma 4.5

(1) *For $O\in\overline{\mathcal{O}_1}$ and $X\in\binom{O}{4}$, X is contained in a unique element of $\mathcal{L}_1$ and in exactly two elements of $\overline{\mathcal{L}_1}$.*

(2) *For $L\in\overline{\mathcal{L}_1}$ and $Y\in\binom{L}{5}$, there exists $X\in\mathfrak{X}$ contained in Y such that a unique ovoid containing X is contained in $\mathcal{O}_1$.*

Proof. (1) By Lemma 2.1, the unique block of $\boldsymbol{D}$ containing X is in $\mathcal{L}_1$, and the other two elements of $\mathcal{L}$ containing X are both in $\overline{\mathcal{L}_1}$.

(2) For distinct elements $X_1,X_2\in\mathfrak{X}$ contained in Y, the triangle $T:=X_1\cap X_2$ is in a unique ovoid $O_1\in\mathcal{O}_1$. Putting $T=\{e_1,e_2,e_3\}$ and $X_1=T\cup\{e_4\}$, we have $X_2=T\cup\{e_3+e_4\}$. By Lemma 2.1(6), O_1 is one of following four ovoids:

$$T\cup\{e_4,e_1+e_2+e_3+e_4\},\quad T\cup\{e_1+e_4,e_2+e_3+e_4\},$$
$$T\cup\{e_2+e_4,e_1+e_3+e_4\},\quad T\cup\{e_3+e_4,e_1+e_2+e_4\}.$$

We assume that $O_1=T\cup\{e_1+e_4,e_2+e_3+e_4\}$ or $T\cup\{e_2+e_4,e_1+e_3+e_4\}$ and will show that this leads to a contradiction. By interchanging e_1 and e_2, we may assume that $O_1=T\cup\{e_1+e_4,e_2+e_3+e_4\}$. Then the other three ovoids containing T are all in $\overline{\mathcal{O}_1}$. Applying (1) to $T\cup\{e_4,e_1+e_2+e_3+e_4\}$ and X_1, we have $L_1:=\{e_1,e_3,e_1+e_3\}\cup\{e_2,e_4,e_2+e_4\}\in\mathcal{L}_1$. Applying (1) to $T\cup\{e_3+e_4,e_1+e_2+e_4\}$ and X_2, we next have $L_2:=\{e_1,e_3+e_4,e_1+e_3+e_4\}\cup\{e_2,e_3,e_2+e_3\}\in\mathcal{L}_1$. We assume that the ovoid containing $X_3:=\{e_1,e_2,e_4,e_3+e_4\}(\subset Y)$ is in $\overline{\mathcal{O}_1}$. Then the other three elements of $\mathcal{L}$ containing X_3 are all in $\overline{\mathcal{L}_1}$ since $O_1\in\mathcal{O}_1$ and $L_1,L_2\in\mathcal{L}_1$, but this contradicts (1). Thus (2) follows. $\square$

We can now prove the uniqueness of an $S(4,\{5,6\},17)$ with 252 blocks of which the derived design at some point is $\boldsymbol{A}$.

Corollary 4.6 *$\boldsymbol{A}$ is uniquely extended to an $S(4,\{5,6\},17)$ with 252 blocks.*

Proof. Let $\boldsymbol{S}$ be an $S(4,\{5,6\},17)$ with 252 blocks of which the derived design at a new point ∞ is $\boldsymbol{A}$ and it is enough to show that $\boldsymbol{D}$ (described above) and $\boldsymbol{S}$ are isomorphic. Since $\lambda = 1$, for any triangle T, $T \cup \{0\}$ is in a unique block B and $\infty \notin B$, and each double triangle is in a unique block B and $|\{0,\infty\} \cap B| = 0$. We define two sets

$$\mathfrak{B} = \{B : \text{block of } \boldsymbol{S} \mid 0 \in B \text{ and } \infty \notin B\},$$

$$\mathfrak{C} = \{C : \text{block of } \boldsymbol{S} \mid |\{0,\infty\} \cap C| = 0\}.$$

For any $B \in \mathfrak{B}$, $B \setminus \{0\}$ must be in $\mathfrak{X}$ if $|B| = 5$ and in $\mathcal{O}$ if $|B| = 6$. For any $C \in \mathfrak{C}$, if $|C| = 5$ then we must have $C \in \mathcal{O}$ or $C = \{x,y,z,w,x+y\}$ for some $\{x,y,z,w\} \in \mathfrak{X}$. If $|C| = 6$ then we will show that $C \in \mathcal{L}$. Suppose first that C contains at least three lines and we take the three lines l,m,n in C, which are mutually meeting and $|l \cap m \cap n| = 0$ since $|C| = 6$. Then C contains an oval, a contradiction. Suppose next that C contains no line. For $p \in C$, from Lemma 2.3 $C \setminus \{p\}$ is an ovoid. Putting $C \setminus \{p\} = \{e_1,e_2,e_3,e_4,e_1+e_2+e_3+e_4\}$ for some $\{e_1,e_2,e_3,e_4\} \in \mathfrak{X}$, we have $p \in \{e_1+e_2+e_3, e_1+e_2+e_4, e_3+e_4\}$, but in any case C contains an oval, a contradiction. Suppose that C contains exactly one line and we take the line l in C. Put $H = \langle x,y,z\rangle \setminus \{0\}$ for the triangle $C \setminus l := \{x,y,z\}$. Since $|l \cap H| \in \{1,3\}$, if $|l \cap H| = 3$ then $l = \{x+y, y+z, z+x\}$, but C contains the line $\{x,y,x+y\}(\neq l)$, a contradiction. If $|l \cap H| = 1$ then it is easily seen that C contains at least two lines, a contradiction. Therefore C contains exactly two lines, which are disjoint since C contains no oval. Hence $C \in \mathcal{L}$.

Set

$$\mathfrak{F} = \{C \in \mathfrak{C} \mid |C| = 5 \text{ and } C \notin \mathcal{O}\}$$

and let

$$b = |\mathfrak{B}|, \quad c = |\mathfrak{C}|, \quad d = \sharp\{B \in \mathfrak{B} \mid |B| = 5\}, \quad e = |\mathfrak{C} \cap \mathcal{O}| \text{ and } f = |\mathfrak{F}|.$$

Then by counting arguments we have the following three equations:

$$\begin{aligned}
&\sharp\{(T,B)\in\{T\mid T \text{ is a triangle}\}\times\mathfrak{B}\mid T\cup\{0\}\subset B\}\\
&\quad = 420 = \binom{4}{3}d+\binom{5}{3}(b-d),\\
&\sharp\{(S,C)\in\{S\mid S \text{ is a double triangle}\}\times\mathfrak{C}\mid S\subset C\}\\
&\quad = 420 = 2f+6(c-e-f),\\
&\sharp\{(X,Y)\in\mathfrak{X}\times(\mathfrak{B}\cup\mathfrak{C})\mid X\subset Y\}\\
&\quad = 840 = d+\binom{5}{4}(b-d)+\binom{5}{4}e+3f+9(c-e-f).
\end{aligned}$$

Moreover we have $b+c=112$. Since the four equations yield

$$\begin{pmatrix} c\\ d\\ e\\ f\end{pmatrix} = \begin{pmatrix} 112\\ -70\\ -14\\ 84\end{pmatrix} + b\begin{pmatrix} -1\\ 5/3\\ 1/3\\ -2\end{pmatrix},$$

it follows that $b=42$, $e=f=0$, and so $c=70$, $d=0$.

By Proposition 3.5 and Theorem 4.2, there exists $\rho\in GL(V)$ such that $\{B\setminus\{0\}\mid B\in\mathfrak{B}\}=\rho\mathcal{O}_1$, and by Lemma 4.5(2) it follows that $\mathfrak{C}\subseteq\rho\mathcal{L}_1$, so equality holds. Thus the map $\rho^*: V\cup\{\infty_1\}\to V\cup\{\infty\}$ defined by $\infty_1\mapsto\infty$ and $x\mapsto\rho(x)$ is clearly an isomorphism. $\square$

Remark 4.7 We can see that $\boldsymbol{D}$ is not 3-wise balanced. In fact, each $\{x,y\}\in\binom{V\setminus\{0\}}{2}$ is in exactly 4 ovoids of $\mathcal{O}_1$. Therefore $\{x,y,0\}$ is in exactly $5(=1+4)$ blocks, whereas $\{x,y,\infty_1\}$ is in exactly $7(=1+6)$ blocks since $\{x,y\}$ is in exactly 6 ovals.

In Östergård and Pottonen [4], it has been shown that an $S(4,5,17)$ does not exist, so that $\boldsymbol{A}$ is not extendable, but the 3-$(16,4,2)$ structure $2.\boldsymbol{A}$ (in which each block of $\boldsymbol{A}$ is repeated 2 times) has a (usual) extension.

Corollary 4.8 $2.\boldsymbol{A}$ *is extendable* (*in the usual meaning*).

Proof. Set

$$\mathcal{O}_1' = \bigcup_{O \in \mathcal{O}_1} \binom{O}{4}$$

$$\mathcal{L}_1' = \bigcup_{L \in \mathcal{L}_1} \binom{L}{5}$$

and let $\mathcal{A}$ be the multi-set of $B \cup \{\infty\}$ for all the blocks B of $2.\boldsymbol{A}$, where ∞ is a new point not in V. We define the pair $(\mathcal{P}, \mathcal{B})$ as follows:

$$\mathcal{P} = V \cup \{\infty\},$$

$$\mathcal{B} = \mathcal{A} \cup \{X \cup \{0\} \mid X \in \mathcal{O}_1'\} \cup \mathcal{L}_1' \cup \mathcal{O}_1.$$

There are $2 \cdot 140 + 210 + 420 + 42 = 952$ blocks since $|\mathcal{O}_1'| = 42 \cdot 5 = 210$ and $|\mathcal{L}_1'| = 70 \cdot 6 = 420$. Since $\binom{17}{4} \cdot 2 = 952 \cdot \binom{5}{4}$, it is enough to show that each $S \in \binom{\mathcal{P}}{4}$ is in at least two blocks. If S contains 0 or ∞, then it is clear that S is in at least two blocks. Thus we may assume that $|\{0, \infty\} \cap S| = 0$ and S is not an oval. If S is a double triangle, then there is a unique element of $\mathcal{L}_1$ containing S, thus S is in at least two blocks. For $X \in \mathfrak{X}$, we denote by $\tilde{X}$ the unique ovoid containing X. Suppose that $S \in \mathfrak{X}$. If $\tilde{S} \in \mathcal{O}_1$, then S is in at least two blocks. If $\tilde{S} \in \overline{\mathcal{O}_1}$, then from Lemma 4.5(1) there exists $L \in \mathcal{L}_1$ containing S, from which S is in at least two blocks. This yields the result. $\square$

Acknowledgments The author is grateful to Professor M. Kitazume and Hiroyuki Nakasora for carefully reading the manuscript and pointing out many mistakes.

References

[1] Cameron P. J. and van Lint J. H., *Designs, Graphs, Codes and their links.* London Mathematical Society Student Text **22**, Cambridge Univ. Press, 1991.

[2] Dembowski P., *Finite Geometries.* Classics in Mathematics, Springer-Verlag, Berlin, 1997.

[3] Kramer E. S. and Mathon R. A., *Proper $S(t, K, v)$'s for $t \geq 3$, $v \leq 16$, $|K| > 1$ and their extensions.* J. Combin. Designs **3** (1995), 411–425.

[4] Östergård P. R. J. and Pottonen O., *There exists no Steiner system*

$S(4,5,17)$. J. Combin. Theory (A) **115** (2008), 1570–1573.

[5] Taylor D. E., *The Geometry of the Classical Groups.* Sigma Series in Pure Mathematics **9**, Heldermann Verlag, Berlin, 1992.

[6] Yucas J. L., *Extending* $AG(4,2)$ *to* $S(4,\{5,6\},17)$. J. Combin. Designs **7** (1999), 113–117.

Graduate School of Science
Chiba University
Chiba 263-8522, Japan
E-mail: inoue@graduate.chiba-u.jp