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Invariant measures for subshifts arising from substitutions of some primitive components

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Abstract. The notion of substitutions of some primitive components is introduced. A bilateral subshift arising from a substitution of some primitive components is decomposed into pairwise disjoint, locally compact, shift-invariant sets, on each of which an invariant Radon measure is unique up to scaling. In terms of eigenvalues of an incidence matrix associated with the substitution, it is completely characterized when the unique invariant measure is finite.

Key words: non-primitive substitution, subshift, invariant measure, ergodicity.

1. Introduction

It was shown by [6], [4] that any "aperiodic", stationary, properly ordered Bratteli diagram gives rise to a Bratteli-Vershik system conjugate to a bilateral subshift arising from an aperiodic, primitive substitution, and vice versa. In [1], this correspondence was extended by successfully removing the hypothesis of both simplicity for Bratteli diagrams and primitivity for substitutions. They showed that if B is a stationary, ordered Bratteli diagram which admits an aperiodic Vershik map λ_B acting on a perfect space X_B, then the following are equivalent:

- Bratteli-Vershik system (X_B, λ_B) is conjugate to a subshift arising from an aperiodic substitution;
- no restriction of λ_B to a minimal set is conjugate to an odometer.

They also showed a converse statement that given an aperiodic substitution with nesting property, there exists a stationary, ordered Bratteli diagram yielding a Bratteli-Vershik system conjugate to a subshift arising from the substitution. In contrast to [6], [4], it is remarkable that ordered Bratteli diagrams with more than one minimal or maximal paths play central roles in the above-mentioned correspondence [1]; see also [9].

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On the other hand, aiming at a similar result in the class of almost simple, ordered Bratteli diagrams [3], the second author [12] introduced the notion of almost primitivity for substitutions, and showed that an almost primitive substitution generates an almost minimal subshift [3] with a unique (up to scaling), nonatomic, invariant Radon measure. Before this work, as pointed out in [2], a concrete almost primitive substitution was studied in [5], which is the so-called Cantor substitution. By [7], [3], any almost minimal system is conjugate to the Vershik map arising from an almost simple, ordered Bratteli diagram. It is still an open question to characterize a class of almost simple, ordered Bratteli diagrams whose Vershik maps conjugate to subshifts arising from almost primitive substitutions. Actually, this question forces us to be in a quite different situation from [6], [4], [1]: there exists a class of *non-stationary*, almost simple, ordered Bratteli diagrams whose Vershik maps are conjugate to subshifts arising from almost primitive substitutions; see for details [12, Remark 5.5].

Applying the correspondence [1] mentioned above and exploiting stationary, ordered Bratteli diagrams, S. Bezuglyi, J. Kwiatkowski, K. Medynets and B. Solomyak [2] studied invariant measures for subshifts arising from aperiodic substitutions. Roughly speaking, one of their results showed the existence of a one-to-one correspondence between the set of ergodic, probability (resp. nonatomic, infinite) measures for the subshift X_σ arising from a given aperiodic substitution σ and the set of "distinguished" eigenvectors (resp. non-distinguished eigenvalues) of the incidence matrix M_σ of σ. One of the goals of this paper is to restructure this correspondence in the class of substitutions of *some primitive components* (Definition 2.1) without using any Bratteli diagrams. The class is so large that it includes all the primitive or almost primitive ones, and the so-called Chacon substitution as well. Some properties required for a substitution to be of some primitive components are stronger, but the other is weaker, than properties of substitutions studied in [2]. We will also show that a bilateral subshift X_σ arising from a given substitution σ of some primitive components is decomposed into finite number of pairwise disjoint, locally compact, shift-invariant sets X_i so that an invariant Radon measure on each X_i is unique up to scaling, and moreover, the orbit of any point in each X_i is dense in X_i. In terms of eigenvalues of M_σ, we will also describe the same criterion as [2] to determine when the unique invariant Radon measure is finite.

All the way to the end of this paper, we will not exploit any Bratteli diagrams but tools within the framework of subshifts. This standing position is quite different from [2]. A characterization (Lemma 5.4) when a locally compact minimal subshift over a finite alphabet has a unique (up to scaling) invariant Radon measure will help us prove Theorem 5.5. In Section 4, auxiliary substitutions developed by [11] play central roles when we estimate how fast the number of the occurrences of a letter in a k-word of a given substitution of some primitive components increases as k tends to infinity. The auxiliary substitutions also make it possible to calculate measures of cylinder sets with respect to invariant measures (Example 5.3).

2. Substitutions in question

We basically follow notation and terminology adopted in [4] concerning combinatorics on words. Let A be a finite alphabet with $\sharp A \geq 2$. Let A^+ denote the set of nonempty words over A. Set $A^* = A^+ \cup \{\Lambda\}$, where Λ is the empty word. We say that $u \in A^+$ *occurs* in $v \in A^+$, or u is a *factor* of v, if there exists an integer i with $1 \leq i \leq |v|$ such that $v_{[i,i+|u|)} := v_i v_{i+1} \dots v_{i+|u|-1} = u$, where $|v|$ denotes the length of v and v_n is the n-th letter of v. We refer to i as an *occurrence* of u in v. Given $u, v \in A^+$, we denote by $N(u,v)$ the number of the occurrences of u in v.

A *substitution* σ on A is a map from A to A^+. By concatenations of words, we may define powers $\sigma^k : A \to A^+$ of σ for $k \in \mathbb{N}$, and may enlarge the domain of the powers to A^+ or $A^{\mathbb{Z}}$. A subshift

$$X_\sigma = \left\{x = (x_i)_{i\in\mathbb{Z}} \in A^{\mathbb{Z}}; x_{[-i,i]} := x_{-i}x_{-i+1}\dots x_i \in \mathcal{L}(\sigma) \text{ for every } i \in \mathbb{N}\right\}$$

is called a *substitution dynamical system*, where

$$\mathcal{L}(\sigma) = \bigcup_{n\in\mathbb{N}, a\in A} \left\{w \in A^*; w \text{ is a factor of } \sigma^n(a)\right\}.$$

A word of the form $\sigma^n(a)$ is called an *n-word.* We set $\mathcal{L}_n(\sigma) = \{w \in \mathcal{L}(\sigma); |w| = n\}$ for $n \in \mathbb{N}$. We denote by T_σ the left shift on X_σ, and let $\mathrm{Orb}_{T_\sigma}(x) = \{{T_\sigma}^n x; n \in \mathbb{Z}\}$ for $x \in X_\sigma$. Given an infinite sequence x over A, we set $\mathcal{L}(x) = \{w \in A^*; w \text{ is a factor of } x\}$, and $\mathcal{L}(X) = \bigcup_{x\in X} \mathcal{L}(x)$ if $X \subset A^{\mathbb{Z}}$. Given $u, v \in A^*$, we denote by $[u.v]$ a *cylinder set* $\{x \in X_\sigma; x_{[-|u|,|v|)} = uv\}$. If $u = \Lambda$, then we use the notation $[v]$ in stead of $[\Lambda.v]$. Given $x \in X_\sigma$, let δ_x denote the point mass concentrated on x. A positive measure on a locally compact metric space is called a *Radon measure* if it is finite on any compact set.

The *incidence matrix* M_σ of σ is an $A \times A$ matrix whose (a, b)-entry is $N(b, \sigma(a))$. Putting a linear order on A, say $a_1 < a_2 < \cdots < a_n$, we also write $(M_\sigma)_{i,j}$ to indicate the (a_i, a_j)-entry $(M_\sigma)_{a_i,a_j}$. Let us recall some basic facts concerning square matrices. Let M be a nonnegative square matrix. The matrix M is said to be *primitive* if there exists $k \in \mathbb{N}$ such that $M^k > 0$, i.e. every entry of M^k is positive. We denote by $\mathrm{Sp}(M)$ the set of eigenvalues of M. If $\lambda \in \mathrm{Sp}(M)$ is such that $|\eta| < \lambda$ for any other $\eta \in \mathrm{Sp}(M)$, then we call λ a *dominant eigenvalue* of M. In this case,

$$\min_i \sum_j M_{i,j} \leq \lambda \leq \max_i \sum_j M_{i,j}.$$

Perron-Frobenius Theory guarantees that any primitive matrix M has a simple, dominant eigenvalue λ which admits a positive eigenvector. Then, letting α and β be positive, right and left eigenvectors of M corresponding to λ, respectively, with $\beta\alpha = 1$, it follows that $\lim_{k\to\infty} \lambda^{-k}(M^k)_{ij} = \alpha_i\beta_j$ for all possible i, j. See for details [8].

Definition 2.1 A substitution $\sigma : A \to A^+$ is said to be *of some primitive components* if there is a sequence $\emptyset \neq A_1 \subsetneqq A_2 \subsetneqq \cdots \subsetneqq A_{n-1} \subsetneqq A_n = A$ such that

(i) for every integer i with $1 \leq i \leq n$, it holds that $\sigma(a) \in {A_i}^+$ if $a \in A_i$;
(ii) there exists $k \in \mathbb{N}$ such that for any integer i with $1 \leq i \leq n$, any $a \in A_i \setminus A_{i-1}$ and any $b \in A_i$, the letter b occurs in $\sigma^k(a)$,

where $A_0 = \emptyset$. We also say that the substitution σ is *of n primitive components*. We call n the *number of primitive components* of σ, and denote it by n_σ.

If a given substitution σ is of some primitive components, then M_σ is written in a form:

$$M_\sigma = \begin{bmatrix} Q_1 & 0 & 0 & \cdots & 0 \\ R_{2,1} & Q_2 & 0 & \cdots & 0 \\ R_{3,1} & R_{3,2} & Q_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{n_\sigma,1} & R_{n_\sigma,2} & R_{n_\sigma,3} & \cdots & Q_{n_\sigma} \end{bmatrix} \tag{2.1}$$

so that all the entries on or below diagonal of some power of M_σ are positive. Conversely, this property implies that a given substitution is of some primitive components.

In the case $n_\sigma = 1$, a substitution σ becomes primitive. Then the subshift X_σ is minimal and uniquely ergodic; see for example [10], [11]. Throughout this paper, we assume $n_\sigma \geq 2$. Any almost primitive substitution is of two primitive components; see for details [12].

Definition 2.1 allows us to define a substitution $\sigma_i : A_i \to {A_i}^+$, $1 \leq i \leq n_\sigma$, by $\sigma_i(a) = \sigma(a)$ for $a \in A_i$. The substitution σ_i is of i primitive components. Since

$$\mathcal{L}(\sigma_1) \subset \mathcal{L}(\sigma_2) \subset \cdots \subset \mathcal{L}(\sigma_{n_\sigma - 1}) \subset \mathcal{L}(\sigma_{n_\sigma}) = \mathcal{L}(\sigma),$$

we have

$$X_{\sigma_1} \subset X_{\sigma_2} \subset \cdots \subset X_{\sigma_{n_\sigma - 1}} \subset X_{\sigma_{n_\sigma}} = X_\sigma,$$

which are all T_σ-invariant closed sets. All of $X_{\sigma_2}, X_{\sigma_3}, \ldots, X_\sigma$ are always nonempty. It holds that $X_{\sigma_1} = \emptyset$ if and only if A_1 is a singleton and $\sigma(s) = s$, where $A_1 = \{s\}$.

The class $\mathcal{S}$ of substitutions of some primitive components is different from the class $\mathcal{T}$ of substitutions studied in [2]. A substitution $\sigma : A \to A^+$ belongs to $\mathcal{T}$ if and only if the following conditions are satisfied:

(1) $\lim_{n\to\infty} |\sigma^n(a)| = \infty$ for any $a \in A$;
(2) σ is aperiodic, that is, X_σ has no periodic points of T_σ;
(3) M_σ is written in a form:

$$M_\sigma = \begin{bmatrix} Q_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & Q_2 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Q_s & 0 & \cdots & 0 \\ R_{s+1,1} & R_{s+1,2} & \cdots & R_{s+1,s} & Q_{s+1} & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ R_{m,1} & R_{m,2} & \cdots & R_{m,s} & R_{m,s+1} & \cdots & Q_m \end{bmatrix} \tag{2.2}$$

so that

(a) for every integer i with $1 \leq i \leq m$, Q_i is a primitive matrix if it is nonzero;
(b) for every integer i with $s < i \leq m$, there exists an integer j with $1 \leq j < i$ such that $R_{i,j} \neq 0$.

Notice that no inclusion relations hold between $\mathcal{S}$ and $\mathcal{T}$. Neither (1) nor (2) is required for a substitution to be of some primitive components. However, the irreducible properties of incidence matrices required in (3) are not as rigid as those required for substitutions of some primitive components.

The following is a key lemma to investigate recurrence property and invariant sets for X_σ.

Lemma 2.2 *Given an integer i with $1 < i \leq n_\sigma$, there exist $a \in A_{i-1}$, $b \in A_i \setminus A_{i-1}$, $k \in \mathbb{N}$, $u \in {A_{i-1}}^*$ and $v \in {A_i}^*$ such that at least one of the following holds*:

(i) $ab \in \mathcal{L}(\sigma_i)$ *and* $\sigma^k(ab) = uabv$;
(ii) $ba \in \mathcal{L}(\sigma_i)$ *and* $\sigma^k(ba) = vbau$.

Proof. Put $r = \sharp A_i$. Find $a_0 \in A_{i-1}$ and $b_0 \in A_i \setminus A_{i-1}$ such that $a_0 b_0 \in \mathcal{L}(\sigma_i)$ or $b_0 a_0 \in \mathcal{L}(\sigma_i)$. It is enough to consider only the case $a_0 b_0 \in \mathcal{L}(\sigma_i)$. Let $1 \leq m_j < |\sigma^j(b_0)|$ be such that $\sigma^j(b_0)_{[1,m_j)} \in {A_{i-1}}^*$ and $b_j := \sigma^j(b_0)_{m_j} \in A_i \setminus A_{i-1}$. Put $a_j = \sigma^j(a_0 b_0)_{|\sigma^j(a_0)|+m_j-1}$. Since $a_{j_1} b_{j_1} = a_{j_2} b_{j_2}$ for some $j_2 > j_1 \geq 0$, (i) holds with $a = a_{j_1}$, $b = b_{j_1}$ and $k = j_2 - j_1$. $\square$

We consider mainly the case where Lemma 2.2 (i) holds, because results below would be verified by means of symmetric arguments also for substitutions satisfying (ii) of the lemma. Since $X_{\sigma^k} = X_\sigma$ for any $k \in \mathbb{N}$, we may assume Lemma 2.2 (i) with $k = 1$.

3. Recurrence property of $\boldsymbol{\sigma}$

Throughout this section, we let a, b, i, u and v be as in Lemma 2.2 (i). We are concerned with structure of $X_{\sigma_i} \setminus X_{\sigma_{i-1}}$. Consider the case $u = \Lambda$. Then $\sigma(a) = a$, and hence $\sigma(b) = bv$, $v \neq \Lambda$, $A_{i-1} = \{a\}$, $i = 2$ and $X_{\sigma_1} = \emptyset$.

Lemma 3.1 ([12, Lemma 2.3]) *Let $\tau : B \to B^+$ be a substitution such that $\tau(s) = s$ for some $s \in B$. Then the following are equivalent*:

(i) $s^p \in \mathcal{L}(\tau)$ *for any* $p \in \mathbb{N}$;

(ii) *there exist $c \in B \setminus \{s\}$, $k, l \in \mathbb{N}$ and $w \in B^*$ such that $\tau^k(c) = s^l cw$ or $\tau^k(c) = wcs^l$.*

Assume Lemma 3.1 (ii) with $\tau = \sigma_2$. Then $B = A_2$ and $s = a$. If $w = \Lambda$, then $c = b$ and $X_{\sigma_2} = \{a^\infty\}$. If $w \neq \Lambda$, then σ_2 is almost primitive, so that X_{σ_2} is almost minimal [12, Theorem 3.8].

Assume the existence of $p_1 \in \mathbb{N}$ such that $a^p \notin \mathcal{L}(\sigma_2)$ if $p \geq p_1$. Then a letter of $A_2 \setminus A_1$ occurs in v, so that b occurs infinitely many times in a fixed point $\omega^+ := bv\sigma(v)\sigma^2(v)\sigma^3(v)\cdots \in {A_2}^{\mathbb{N}}$ of σ. This implies that there exists a periodic point $\omega \in X_{\sigma_2}$ of σ such that $\omega_{[0,\infty)} = \omega^+$. It follows therefore that $\mathcal{L}(X_{\sigma_2}) = \mathcal{L}(\omega)$ and hence $X_{\sigma_2} = \overline{\mathrm{Orb}_{T_\sigma}(\omega)}$.

Proposition 3.2 *X_{σ_2} is minimal and uniquely ergodic.*

Proof. Let $w \in \mathcal{L}(\omega)$. We may assume $\sigma(\omega) = \omega$. Take $k \in \mathbb{N}$ so that w occurs in $\sigma^k(c)$ for any $c \in A_2 \setminus A_1$. Since $\omega = \ldots \sigma^k(\omega_{-2})\sigma^k(\omega_{-1}).\sigma^k(\omega_0)\sigma^k(\omega_1)\ldots$, w occurs in any factor of ω whose length is $2\max_{c\in A_2\setminus A_1} |\sigma^k(c)| + p_1$. This means the minimality of X_{σ_2}, since w occurs infinitely often in ω with a bounded gap.

The unique ergodicity will be proved below by using Theorem 5.5. □

The Chacon substitution: $a \mapsto a$, $b \mapsto bbab$ satisfies the hypothesis of Proposition 3.2. The subshift X_{σ_2} may be the orbit of a shift-periodic point. If $A_2 = \{a, b\}$ and σ_2 is defined by $a \mapsto a$ and $b \mapsto bab$, then $X_{\sigma_2} = \{(ab)^\infty.(ab)^\infty, (ba)^\infty.(ba)^\infty\}$.

We assume $u \neq \Lambda$ until Proposition 3.13. Consider the case $v = \Lambda$. Denoting ua by u afresh, we may write $\sigma(b) = ub$. Observe $A_i \setminus A_{i-1} = \{b\}$. In view of a configuration of words:

$$\begin{aligned}
\sigma(b) &= & ub \\
\sigma^2(b) &= & \sigma(u)ub \\
\sigma^3(b) &= & \sigma^2(u)\sigma(u)ub \\
&\vdots & \vdots
\end{aligned}$$

we define a fixed point $\omega \in {A_i}^{-\mathbb{N}}$ of σ by $\omega = \ldots\sigma^5(u)\sigma^4(u)\sigma^3(u)\sigma^2(u)\sigma(u)ub$. Observe $\mathcal{L}(\omega) = \mathcal{L}(\sigma_i)$.

Assume $u = a^{|u|}$. Then $A_{i-1} = \{a\}$, so that $i = 2$. If $\sigma(a) = a$, then $X_{\sigma_2} = \{a^\infty\}$ and $X_{\sigma_1} = \emptyset$. If $\sigma(a) = a^p$ with $p \geq 2$, then $X_{\sigma_2} = X_{\sigma_1} =$

$\{a^\infty\}$. Throughout the remainder of this section, we assume $u \neq a^{|u|}$. Then one of the following holds:

- $i = 2$ and $\sharp A_{i-1} \geq 2$;
- $i \geq 3$.

Lemma 3.3 *Any word in $\mathcal{L}(X_{\sigma_i})$ occurs infinitely often in ω. Consequently, $X_{\sigma_i} \subset {A_{i-1}}^{\mathbb{Z}}$.*

Proof. Assume that some $v \in \mathcal{L}(X_{\sigma_i})$ occurs only finitely many times in ω. Take $L \in \mathbb{N}$ so that v does not occur in $\omega_{(-\infty,-L)}$. It follows that if $vw \in \mathcal{L}(X_{\sigma_i})$ then $|w| \leq L$, which is a contradiction. □

If $\{u_n\}_{n\in\mathbb{N}}, \{v_n\}_{n\in\mathbb{N}} \subset A^+$ satisfy the properties:

- u_n is a suffix of u_{n+1} for each $n \in \mathbb{N}$;
- v_n is a prefix of v_{n+1} for each $n \in \mathbb{N}$;
- $\lim_{n\to\infty} |u_n| = \lim_{n\to\infty} |v_n| = \infty$,

we denote by $\lim_{n\to\infty} u_n.v_n$ a point $x \in A^{\mathbb{Z}}$ defined by $x_{[-|u_n|,|v_n|)} = u_n v_n$ for each $n \in \mathbb{N}$.

Proposition 3.4 *Let $x \in X_{\sigma_i}$. Then the following are equivalent*:

(i) *$x \in X_{\sigma_i} \setminus X_{\sigma_{i-1}}$;*
(ii) *x belongs to the orbit of a periodic point $y \in X_{\sigma_i} \setminus X_{\sigma_{i-1}}$ of σ, which is aperiodic under T_σ, such that for some $n \in \mathbb{N}$, $y_{[-n,n)} \notin \mathcal{L}(X_{\sigma_{i-1}})$ and $y_{(-\infty,-n-1]}, y_{[n,\infty)} \in \mathcal{L}(X_{\sigma_{i-1}})$.*

The point y would occur in one of the following fashions. If $\lim_{n\to\infty} |\sigma^n(c)| = \infty$ for any $c \in A$, then there exist $\gamma, \delta \in A_{i-1}$ and $q \in \mathbb{N}$ such that

- *$\gamma\delta \in \mathcal{L}(X_{\sigma_i}) \setminus \mathcal{L}(X_{\sigma_{i-1}})$;*
- *$\sigma^{qj}(\gamma)_{|\sigma^{qj}(\gamma)|} = \gamma$ and $\sigma^{qj}(\delta)_1 = \delta$ for any $j \in \mathbb{N}$;*
- *$y = \lim_{j\to\infty} \sigma^{qj}(\gamma).\sigma^{qj}(\delta)$.*

If A_1 is a singleton, say $\{s\}$, then there exist $\gamma, \delta \in A_{i-1} \setminus A_1$, $p \in \mathbb{N} \cup \{0\}$ and $r \in \mathbb{N}$ such that

- *$\sigma^{rj}(\delta)_{|\sigma^{rj}(\delta)|} = \delta$ and $\sigma^{rj}(\gamma)_1 = \gamma$ for any $j \in \mathbb{N}$;*
- *y is written in one of the following forms*:

$$y = \lim_{j\to\infty} s^j.\sigma^{rj}(\gamma), \quad y = \lim_{j\to\infty} \sigma^{rj}(\delta).s^j \text{ and } y = \lim_{j\to\infty} \sigma^{rj}(\delta).s^p\sigma^{rj}(\gamma).$$

Proof. Assuming (i), we see (ii) in each of the cases:

(A) $\lim_{n\to\infty} |\sigma^n(c)| = \infty$ for all $c \in A_1$;
(B) A_1 is a singleton, say $A_1 = \{s\}$, and $\sigma(s) = s$.

Case (A). Take strictly increasing sequences $\{h_j\}_{j\in\mathbb{N}}, \{n_j\}_{j\in\mathbb{N}} \subset \mathbb{N}$ so that for each $j \in \mathbb{N}$, $x_{[-h_j,h_j)} \notin \mathcal{L}(X_{\sigma_{i-1}})$ and $\min_{c\in A_{i-1}} |\sigma^{n_j}(c)| \geq 2h_j$. Set

$$K_j = \big\{k \in -\mathbb{N}; x_{[-h_j,h_j)} \text{ occurs in } \sigma^{n_j}(\omega_{k-1})\sigma^{n_j}(\omega_k)\big\}.$$

Choose $c, d \in A_{i-1}$ so that there exist infinitely many j's such that $\omega_{k_j-1}\omega_{k_j} = cd$ for some $k_j \in K_j$. By replacing $\{n_j\}_{j\in\mathbb{N}}$ with its appropriate subsequence, we may assume that for every $j \in \mathbb{N}$, $x_{[-h_j,h_j)}$ is a factor of $\sigma^{n_j}(c)\sigma^{n_j}(d)$, and $\sigma^{n_j}(c)_{|\sigma^{n_j}(c)|}$ and $\sigma^{n_j}(d)_1$ are constant, say γ and δ, respectively. We may assume furthermore that $n_{j+1} - n_j$ is constant, say q, which might be the least common multiple of $\min\{n \in \mathbb{N}; \sigma^n(\gamma)_{|\sigma^n(\gamma)|} = \gamma\}$ and $\min\{n \in \mathbb{N}; \sigma^n(\delta)_1 = \delta\}$. There exists a sequence $\{w_j \in {A_{i-1}}^+; w_j \text{ is a factor of } \sigma^{qj}(\gamma\delta)\}_{j\in\mathbb{N}}$ which approximates x arbitrarily close, because (A) is assumed. Since $x_{[-h_j,h_j)} \notin \mathcal{L}(X_{\sigma_{i-1}})$, $\gamma\delta \notin \mathcal{L}(X_{\sigma_{i-1}})$ and each factor $x_{[-h_j,h_j)}$ of $\sigma^{qj}(\gamma)\sigma^{qj}(\delta)$ necessarily contains $\gamma\delta$ as a factor. Observe that $\gamma\delta$ occurs just once in $x_{[-h_j,h_j)}$, which would imply the shift-aperiodicity of x. Then, ${T_\sigma}^k x = \lim_{j\to\infty} \sigma^{qj}(\gamma).\sigma^{qj}(\delta)$ for some $k \in \mathbb{Z}$, which is a periodic point of σ.

Case (B). We have $i \geq 3$. By Lemma 3.1, $s^\infty \notin X_{\sigma_i} \setminus X_{\sigma_{i-1}}$. Take strictly increasing sequences $\{h_j\}_j, \{n_j\}_j \subset \mathbb{N}$ such that for every $j \in \mathbb{N}$, $x_{[-h_j,h_j)} \notin \mathcal{L}(X_{\sigma_{i-1}})$ and $\min_{c\in A_{i-1}\setminus A_1} |\sigma^{n_j}(c)| \geq 2h_j$. For each $j \in \mathbb{N}$, there exist $k_j \in -\mathbb{N}$ and $p_j, q_j \geq 0$ such that

- $x_{[-h_j,h_j)}$ occurs in $\sigma^{n_j}(\omega_{[k_j-q_j,k_j+p_j+1]})$;
- $\omega_{k_j}, \omega_{k_j+p_j+1} \in A_{i-1} \setminus A_1$;
- $\omega_{[k_j-q_j,k_j)}$ and $\omega_{[k_j+1,k_j+p_j]}$ are powers of s.

Similar arguments to those in Case (A) may allow us to assume the existence of $w, w' \in \mathcal{L}(\sigma_{i-1})$, $\delta, \gamma, \epsilon \in A_{i-1} \setminus A_1$ and $r \in \mathbb{N}$ such that

- for each $j \in \mathbb{N}$, $x_{[-h_j,h_j)}$ is a factor of $s^{q_j}\sigma^{rj}(w)s^{p_j}\sigma^{rj}(w')$;
- for each $j \geq 0$, the first and the last letters of $A_{i-1} \setminus A_1$ to occur in $\sigma^{rj}(w)$ are δ and γ, respectively;
- for each $j \geq 0$, the first letter of $A_{i-1} \setminus A_1$ to occur in $\sigma^{rj}(w')$ is ϵ.

Let m_j be an occurrence of $x_{[-h_j,h_j)}$ in $s^{q_j}\sigma^{rj}(w)s^{p_j}\sigma^{rj}(w')$. Put $m'_j = m_j + 2h_j - 1$. It might be sufficient to consider the following cases:

(I) $\sharp\{j \in \mathbb{N}; 1 \le m_j \le q_j\} = \infty$;
(II) $\sharp\{j \in \mathbb{N}; 0 < m_j - q_j \le |\sigma^{rj}(w)|\} = \infty$.

Let us first consider Case (I). Let $l_j = \max\{l \ge 0; s^l$ is a prefix of $\sigma^{rj}(w')\}$. Assume $\sharp\{j \in \mathbb{N}; m'_j \le q_j + |\sigma^{rj}(w)| + p_j + l_j\} = \infty$. Since $x \notin X_{\sigma_{i-1}} \cup \{s^\infty\}$, there exists $\{j_k\}_k \subset \mathbb{N}$ such that $\lim_{k\to\infty}(q_{j_k} - m_{j_k}) = \infty$ or $\lim_{k\to\infty}(m'_{j_k} - q_{j_k} - |\sigma^{rj_k}(w)|) = \infty$. Hence, we may assume that either for every k, the first letter of $\sigma^{rj_k}(w)$ is γ, or for every k, the last letter of $\sigma^{rj_k}(w)$ is δ. Then, one of the following holds:

- ${T_\sigma}^k x = \lim_{j\to\infty} s^j.\sigma^{rj}(\gamma)$ for some $k \in \mathbb{Z}$, and $s^p\gamma \notin \mathcal{L}(X_{\sigma_{i-1}})$ for some $p \in \mathbb{N}$;
- ${T_\sigma}^k x = \lim_{j\to\infty} \sigma^{rj}(\delta).s^j$ for some $k \in \mathbb{Z}$, and $\delta s^p \notin \mathcal{L}(X_{\sigma_{i-1}})$ for some $p \in \mathbb{N}$.

This shows the conclusion.

Let us assume $\sharp\{j \in \mathbb{N}; m'_j > p_j + |\sigma^{rj}(w)| + q_j + l_j\} = \infty$. If for any $p \in \mathbb{N}$, $\delta s^p \epsilon$ occurs in x, then each of them occurs infinitely many times in ω. However, it is impossible, because the last (resp. first) letter of $A_{i-1} \setminus A_1$ to occur in $\sigma^{rj}(\delta)$ (resp. $\sigma^{rj}(\epsilon)$) is δ (resp. ϵ). Hence, $\{p_j\}_j$ is bounded, and the last (resp. first) letter of $\sigma^{rj}(\delta)$ (resp. $\sigma^{rj}(\epsilon)$) is δ (resp. ϵ). Then, ${T_\sigma}^k x = \lim_{j\to\infty} \sigma^{rj}(\gamma).\sigma^{rj}(s^p).\sigma^{rj}(\delta)$ for some $k \in \mathbb{Z}$ and some $p \in \mathbb{N}$, and we have $\gamma s^p \delta \notin \mathcal{L}(X_{\sigma_{i-1}})$. The same argument works also in Case (II). This completes the proof. □

The following is an immediate consequence of Proposition 3.4.

Corollary 3.5 *There is a possibly empty set* $\{x_j \in X_{\sigma_i} \backslash X_{\sigma_{i-1}}; 1 \le j \le N\}$ *of periodic points of* σ *such that*

(i) $\overline{\mathrm{Orb}_{T_\sigma}(x_j)} = X_{\sigma_{i-1}} \cup \mathrm{Orb}_{T_\sigma}(x_j)$ (*a disjoint union*) *for any integer* j *with* $1 \le j \le N$;
(ii) $X_{\sigma_i} \setminus X_{\sigma_{i-1}} = \bigcup_{j=1}^N \mathrm{Orb}_{T_\sigma}(x_j)$ (*a disjoint union*).

Example 3.6 We shall see substitutions satisfying the hypothesis of Proposition 3.4.

(i) Set $A = \{a, b, c\}$. Let $w \in \{a, b\}^+$. Define $\sigma : A \to A^+$ by $a \mapsto ab$, $b \mapsto a$, $c \mapsto wc$. Since $\{aa, ab, ba\} \subset \mathcal{L}(X_{\sigma_1})$ and $bb \notin \mathcal{L}(\sigma_2)$, we have

$X_{\sigma_2} \setminus X_{\sigma_1} = \emptyset$.

(ii) Set $A = \{a, b, c, d\}$. Define $\sigma : A \to A^+$ by $a \mapsto abca$, $b \mapsto bacb$, $c \mapsto cbac$, $d \mapsto abbcad$. Since $\{aa, bb\} \subset \mathcal{L}(\sigma_2) \setminus \mathcal{L}(X_{\sigma_1})$,

$$X_{\sigma_2} \setminus X_{\sigma_1} = \mathrm{Orb}_{T_\sigma}\Big(\lim_{n\to\infty} \sigma^n(a).\sigma^n(a)\Big) \cup \mathrm{Orb}_{T_\sigma}\Big(\lim_{n\to\infty} \sigma^n(b).\sigma^n(b)\Big).$$

(iii) Set $A = \{a, b, c, d, e\}$. Define $\sigma : A \to A^+$ by $a \mapsto a$, $b \mapsto cba$, $c \mapsto cbc$, $d \mapsto dc$, $e \mapsto bde$. It follows that $X_{\sigma_1} = \emptyset$, X_{σ_2} is almost minimal with a unique fixed point a^∞,

$$X_{\sigma_3} \setminus X_{\sigma_2} = \mathrm{Orb}_{T_\sigma}\Big(\lim_{n\to\infty} \sigma^n(c).\sigma^n(c)\Big) \text{ and}$$

$$X_{\sigma_4} \setminus X_{\sigma_3} = \mathrm{Orb}_{T_\sigma}\Big(\lim_{n\to\infty} a^n.\sigma^n(d)\Big).$$

We assume $v \neq \Lambda$ until Proposition 3.13. In view of a configuration of words:

$$\begin{aligned}
\sigma(ab) &= && uabv \\
\sigma^2(ab) &= && \sigma(u)uabv\sigma(v) \\
\sigma^3(ab) &= && \sigma^2(u)\sigma(u)uabv\sigma(v)\sigma^2(v) \\
&\vdots && \vdots
\end{aligned}$$

we define a point $\omega \in X_{\sigma_i} \setminus X_{\sigma_{i-1}}$ by

$$\omega = \ldots \sigma^4(u)\sigma^3(u)\sigma^2(u)\sigma(u)ua.bv\sigma(v)\sigma^2(v)\sigma^3(v)\sigma^4(v)\ldots$$

Following [12, Definition 2.5], we make a definition:

Definition 3.7 We call the point ω a *quasi-fixed point* of the substitution σ. We refer to a quasi-fixed point of a power σ^k as a *quasi-periodic point* of σ. A quasi-periodic point $x \in X_{\sigma_i} \setminus X_{\sigma_{i-1}}$ is said to be of a *primitive type* if it holds that $x_k \in A_i \setminus A_{i-1} \Leftrightarrow k = 0$.

It follows from the construction of ω that

(i) for every $k \in \mathbb{N}$, there exists an integer l_k with $0 \leq l_k < |\sigma^k(b)|$ such that ${T_\sigma}^{l_k}\sigma^k(\omega) = \omega$;

(ii) $\mathcal{L}(\omega) = \mathcal{L}(\sigma_i) = \mathcal{L}(X_{\sigma_i})$;

(iii) $\overline{\mathrm{Orb}_{T_\sigma}(\omega)} = X_{\sigma_i}$.

Recall that $x \in A^{\mathbb{Z}}$ is said to be *positively recurrent* if for every $n \in \mathbb{N}$, there is $i \in \mathbb{N}$ such that $x_{[i-n,i+n)} = x_{[-n,n)}$.

Lemma 3.8

(i) *ω is aperiodic under T_σ.*

(ii) *ω is positively recurrent if and only if $v \in {A_i}^+ \setminus {A_{i-1}}^+$.*

(iii) *Put $K_n = \{k \in \mathbb{Z}; \omega_{[k,k+n)} \in {A_{i-1}}^+\}$ for $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $\{l \le k < l+m; k \in K_n\} \neq \emptyset$ for any $l \in \mathbb{Z}$.*

Proof. Since $\omega_{(-\infty,-1]} \in {A_{i-1}}^{-\mathbb{N}}$ and $\omega_0 \notin A_{i-1}$, ω is aperiodic under T_σ.

If $v \in {A_{i-1}}^+$, then ω is not positively recurrent, because $\omega_n \neq \omega_0$ for every $n > 0$. Suppose $v_j \in A_i \setminus A_{i-1}$ with $1 \le j \le |v|$. Take $k \in \mathbb{N}$ so that for any $c \in A_i \setminus A_{i-1}$, ab occurs in $\sigma^k(c)$. Since for every $n \in \mathbb{N}$ and any $c \in A_i \setminus A_{i-1}$, $\sigma^n(ab)$ occurs in $\sigma^{n+k}(c)$, ω is positively recurrent.

If $v \in {A_{i-1}}^+$, then (iii) is trivial. Assume $v \in {A_i}^+ \setminus {A_{i-1}}^+$. Given $n \in \mathbb{N}$, choose $p \in \mathbb{N}$ so that any word belonging to $\mathcal{L}_n(\sigma_i)$ occurs in $\sigma^p(c)$ for any $c \in A_i \setminus A_{i-1}$. Then, (iii) holds with $m = 2\max\{\max_{c \in A_i \setminus A_{i-1}} |\sigma^p(c)|, n\}$. □

If $u = a^{|u|}$, then $\sigma(a) = a^p$ for some $p \in \mathbb{N}$, which forces that $i = 2$, $A_1 = \{a\}$ and σ_2 is almost primitive. From now on, we assume $u \neq a^{|u|}$.

Consider the case $v \in {A_{i-1}}^+$. It follows that $A_i \setminus A_{i-1} = \{b\} = \{\omega_0\}$, ω is of a primitive type, and $\overline{\mathrm{Orb}_{T_\sigma}(\omega)} = X_{\sigma_{i-1}} \cup \mathrm{Orb}_{T_\sigma}(\omega)$ (a disjoint union).

Proposition 3.9 *There is a possibly empty set $\{x_j \in (X_{\sigma_i} \setminus X_{\sigma_{i-1}}) \cap {A_{i-1}}^{\mathbb{Z}}; 1 \le j \le N\}$ of periodic points of the substitution σ such that*

(i) $X_{\sigma_i} \setminus X_{\sigma_{i-1}} = \mathrm{Orb}_{T_\sigma}(\omega) \cup \bigcup_{j=1}^{N} \mathrm{Orb}_{T_\sigma}(x_j)$ *(a disjoint union)*;

(ii) *if x_j is periodic under T_σ, then A_1 is a singleton, say $\{s\}$, and $x_j = s^\infty$;*

(iii) *if x_j is aperiodic under T_σ, then $\overline{\mathrm{Orb}_{T_\sigma}(x_j)} = X_{\sigma_{i-1}} \cup \mathrm{Orb}_{T_\sigma}(x_j)$.*

Proof. Assume $x \in X_{\sigma_i} \setminus X_{\sigma_{i-1}}$. If $x_k \in A_i \setminus A_{i-1}$ for some $k \in \mathbb{Z}$, then $x = {T_\sigma}^k\omega$. If $x \in {A_{i-1}}^{\mathbb{Z}}$, then arguments in the proof of Proposition 3.4 work. □

Example 3.10 We shall see substitutions satisfying the hypothesis of Proposition 3.9.

(i) Set $A = \{a, b, c, d\}$. Define $\sigma : A \to A^+$ by $a \mapsto abca$, $b \mapsto bacb$, $c \mapsto cbac$, $d \mapsto abadcac$. It follows that $\{aa, cc\} \subset \mathcal{L}(\sigma_2) \setminus \mathcal{L}(\sigma_1)$. Then,

$$X_{\sigma_2} \setminus X_{\sigma_1} = \cup \operatorname{Orb}_{T_\sigma}(\omega) \cup \bigcup_{j=1}^{2} \operatorname{Orb}_{T_\sigma}(x_j),$$

where $x_1 = \lim_{n\to\infty} \sigma^n(a).\sigma^n(a)$ and $x_2 = \lim_{n\to\infty} \sigma^n(c).\sigma^n(c)$.

(ii) Set $A = \{a, b, c, d\}$. Define $\sigma : A \to A^+$ by $a \mapsto ab$, $b \mapsto ab$, $c \mapsto acb$, $d \mapsto cdc$. Then, $X_{\sigma_1} = \operatorname{Orb}_{T_\sigma}(x) = \{x, T_\sigma x\}$, $X_{\sigma_2} \setminus X_{\sigma_1} = \operatorname{Orb}_{T_\sigma}(\omega)$ and $X_{\sigma_3} \setminus X_{\sigma_2} = \operatorname{Orb}_{T_\sigma}(\omega')$, where $x = (bc)^\infty.(bc)^\infty$, $\omega = \dots\sigma^2(a)\sigma(a)a.cb\sigma(b)\sigma^2(b)\dots$, and $\omega' = \dots\sigma^2(c)\sigma(c)c.dc\sigma(c)\sigma^2(c)\dots$.

(iii) Set $A = \{a, b, c, d, e\}$. Define $\sigma : A \to A^+$ by $a \mapsto a$, $b \mapsto cbab$, $c \mapsto cbc$, $d \mapsto adc$, $e \mapsto bdea$. Then, $X_{\sigma_1} = \emptyset$, X_{σ_2} is minimal,

$$X_{\sigma_3} \setminus X_{\sigma_2} = \{a^\infty\} \cup \operatorname{Orb}_{T_\sigma}\left(\lim_{n\to\infty} \sigma^n(c).\sigma^n(c)\right) \cup \operatorname{Orb}_{T_\sigma}(\omega),$$

and $X_{\sigma_4} \setminus X_{\sigma_3} = \emptyset$, where $\omega = a^\infty.dc\sigma(c)\sigma^2(c)\sigma^3(c)\dots$.

We next consider the case $v \in {A_i}^+ \setminus {A_{i-1}}^+$.

Definition 3.11

(i) Let $w \in {A_i}^+$. We refer to $w_{[m,n]} \in {A_{i-1}}^+$ as a *possible word* in w if

$$w_{[m',n']} \in {A_{i-1}}^+, \quad 1 \le m' \le m,\ n \le n' \le |u| \Rightarrow m' = m,\ n' = n.$$

(ii) Let $k' \ge k \ge 1$ be integers and let $c \in A_i$. Suppose that $\sigma^k(c)_{[m,n]}$ (resp. $\sigma^{k'}(c)_{[m',n']}$) is a possible word in $\sigma^k(c)$ (resp. $\sigma^{k'}(c)$). We call $\sigma^k(c)_{[m,n]}$ an *ancestor* of $\sigma^{k'}(c)_{[m',n']}$ if

$$\left|\sigma^{k'-k}(\sigma^k(c)_{[1,m)})\right| + 1 \ge m' \text{ and } \left|\sigma^{k'-k}(\sigma^k(c)_{[m,n]})\right| \le n'.$$

Lemma 3.12 *Set* $M = \max_{c,d\in A_i}\{|w|; w$ *is a possible word in* $\sigma(cd)\}$. *Let* $p \in \mathbb{N}$. *Suppose that* $\sigma^k(c)_{[j,j+n)} \in {A_{i-1}}^+$ *and* $\sigma^k(c)_{j+n} \in A_i \setminus A_{i-1}$ *for some* $(c, k, j, n) \in A_i \setminus A_{i-1} \times \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ *with* $1 \le j \le |\sigma^k(c)|$ *and*

$n \geq (p+M)M$. Then, there exists $c' \in A_i \setminus A_{i-1}$ such that

$$\sigma^k(c)_{[j+n-n',j+n-n'+|\sigma^p(c')|)} = \sigma^p(c'),$$

where $n' = \max\{|w|; w \in {A_{i-1}}^+, w$ is a prefix of $\sigma^p(c')\}$.

Proof. Let j' be such that $\sigma^k(c)_{[j',j+n)}$ is a possible word in $\sigma^k(c)$. There exists an integer k' with $1 \leq k' \leq k$ such that $\sigma^k(c)_{[j',j+n)}$ does not have any ancestor in $\sigma^{k'-1}(c)$ but does in $\sigma^{k'}(c)$. For each integer l with $k' \leq l \leq k$, let $\sigma^l(c)_{[j_l,j_l+n_l)}$ denote the ancestor of $\sigma^k(c)_{[j',j+n)}$. Since

$$\begin{aligned}(p+M)M \leq n+j-j' &\leq n_{k'}M + \sum_{l=k'}^{k-1}\left(n_{l+1} - |\sigma(\sigma^l(c)_{[j_l,j_l+n_l)})|\right)\\ &\leq M^2 + (k-k')M,\end{aligned}$$

we obtain $k-k' \geq p$. The conclusion holds by taking c' to be the first letter of $A_i \setminus A_{i-1}$ to occur in $\sigma^{k-k'-p}(\sigma^{k'}(c)_{j_{k'}+n_{k'}})$. □

Proposition 3.13 *There is a possibly empty set $\{x_j \in (X_{\sigma_i} \setminus X_{\sigma_{i-1}}) \cap {A_{i-1}}^{\mathbb{Z}}; 1 \leq j \leq N\}$ of periodic points of σ such that*

- (i) $\mathrm{Orb}_{T_\sigma}(x_j) \cap \mathrm{Orb}_{T_\sigma}(x_{j'}) = \emptyset$ *if $j \neq j'$;*
- (ii) *if x_j is periodic under T_σ, then A_1 is a singleton, say $\{s\}$, and $x_j = s^\infty$;*
- (iii) *if x_j is aperiodic under T_σ, then $\overline{\mathrm{Orb}_{T_\sigma}(x_j)} = X_{\sigma_{i-1}} \cup \mathrm{Orb}_{T_\sigma}(x_j)$ (a disjoint union);*
- (iv) *the orbit of any point in a T_σ-invariant, locally compact set:*

$$X_i := X_{\sigma_i} \setminus \left(X_{\sigma_{i-1}} \cup \bigcup_{j=1}^{N} \mathrm{Orb}_{T_\sigma}(x_j)\right) \tag{3.1}$$

is dense in X_{σ_i}, where we let $X_{\sigma_0} = \emptyset$.

Proof. Properties (i)∼(iii) are verified by the same argument as in the proof of Proposition 3.9. Let $x' \in X_i$. Let $w \in \mathcal{L}(X_{\sigma_i})$. Take $p \in \mathbb{N}$ so that w is a factor of $\sigma^p(c)$ for all $c \in A_i \setminus A_{i-1}$. Fix $n \in \mathbb{N}$ with $n \geq \max\{(p+M)M, |\sigma^p(c)|; c \in A_i \setminus A_{i-1}\}$, where M is as in Lemma 3.12. Lemma 3.8 (iii) together with the fact that $\overline{\mathrm{Orb}_{T_\sigma}(\omega)} = X_{\sigma_i}$ enables us to find $l \in \mathbb{Z}$ such that $x'_{[l,l+n)} \in {A_{i-1}}^+$ and $x'_{l+n} \in A_i \setminus A_{i-1}$. Since $x'_{[l,l+2n)}$ is a factor of $\sigma^k(c)$ for some $k \in \mathbb{N}$ and some $c \in A_i \setminus A_{i-1}$, Lemma 3.12 ensures the existence of $d \in A_i \setminus A_{i-1}$ such that $\sigma^p(d)$ is a factor of $x'_{[l,l+2n)}$. Hence w is a factor of $x'_{[l,l+2n)}$. This completes the proof. □

Example 3.14 The following substitutions satisfy the hypothesis of Proposition 3.13.

(i) Set $A = \{a, b, c\}$. Define $\sigma : A \to A^+$ by $a \mapsto ab$, $b \mapsto a$, $c \mapsto acc$. Since σ_1 is primitive, X_{σ_1} is minimal. The set $X_{\sigma_2} \setminus X_{\sigma_1}$ contains no periodic points of σ.

(ii) Set $A = \{a, b, c, d\}$. Define $\sigma : A \to A^+$ by $a \mapsto a$, $b \mapsto bbab$, $c \mapsto bcca$. Then, $X_{\sigma_1} = \emptyset$, X_{σ_2} is minimal, and $X_{\sigma_3} \setminus X_{\sigma_2}$ contains a^∞.

Summarizing all the facts obtained above, we achieve the following.

Theorem 3.15 *Let $\sigma : A \to A^+$ be a substitution of some primitive components. For each integer i with $1 \le i \le n_\sigma$, we have a decomposition*:

$$X_{\sigma_i} \setminus X_{\sigma_{i-1}} = X_i \cup \mathrm{Orb}_{T_\sigma}(y_i) \cup \bigcup_{j=1}^{N_i} \mathrm{Orb}_{T_\sigma}(x_{ij})$$

of $X_{\sigma_i} \setminus X_{\sigma_{i-1}}$ into possibly empty, locally compact, T_σ-invariant sets X_i, $\mathrm{Orb}_{T_\sigma}(y_i)$ and $\mathrm{Orb}_{T_\sigma}(x_{ij})$ so that

(i) *X_i is as in* (3.1) *if it is nonempty, and hence the orbit of any point in X_i is dense in X_{σ_i}*;

(ii) *each y_i is a quasi-periodic point of a primitive type*;

(iii) *each x_{ij} is a periodic point of σ such that if it is periodic under T_σ, then A_1 is a singleton, say $\{s\}$, and $x_{ij} = s^\infty$; otherwise, $\overline{\mathrm{Orb}_{T_\sigma}(x_{ij})} = X_{\sigma_i} \cup \mathrm{Orb}_{T_\sigma}(x_{ij})$.*

As a consequence,

$$X_\sigma = \bigcup_{i=1}^{n_\sigma} X_i \cup \bigcup_{i=2}^{n_\sigma} \mathrm{Orb}_{T_\sigma}(y_i) \cup \bigcup_{i=2}^{n_\sigma} \bigcup_{j=1}^{N_i} \mathrm{Orb}_{T_\sigma}(x_{ij}). \tag{3.2}$$

The number of minimal sets of X_σ is at most two. The two minimal sets are X_{σ_2} and $\{s^\infty\}$, where $A_1 = \{s\}$. The minimal set is unique if and only if one of the following holds:

(i) $\lim_{n\to\infty} |\sigma^n(a)| = \infty$ *for any* $a \in A_1$;
(ii) A_1 *is a singleton, say* $\{s\}$, *and* $s^\infty \notin X_\sigma$;
(iii) A_1 *is a singleton and* σ_2 *is almost primitive.*

In these cases, the unique minimal set is X_{σ_1}, X_{σ_2} *and* $\{s^\infty\}$, *respectively, where* $A_1 = \{s\}$.

4. Perron-Frobenius Theory for auxiliary substitutions

Let $\sigma : A \to A^+$ be a substitution of some primitive components. Let i be an integer with $1 \le i \le n_\sigma$. Let $Q_1(i)$ be Q_i in (2.1). Let θ_i denote a dominant eigenvalue of $Q_1(i)$. Given $m \in \mathbb{N}$, define a substitution $\sigma^{(m)} : \mathcal{L}_m(\sigma) \to \mathcal{L}_m(\sigma)^+$ by for $u \in \mathcal{L}_m(\sigma)$,

$$\sigma^{(m)}(u) = \sigma(u)_{[1,m]}, \sigma(u)_{[2,m+1]}, \sigma(u)_{[3,m+2]}, \dots, \sigma(u)_{[|\sigma(u_1)|,|\sigma(u_1)|+m-1]},$$

where the commas between consecutive $\sigma(u)_{[i,m+i-1]}$'s are not new letters but just mean the separation between letters. Observe that there exists $k_0 \in \mathbb{N}$ such that any word in $\mathcal{L}(\sigma_i)$ occurs in $\sigma^k(a)$ for any $a \in A_i \setminus A_{i-1}$, any integer i with $1 \le i \le n_\sigma$ and any integer $k \ge k_0$. Set $\mathcal{B}_m(i) = \{u \in \mathcal{L}_m(\sigma_{i+1}); u_1 \in A_i\}$ for $0 \le i < n_\sigma$. Set $\lambda_i = \max_{1\le j\le i} \theta_j$ and $\eta_i = \max_{i\le j\le n_\sigma} \theta_j$ for $1 \le i \le n_\sigma$, and $\lambda = \lambda_{n_\sigma}$.

We devote this section to analyzing how fast entries of ${M_{\sigma^{(m)}}}^k$ increase as k tends to infinity.

Lemma 4.1 *With possibly empty matrices* $F_{m,k}(i)$, $G_m(i)$, $Q_m(i)$ *and* $R_{m,k}(i)$, *we may write that for every* $k \in \mathbb{N}$,

$${M_{\sigma^{(m)}}}^k =$$
$$\begin{bmatrix}
Q_m(1)^k & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
F_{m,k}(1) & G_m(1)^k & 0 & 0 & 0 & \cdots & 0 & 0 \\
\multicolumn{2}{c}{R_{m,k}(1)} & Q_m(2)^k & 0 & 0 & \cdots & 0 & 0 \\
\multicolumn{3}{c}{F_{m,k}(2)} & G_m(2)^k & 0 & \cdots & 0 & 0 \\
\multicolumn{4}{c}{R_{m,k}(2)} & Q_m(3)^k & \cdots & 0 & 0 \\
 & & \vdots & & & \ddots & \vdots & \vdots \\
\multicolumn{6}{c}{F_{m,k}(n_\sigma - 1)} & G_m(n_\sigma - 1)^k & 0 \\
\multicolumn{7}{c}{R_{m,k}(n_\sigma - 1)} & Q_m(n_\sigma)^k
\end{bmatrix},$$

where

- $Q_m(i)$ *is an* $\mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1) \times \mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1)$ *matrix*;
- $G_m(i)$ *is a* $\mathcal{B}_m(i) \setminus \mathcal{L}_m(\sigma_i) \times \mathcal{B}_m(i) \setminus \mathcal{L}_m(\sigma_i)$ *matrix*;
- $F_{m,k}(i)$ *is a* $\mathcal{B}_m(i) \setminus \mathcal{L}_m(\sigma_i) \times \mathcal{L}_m(\sigma_i)$ *matrix*;
- $R_{m,k}(i)$ *is an* $\mathcal{L}_m(\sigma_{i+1}) \setminus \mathcal{B}_m(i) \times \mathcal{B}_m(i)$ *matrix.*

Then,

(i) *the following are equivalent*:
  (a) $\theta_i = 1$;
  (b) $Q_1(i) = [1]$;
  (c) $$X_{\sigma_i} \setminus X_{\sigma_{i-1}} = \mathrm{Orb}_{T_\sigma}(y) \cup \bigcup_{j=1}^{N} \mathrm{Orb}_{T_\sigma}(x_j) \text{ (a disjoint union)}$$

  for a quasi-periodic point $y \in X_{\sigma_i} \setminus X_{\sigma_{i-1}}$ *of a primitive type and for some periodic points* $x_1, x_2, \ldots, x_N \in X_{\sigma_i} \setminus X_{\sigma_{i-1}}$ *of* σ, *some of which are possibly nonexistent*;

(ii) *the following are equivalent*:
  (a) $\mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1) = \emptyset$;
  (b) $m > 1$, $A_i \setminus A_{i-1}$ *is a singleton, say* $\{s\}$, *and* $\sigma(s) = us$ *for some* $u \in {A_{i-1}}^+$;

(iii) $Q_m(i)$ *is a primitive matrix with a dominant eigenvalue* θ_i, *if* $Q_m(i)$ *is nonempty*;

(iv) *the absolute value of no eigenvalue of* $G_m(i)$ *is greater than one*;

(v) *there exists a set* $\{c_i > 0; 1 \le i \le n_\sigma\}$ *such that if* $u \in \mathcal{L}_m(\sigma) \setminus \mathcal{B}_m(n_\sigma - 1)$ *and* $v \in \mathcal{L}_m(\sigma_i) \setminus \mathcal{L}_m(\sigma_{i-1})$, *then* $({M_{\sigma^{(m)}}}^k)_{u,v} \ge c_i {\eta_i}^k$ *for all sufficiently large* $k \in \mathbb{N}$.

Proof. Assume that the size of $Q_1(i)$ is greater than one. Then, for a sufficiently large $k \in \mathbb{N}$,

$$\theta_i{}^k = v_i{}^{-1} \sum_j (Q_1(i)^k)_{ij} v_j > 1,$$

where v is a positive, right eigenvector corresponding to θ_i. Hence, $\theta_i = 1$ implies $Q_1(i) = [1]$. The other parts in (i) are straightforward by using facts from Section 3. Similarly, Statement (ii) is readily verified.

Assuming $Q_m(i)$ is nonempty, choose $k_0 \in \mathbb{N}$ so that any word in $\mathcal{L}_m(\sigma_i)$

occurs in $\sigma^k(a)$ for any integer $k \geq k_0$ and any $a \in A_i \setminus A_{i-1}$. This implies $Q_m(i)^k > 0$ if $k \geq k_0$. Define a $\mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1) \times A_i \setminus A_{i-1}$ matrix N by $N_{u,a} = (M_\sigma)_{u_1,a}$ for each (u,a). Then, $Q_m(i)N = NQ_1(i)$ because their (u,a)-entries are $N(a, \sigma^2(u_1))$. If $Q_1(i)v = \theta_i v$ and $v > 0$, then $Q_m(i)Nv = \theta_i Nv$ and $Nv > 0$. This implies (iii).

If $(G_m(i)^k)_{u,v} > 0$, then $v \in \{\sigma^k(u)_{[j,j+m)}; |\sigma^k(u_1)| - m + 2 \leq j \leq |\sigma^k(u_1)|\}$. Hence, for any $k \in \mathbb{N}$, each row sum of $G_m(i)^k$ is not greater than $m-1$, which shows (iv).

In the remainder of this proof, let us show (v). Take $k_0 \in \mathbb{N}$ so that $R_{m,k}(i) > 0$ for any integer $k \geq k_0$ and any integer i with $1 \leq i < n_\sigma$. Put $M_i = M_{\sigma_i(m)}$ for $1 \leq i \leq n_\sigma$. Put

$$M_i = \begin{bmatrix} M_{i-1} & 0 & 0 \\ F & G & 0 \\ R & R' & Q \end{bmatrix},$$

where $G = G_m(i-1)$ and $Q = Q_m(i)$. Define F', R_{k_0}, R'_{k_0} by

$${M_i}^{k_0} = \begin{bmatrix} {M_{i-1}}^{k_0} & 0 & 0 \\ F' & G^{k_0} & 0 \\ R_{k_0} & R'_{k_0} & Q^{k_0} \end{bmatrix}.$$

Reducing ${M_i}^{k_0+k} = {M_i}^{k_0}{M_i}^k = {M_i}^k{M_i}^{k_0}$, we obtain that for every integer $k \in \mathbb{N}$,

$${M_i}^{k_0+k} \geq \begin{bmatrix} {M_{i-1}}^{k_0+k} & 0 & 0 \\ 0 & G^{k_0+k} & 0 \\ Q^k R_{k_0} & Q^k R'_{k_0} & Q^{k_0+k} \end{bmatrix} \text{ and}$$

$${M_i}^{k_0+k} \geq \begin{bmatrix} {M_{i-1}}^{k_0+k} & 0 & 0 \\ 0 & G^{k_0+k} & 0 \\ R_{k_0}{M_{i-1}}^k & {R_{k_0}}'G & Q^{k_0+k} \end{bmatrix}.$$

This shows the conclusion in the case $i = n_\sigma$. Applying this argument to M_{i-2} instead of M_i, we obtain the conclusion in the case $i = n_\sigma - 1$. Repeating the argument, we may obtain (v). $\square$

Lemma 4.2 *Set $i_{\min} = \min_{\theta_i=\lambda} i$ and $i_{\max} = \max_{\theta_i=\lambda} i$. Suppose $\lambda > 1$. Then,*

(i) *a right eigenvector $\alpha = (\alpha_u)_{u\in\mathcal{L}_m(\sigma)}$ of $M_{\sigma^{(m)}}$ corresponding to λ may be chosen so that $(\alpha_u)_{u\in\mathcal{B}_m(i_{\max}-1)} = 0$ and $(\alpha_u)_{u\in\mathcal{L}_m(\sigma)\setminus\mathcal{B}_m(i_{\max}-1)} > 0$;*

(ii) *a left eigenvector $\beta = (\beta_u)_{u\in\mathcal{L}_m(\sigma)}$ of $M_{\sigma^{(m)}}$ corresponding to λ may be chosen so that $(\beta_u)_{u\in\mathcal{L}_m(\sigma_{i_{\min}})} > 0$ and $(\beta_u)_{u\in\mathcal{L}_m(\sigma)\setminus\mathcal{L}_m(\sigma_{i_{\min}})} = 0$;*

(iii) *λ is a simple, dominant eigenvalue of $M_{\sigma^{(m)}}$.*

Proof. Put $\alpha' = (\alpha_u)_{u\in\mathcal{B}_m(n_\sigma-1)}$, $\alpha'' = (\alpha_u)_{u\in\mathcal{L}_m(\sigma)\setminus\mathcal{B}_m(n_\sigma-1)}$ and

$$P_m(i) = \begin{bmatrix} M_{\sigma_i{}^{(m)}} & 0 \\ F_{m,1}(i) & G_m(i) \end{bmatrix}.$$

In order to prove (i), it is sufficient to show the following statements:

(a) if $\theta_{n_\sigma} > \lambda_{n_\sigma-1}$, then $\alpha' = 0$ and α'' may be chosen to be positive;

(b) if $\lambda_i = \lambda$ and ξ is a right eigenvector of $P_m(i)$ corresponding to λ_i such that $\xi' := (\xi_u)_{u\in\mathcal{L}_m(\sigma_i)} \geq 0$ and $\xi' \neq 0$, then $(\xi_u)_{u\in\mathcal{B}_m(i)\setminus\mathcal{L}_m(\sigma_i)} > 0$;

(c) if $\theta_{n_\sigma} = \lambda_{n_\sigma-1}$ and $\alpha' \geq 0$, then $\alpha' = 0$ and α'' may be chosen to be positive;

(d) if $\theta_{n_\sigma} < \lambda_{n_\sigma-1}$, $\alpha' \geq 0$ and $\alpha' \neq 0$, then α'' may be chosen to be positive.

If $\theta_{n_\sigma} > \lambda_{n_\sigma-1}$, then clearly $\alpha' = 0$, and hence we may choose α'' to be positive. Assuming the hypothesis of (b), since for a sufficiently large $k \in \mathbb{N}$,

$$(\xi_u)_{u\in\mathcal{B}_m(i)\setminus\mathcal{L}_m(\sigma_i)} = \lambda_i{}^{-k} \sum_{j=0}^{\infty} \left\{\lambda_i{}^{-k} G_m(i)^k\right\}^j F_{m,k}(i)\xi' > 0, \tag{4.1}$$

we obtain (b). Assume the hypothesis of (c). If $\alpha' \neq 0$, then reducing $M_{\sigma^{(m)}}{}^k\alpha = \theta_{n_\sigma}{}^k\alpha$, we obtain a contradiction that for a sufficiently large $k \in \mathbb{N}$,

$$0 = \delta\left\{\theta_{n_\sigma}{}^k I - Q_m(n_\sigma)^k\right\}\alpha'' = \delta R_{m,k}(n_\sigma - 1)\alpha' > 0, \tag{4.2}$$

where δ is a positive, left eigenvector of $Q_m(n_\sigma)$ corresponding to θ_{n_σ}. Statement (d) is obtained in the same manner as used to obtain (4.1).

Put $\beta' = (\beta_u)_{u\in\mathcal{B}_m(n_\sigma-1)}$ and $\beta'' = (\beta_u)_{u\in\mathcal{L}_m(\sigma)\setminus\mathcal{B}_m(n_\sigma-1)}$. In order to prove (ii), it might be sufficient to show the following statements:

(1) if $\lambda_{n_\sigma-1} \geq \theta_{n_\sigma}$, then $\beta'' = 0$;
(2) if $\lambda_{n_\sigma-1} < \theta_{n_\sigma}$, then β' may be chosen to be positive;
(3) if $\lambda_{n_\sigma-1} = \lambda$ and ξ is a left eigenvector of $P_m(n_\sigma - 1)$ corresponding to λ, then $\xi_u = 0$ for any $u \in \mathcal{B}_m(n_\sigma - 1) \setminus \mathcal{L}_m(\sigma_{n_\sigma-1})$.

Statement (3) follows from Lemma 4.1 (iv). If $\lambda_{n_\sigma-1} > \theta_{n_\sigma}$, then $\beta'' = 0$. Assume $\lambda_{n_\sigma-1} = \theta_{n_\sigma}$. If $\beta'' \neq 0$, then we may assume $\beta'' > 0$. This yields a contradiction similar to (4.1). Assume $\lambda_{n_\sigma-1} < \theta_{n_\sigma}$. We may assume $\beta'' > 0$. In a similar way to obtaining (4.2), we may see (2). Statement (iii) is a consequence of the above argument. □

Remark 4.3 Perron-Frobenius Theory for matrices in (2.2) is discussed also in [2]. In fact, more general facts are stated in Theorem 3.1 therein.

Lemma 4.4 *Suppose $\theta_i > 1$. If $\theta_i > \lambda_{i-1}$, then for any $u \in \mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1)$ and any $v \in \mathcal{L}_m(\sigma_i)$,*

$$\lim_{k\to\infty} \left({\theta_i}^{-k} {M_{\sigma^{(m)}}}^k\right)_{u,v} = \alpha_u \beta_v > 0. \tag{4.3}$$

If $\theta_i \leq \lambda_{i-1}$, then there exist $\{\gamma_u > 0; u \in \mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1)\}$ and $\{\delta_v > 0; v \in \mathcal{L}_m(\sigma_i) \setminus \mathcal{L}_m(\sigma_{i'-1})\}$ such that

$$\lim_{k\to\infty} \left({\theta_i}^{-k} {M_{\sigma^{(m)}}}^k\right)_{u,v} = \begin{cases} \infty & \text{if } v \in \mathcal{L}_m(\sigma_{i'-1}); \\ \gamma_u \delta_v & \text{otherwise,} \end{cases} \tag{4.4}$$

where, putting $I = \{1 \leq i_0 < i; \theta_{i_1} < \theta_i$ for any integer i_1 with $i_0 \leq i_1 < i\}$,

$$i' = \begin{cases} i & \text{if } I = \emptyset; \\ \min I & \text{otherwise.} \end{cases}$$

Proof. We may assume $i = n_\sigma$. Let s denote the size of $M_{\sigma^{(m)}}$.

Suppose $\theta_i > \lambda_{i-1}$. Let N be a matrix which puts ${\theta_i}^{-1} M_{\sigma^{(m)}}$ into a Jordan normal form:

$$N^{-1}({\theta_i}^{-1}M_{\sigma^{(m)}})N = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \epsilon_1/\lambda & * & \cdots & 0 \\ 0 & 0 & \epsilon_2/\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \epsilon_{s-1}/\lambda \end{bmatrix},$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_{s-1}$ are eigenvalues of $M_{\sigma^{(m)}}$ other than λ. We obtain (4.3), since

$$N^{-1}\Big(\lim_{k\to\infty} {\theta_i}^{-k}{M_{\sigma^{(m)}}}^{k}\Big)N = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and since the first column and the first row of N and N^{-1} are α and β, respectively.

Assume $\theta_i = \lambda_{i-1}$. In this case, $\eta_j = \theta_i$ for any integer j with $1 \le j < i$. Let N be a matrix which puts ${\theta_i}^{-1}M_{\sigma^{(m)}}$ into a Jordan normal form:

$$N^{-1}({\theta_i}^{-1}M_{\sigma^{(m)}})N = J := \begin{bmatrix} 1 & 1/\lambda & \cdots & 0 & 0 & 0 & \ldots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 1/\lambda & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \epsilon_{r+1}/\lambda & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & \epsilon_s/\lambda \end{bmatrix},$$

where $\epsilon_{r+1}, \epsilon_{r+2}, \ldots, \epsilon_s$ are eigenvalues of $M_{\sigma^{(m)}}$ other than λ. By Lemma 4.2, the multiplicity r of the eigenvalue λ is greater than one.

Set $\{1 \le p \le n_\sigma; \theta_p = \lambda\} = \{i_{\min} = i_1 < i_2 < \cdots < i_r = n_\sigma\}$. Set $s_p = \sharp\mathcal{L}_m(\sigma_{i_p})$ for $1 \le p \le r$. Let ξ be such that $\xi(M_{\sigma^{(m)}} - \lambda I) = \beta$. Put $\xi' = (\xi_j)_{j=1}^{s_1}$ and $\xi'' = (\xi_j)_{j>s_1}$. If $\xi'' = 0$, then $\xi'(M_{\sigma_{i_1}^{(m)}} - \lambda I) = \beta' := (\beta_j)_{j=1}^{s_1}$. This yields a contradiction that $0 = \xi'(M_{\sigma_{i_1}^{(m)}} - \lambda I)\zeta = \beta'\zeta > 0$, where ζ is a nonnegative, right eigenvector of $M_{{\sigma_{i_1}}^{(m)}}$ corresponding to λ. Hence, ξ'' is a left eigenvector of the matrix:

$$\begin{bmatrix} G_m(i_1) & 0 & \cdots & 0 \\ * & Q_m(i_1+1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & Q_m(n_\sigma) \end{bmatrix}$$

corresponding to λ. Using techniques developed in the proof of Lemma 4.2, we may verify that given an integer j with $s_1 < j \leq s$, $\xi_j \neq 0$ if and only if $s_1 < j \leq s_2$. Let ρ be such that $\rho(M_{\sigma^{(m)}} - \lambda I) = \xi$. The same argument shows that given an integer j with $s_2 < j \leq s$, $\rho_j \neq 0$ if and only if $s_2 < j \leq s_3$.

Repeating this argument, we may see that given an integer p with $1 \leq p \leq r$, the p-th row ξ of N^{-1} satisfies the properties:

- $\xi_j \neq 0$ if $s_{r-p} < j \leq s_{r-p+1}$;
- $\xi_j = 0$ if $s_{r-p+1} < j \leq s$,

where $s_0 = 0$. Since given integers p, q with $1 \leq p < r$ and $1 \leq q \leq r - p$, $\lim_{k\to\infty}(J^k)_{p,p+q}/k^t > 0$ if and only if $t = q$, it follows that under the extended arithmetics,

$$\lim_{k\to\infty} \theta_i{}^{-k} M_{\sigma^{(m)}}{}^k = N(\lim_{k\to\infty} J^k)N^{-1}$$

$$= \begin{bmatrix} 0 & \\ \vdots & \\ 0 & \\ \alpha_{r'+1} & * \\ \alpha_{r'+2} & \\ \vdots & \\ \alpha_s & \end{bmatrix} \begin{bmatrix} 1 & \infty & \cdots & \infty & \\ 0 & 1 & \cdots & \infty & \\ \vdots & \vdots & \ddots & \vdots & 0 \\ 0 & 0 & \cdots & 1 & \\ & & 0 & & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} * & \cdots & * & * & \cdots & * & \cdots & \xi_{s_{r-1}+1} & \cdots & \xi_s \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \xi_{s_2+1} & \cdots & \xi_{s_3} & \cdots & 0 & \cdots & 0 \\ \beta_1 & \cdots & \beta_{s_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & & & & & * & & & & \end{bmatrix}$$

$$
= \begin{bmatrix} & & & * & & & \\ \alpha_{r'+1} & \infty & \cdots & \infty & 0 & \cdots & 0 \\ \alpha_{r'+2} & \infty & \cdots & \infty & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_s & \infty & \cdots & \infty & 0 & \cdots & 0 \end{bmatrix}
$$

$$
\cdot \begin{bmatrix} * & \cdots & * & * & \cdots & * & \cdots & \xi_{s_{r-1}+1} & \cdots & \xi_s \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ * & \cdots & * & \xi_{s_2+1} & \cdots & \xi_{s_3} & \cdots & 0 & \cdots & 0 \\ \beta_1 & \cdots & \beta_{s_1} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ & & & & & * & & & & \end{bmatrix}
$$

$$
= \begin{bmatrix} & & & & * & & & \\ \infty & \infty & \cdots & \infty & \alpha_{r'+1}\xi_{s_{r-1}+1} & \alpha_{r'+1}\xi_{s_{r-1}+2} & \cdots & \alpha_{r'+1}\xi_s \\ \infty & \infty & \cdots & \infty & \alpha_{r'+2}\xi_{s_{r-1}+1} & \alpha_{r'+2}\xi_{s_{r-1}+2} & \cdots & \alpha_{r'+2}\xi_s \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \infty & \infty & \cdots & \infty & \alpha_s\xi_{s_{r-1}+1} & \alpha_s\xi_{s_{r-1}+2} & \cdots & \alpha_s\xi_s \end{bmatrix},
$$

where $r' = \sharp\mathcal{B}_m(n_\sigma - 1)$.

Assume $\theta_i < \lambda_{i-1}$. Put $M = \{(M_{\sigma^{(m)}})_{u,v}\}_{u,v\in\mathcal{L}_m(\sigma)\setminus\mathcal{L}_m(\sigma_{i'-1})}$. It follows that θ_i is a simple, dominant eigenvalue of M. Since similar statements to Lemma 4.2 holds for this matrix M, arguments used to show (4.3) of this lemma show the second half of (4.4). The other half is obtained by using Lemma 4.1 (v). □

Example 4.5

(i) Set $A = \{a, b, c\}$. Define $\sigma : A \to A^+$ by $a \mapsto a^4$, $b \mapsto ab^3$, $c \mapsto cbc$. Then

$$
M_\sigma = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix},\ \theta_1 = 4,\ \theta_2 = 3,\ \theta_3 = 2,\ \alpha = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},\ \beta = [1\ \ 0\ \ 0].
$$

It follows that $\mathcal{L}_2(\sigma) = \{aa, ab, ba, bb, bc, ca, cb\}$, $\mathcal{L}_2(\sigma_1) = \{aa\}$, $\mathcal{B}_2(1) = \{aa, ab\}$, $\mathcal{L}_2(\sigma_2) = \{aa, ab, ba, bb\}$, $\mathcal{B}_2(2) = \{aa, ab, ba, bb, bc\}$. We have

$$\sigma^{(2)}(aa) = aa, aa, aa, aa; \qquad \sigma^{(2)}(ab) = aa, aa, aa, aa;$$
$$\sigma^{(2)}(ba) = ab, bb, bb, ba; \qquad \sigma^{(2)}(bb) = ab, bb, bb, ba;$$
$$\sigma^{(2)}(bc) = ab, bb, bb, bc; \qquad \sigma^{(2)}(ca) = cb, bc, ca;$$
$$\sigma^{(2)}(cb) = cb, bc, ca,$$

and

$$M_{\sigma^{(2)}} = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \quad \beta = [1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0].$$

(ii) Set $A = \{a, b, c, d\}$. Define $\sigma : A \to A^+$ by $a \mapsto aa$, $b \mapsto ab^3c^3$, $c \mapsto abc^5$, $d \mapsto abcd^2$. Then

$$M_\sigma = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 \\ 1 & 1 & 5 & 0 \\ 1 & 1 & 1 & 2 \end{bmatrix}, \ \theta_1 = 2, \ \theta_2 = 6, \ \theta_3 = 2, \ \alpha = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \ \beta = [1 \ 1 \ 3 \ 0].$$

It follows that $\mathcal{L}_2(\sigma) = \{aa, ab, bb, bc, ca, cc, cd, da, dd\}$, $\mathcal{L}_2(\sigma_1) = \{aa\}$, $\mathcal{B}_2(1) = \{aa, ab\}$, $\mathcal{L}_2(\sigma_2) = \{aa, ab, bb, bc, ca, cc\}$, $\mathcal{B}_2(2) = \{aa, ab, bb, bc, ca, cc, cd\}$. We have

$$\sigma^{(2)}(aa) = aa, aa; \qquad \sigma^{(2)}(ab) = aa, aa;$$
$$\sigma^{(2)}(bb) = ab, bb, bb, bc, cc, cc, ca; \qquad \sigma^{(2)}(bc) = ab, bb, bb, bc, cc, cc, ca;$$
$$\sigma^{(2)}(ca) = ab, bc, cc, cc, cc, cc, ca; \qquad \sigma^{(2)}(cc) = ab, bc, cc, cc, cc, cc, ca;$$
$$\sigma^{(2)}(cd) = ab, bc, cc, cc, cc, cc, ca; \qquad \sigma^{(2)}(da) = ab, bc, cd, dd, da;$$
$$\sigma^{(2)}(dd) = ab, bc, cd, dd, da,$$

and

$$M_{\sigma^{(2)}} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix},$$

$$\beta = [1 \quad 2 \quad 1 \quad 2 \quad 2 \quad 7 \quad 0 \quad 0 \quad 0].$$

5. Invariant measures for X_σ

Let $\sigma : A \to A^+$ be a substitution of some primitive components. Let $i \in \mathbb{N}$ be $1 < i \leq n_\sigma$.

Corollary 5.1 *Suppose $\theta_i > 1$. Let $a \in A_i \setminus A_{i-1}$, $v \in \mathcal{L}(\sigma_i)$, $m = |v|$ and $u \in \mathcal{L}_m(\sigma_i)$ with $u_1 = a$.*

(i) *If $\theta_i > \lambda_{i-1}$, then*

$$\lim_{k\to\infty} \frac{N(v, \sigma^k(a))}{|\sigma^k(a)|} = \frac{\beta_v}{\sum\limits_{w \in \mathcal{L}_m(\sigma_i)} \beta_w}.$$

(ii) *If $\theta_i \leq \lambda_{i-1}$, then*

$$\lim_{k\to\infty} \frac{1}{{\theta_i}^k} N(v, \sigma^k(a)) = \begin{cases} \infty & \text{if } v \in \mathcal{L}_m(\sigma_{i'-1}); \\ \gamma_u \delta_v & \text{otherwise.} \end{cases}$$

(iii) *If $v \notin \mathcal{L}(\sigma_{i'-1})$, then*

$$\lim_{k\to\infty} \frac{N(v, \sigma^k(a))}{\sum\limits_{b \in A_i \setminus A_{i-1}} N(b, \sigma^k(a))} = \frac{\delta_v}{\sum\limits_{w \in \mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1)} \delta_w}.$$

Proof. Put $\tau = \sigma^{(m)}$. Assume $\theta_i > \lambda_{i-1}$. Since for every $k \in \mathbb{N}$,

$$|\sigma^k(a)| = |\tau^k(u)|;$$
$$N(v, \tau^k(u)) - (m-1) \leq N(v, \sigma^k(a)) \leq N(v, \tau^k(u)),$$

it follows from Lemma 4.4 that

$$\lim_{k\to\infty} \frac{N(v, \sigma^k(a))}{|\sigma^k(a)|} = \lim_{k\to\infty} \frac{N(v, \tau^k(u))}{|\tau^k(u)|}$$
$$= \lim_{k\to\infty} \frac{({M_\tau}^k)_{u,v}}{\sum\limits_{w\in\mathcal{L}_m(\sigma_i)} ({M_\tau}^k)_{u,w}} = \frac{\beta_v}{\sum\limits_{w\in\mathcal{L}_m(\sigma_i)} \beta_w}.$$

Assume $\theta_i \leq \lambda_{i-1}$. Since

$$\lim_{k\to\infty} \frac{1}{{\theta_i}^k} N(v, \sigma^k(a)) = \lim_{k\to\infty} \frac{1}{{\theta_i}^k} ({M_\tau}^k)_{u,v},$$

(ii) follows from Lemma 4.4. It follows from Lemma 4.4 again that

$$\lim_{k\to\infty} \frac{N(v, \sigma^k(a))}{\sum\limits_{b\in A_i\setminus A_{i-1}} N(b, \sigma^k(a))} = \lim_{k\to\infty} \frac{({M_\tau}^k)_{u,v}}{\sum\limits_{w\in\mathcal{L}_m(\sigma_i)\setminus\mathcal{B}_m(i-1)} ({M_\tau}^k)_{u,w}}$$
$$= \frac{\delta_v}{\sum\limits_{w\in\mathcal{L}_m(\sigma_i)\setminus\mathcal{B}_m(i-1)} \delta_w}. \qquad \square$$

Suppose that X_i in (3.2) is nonempty. Lemma 4.1 (i) allows us to assume $\theta_i > 1$. Let $\omega \in X_i$ be a quasi-fixed point of the substitution σ, so that for each $k \in \mathbb{N}$, there are integers $m_k \geq 0$ and $n_k > 0$ such that $\omega_{[-m_k, n_k)} = \sigma^k(\omega_0)$. Then, the following holds.

Proposition 5.2

(i) *If $\lambda_{i-1} < \theta_i$, then the weak* limit*

$$\mu_i = \lim_{k\to\infty} \frac{1}{m_k + n_k} \sum_{j=-m_k}^{n_k - 1} \delta_{{T_\sigma}^j \omega}$$

exists.

(ii) *If $\lambda_{i-1} \geq \theta_i$, then X_i has an infinite, invariant Radon measure ν_i*

characterized by the fact that for any $v \in \mathcal{L}(\sigma_i)$,

$$\nu_i([v]) = \lim_{k\to\infty} \frac{1}{{\theta_i}^k} \sum_{j=-m_k}^{n_k-1} \delta_{T_\sigma{}^j\omega}([v]).$$

Proof. Let $v \in \mathcal{L}(\sigma_i)$. Put $m = |v|$.

Assuming $\lambda_{i-1} < \theta_i$, it follows from Corollary 5.1 (i) that

$$\mu_i([v]) = \lim_{k\to\infty} \frac{1}{|\sigma^k(\omega_0)|} N(v, \sigma^k(\omega_0)) = \frac{\beta_v}{\sum\limits_{w\in\mathcal{L}_m(\sigma_i)} \beta_w}.$$

Assume $\lambda_{i-1} \geq \theta_i$. Define an extended, real-valued, set function $\tilde{\nu_i}$ on the ring $\mathscr{C} = \{[u.v]; uv \in \mathcal{L}(\sigma_i)\}$ of cylinder sets by

$$\tilde{\nu_i}([u.v]) = \lim_{k\to\infty} \frac{1}{{\theta_i}^k} \sum_{j=-m_k}^{n_k-1} \delta_{T_\sigma{}^j\omega}([u.v]) = \lim_{k\to\infty} \frac{1}{{\theta_i}^k} N(uv, \sigma^k(\omega_0)) = \gamma_w \delta_{uv},$$

where $w \in \mathcal{L}_{|uv|}(\sigma_i)$ with $w_1 = \omega_0$. The set function $\tilde{\nu_i}$ is countably additive, and also, finite on any compact open subset of X_i. Hence, $\tilde{\nu_i}$ is uniquely extended to a T_σ-invariant, Radon measure ν_i on X_i. It follows from Corollary 5.1 that ν is infinite. □

It might be worthwhile noticing that given an integer j with $1 \leq j \leq i$, $\nu_i(X_{\sigma_j} \setminus X_{\sigma_{j-1}}) = \infty$ iff $1 \leq j < i'$.

Example 5.3

(i) Let σ be as in Example 4.5 (i). Then, $X_{\sigma_1} = \{a^\infty\}$ has an invariant probability measure $\mu_1 = \delta_{a^\infty}$, and $X_{\sigma_2} \setminus X_{\sigma_1}$ and $X_{\sigma_3} \setminus X_{\sigma_2}$ have infinite invariant measures ν_2 and ν_3, respectively. The measures of cylinder sets with respect to ν_2 or ν_3 can be calculated as follows:

$$\nu_2([a]) = \infty, \quad \nu_2([b]) = 1, \quad \nu_2([aa]) = \infty, \quad \nu_2([ab]) = 1/3,$$
$$\nu_2([ba]) = 1/3, \quad \nu_2([bb]) = 2/3, \quad \nu_3([a]) = \infty, \quad \nu_3([b]) = \infty,$$
$$\nu_3([c]) = 1, \quad \nu_3([aa]) = \infty, \quad \nu_3([aa]) = \infty, \quad \nu_3([ab]) = \infty,$$
$$\nu_3([ba]) = \infty, \quad \nu_3([bb]) = \infty, \quad \nu_3([bc]) = 1, \quad \nu_3([ca]) = 1/2,$$
$$\nu_3([cb]) = 1/2.$$

(ii) Set $A = \{a, b, c, d, e\}$. Define $\sigma : A \to A^+$ by $a \mapsto ab, b \mapsto a, c \mapsto acd, d \mapsto adc, e \mapsto dece$. Then, $\theta_1 = (1+\sqrt{5})/2$, $\theta_2 = \theta_3 = 2$. It follows from Proposition 5.2 that X_{σ_1} and X_{σ_2} have invariant probability measures μ_1 and μ_2, respectively, and that X_{σ_3} has an infinite invariant measure ν_3. The measures of cylinder sets with respect to μ_1, μ_2 and ν_3 are calculated as follows:

$$
\begin{array}{rrr}
\mu_1([a]) = (\sqrt{5}-1)/2, & \mu_1([b]) = (3-\sqrt{5})/2, & \mu_1([aa]) = \sqrt{5}-2, \\
\mu_1([ab]) = (3-\sqrt{5})/2, & \mu_1([ba]) = (3-\sqrt{5})/2, & \mu_2([a]) = 1/2, \\
\mu_2([b]) = 1/4, & \mu_2([c]) = 1/8, & \mu_2([d]) = 1/8 \\
\mu_2([aa]) = 1/8, & \mu_2([ab]) = 1/4, & \mu_2([ba]) = 1/4, \\
\mu_2([ac]) = 1/16, & \mu_2([ad]) = 1/16, & \mu_2([ca]) = 1/16, \\
\mu_2([cd]) = 1/16, & \mu_2([da]) = 1/16, & \mu_2([dc]) = 1/16, \\
\nu_3([a]) = \infty, & \nu_3([b]) = \infty, & \nu_3([c]) = \infty, \\
\nu_3([d]) = \infty, & \nu_3([e]) = 1, & \nu_3([aa]) = \infty, \\
\nu_3([ab]) = \infty, & \nu_3([ba]) = \infty, & \nu_3([ac]) = \infty, \\
\nu_3([ad]) = \infty, & \nu_3([ca]) = \infty, & \nu_3([cd]) = \infty, \\
\nu_3([da]) = \infty, & \nu_3([dc]) = \infty, & \nu_3([dd]) = 1/4, \\
\nu_3([ce]) = 1/2, & \nu_3([de]) = 1/2, & \nu_3([ea]) = 1/2, \\
\nu_3([ec]) = 1/2. & &
\end{array}
$$

The following lemma plays a crucial role in showing that μ_i or ν_i is a *unique* invariant measure for X_i.

Lemma 5.4 *Let $X \subset A^{\mathbb{Z}}$ be a locally compact, minimal subshift. Let T denote the left shift on X. Let $K \subset X$ be a nonempty, compact open set. Choose a point $\omega \in K$ which returns to K infinitely many times, say at $0 = k_0 < k_1 < k_2 < \cdots$. Then, the following are equivalent*:

(i) *X has a unique (up to scaling), invariant Radon measure*;
(ii) *for any $v \in \mathcal{L}(X)$ such that $[v] \subset K$,*

$$\lim_{n\to\infty} \frac{1}{n} N\big(v, \omega_{[k_j, k_{j+n}]}\big)$$

converges to a constant uniformly in $j \geq 0$.

Furthermore, if these conditions hold, then the unique invariant measure is ergodic.

Proof. If X is non-compact, then this is a consequence of [12, Theorem 4.5]; see also the proof of [12, Corollary 4.6].

If X is compact, then it is sufficient to consider when the first return map T_K induced on K by T is uniquely ergodic, because there exists a one-to-one correspondence between the set of T-invariant probability measures and the set of T_K-invariant probability measures. It follows from [11, Theorem IV. 13] that given a minimal homeomorphism S on a totally disconnected, compact metric space Y, S is uniquely ergodic if and only if for an arbitrarily chosen point $y \in Y$, it holds that for any nonempty, clopen set $F \subset Y$, there exists a constant c such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_F(S^{i+j}y) \to c \text{ as } n \to \infty, \text{ uniformly in } j \geq 0,$$

where χ_F is the characteristic function of F. We obtain the conclusion by applying this criterion with $S = T_K$, $y = \omega$ and $F = [v]$. □

Theorem 5.5 *Let*

$$X_\sigma = \bigcup_{i=1}^{n_\sigma} X_i \cup \bigcup_{i=2}^{n_\sigma} \mathrm{Orb}_{T_\sigma}(y_i) \cup \bigcup_{i=2}^{n_\sigma} \bigcup_{j=1}^{N_i} \mathrm{Orb}_{T_\sigma}(x_{ij})$$

be the decomposition (3.2). *The counting measure on each* $\mathrm{Orb}_{T_\sigma}(x_{ij})$ *is ergodic. The measure is finite if and only if the point* x_{ij} *is a fixed point of* T_σ. *The counting measure on each* $\mathrm{Orb}_{T_\sigma}(y_i)$ *is an ergodic, infinite measure.*

If $X_i \neq \emptyset$, *then an invariant Radon measure on* X_i *provided with the relative topology is unique up to scaling, and ergodic. This measure is finite if* $\theta_i > \lambda_{i-1}$, *and infinite if* $\theta_i \leq \lambda_{i-1}$.

Proof. In view of Lemma 4.1 (i) and Proposition 5.2, it is enough for us to show the uniqueness of an invariant measure under the assumption $\theta_i > 1$. Put $\{0 = k_0 < k_1 < k_2 < \cdots\} = \{k \geq 0; \omega_k \in A_i \setminus A_{i-1}\}$. Suppose that a word $v \in \mathcal{L}(\sigma_i)$ contains a letter in $A_i \setminus A_{i-1}$ as a factor. Put

$$\Delta_v = \frac{\delta_v}{\sum\limits_{w \in \mathcal{L}_m(\sigma_i) \setminus \mathcal{B}_m(i-1)} \delta_w}.$$

It is sufficient to prove that $N(v, \omega_{[k_j, k_{j+n}]})/n \to \Delta_v$ as $n \to \infty$, uniformly in $j \geq 0$. Let $\epsilon > 0$. Choose $p \in \mathbb{N}$ so that

$$3\Big\{\min_{a,b \in A_i \setminus A_{i-1}} N(a, \sigma^p(b))\Big\}^{-1} < \frac{1}{4}|v|^{-1}\epsilon;$$

$$\Big|N(v, \sigma^p(b)) - \Delta_v \sum_{a \in A_i \setminus A_{i-1}} N(a, \sigma^p(b))\Big|$$
$$< \frac{1}{4}\epsilon \sum_{a \in A_i \setminus A_{i-1}} N(a, \sigma^p(b)) \text{ for any } b \in A_i \setminus A_{i-1}.$$

Choose an integer $n_0 \geq 2\max_{a \in A} |\sigma^p(a)|$ so that for all integers $n \geq n_0$,

$$2\Delta_v n^{-1} \max_{a \in A} |\sigma^p(a)| + 2n^{-1} \max_{a \in A} |\sigma^p(a)| < \frac{1}{4}\epsilon.$$

Since ω and $\sigma^p(\omega)$ coincide up to a shift by some digits, for every integer $j \geq 0$ there exist $q \in \mathbb{N}$, $r \in \mathbb{N}$ and $s, t \in A^*$ such that

- s is a suffix of $\sigma^p(\omega_{q-1})$;
- t is a prefix of $\sigma^p(\omega_{q+r+1})$;
- $\omega_{[k_j, k_{j+n}]} = s\sigma^p(\omega_q)\sigma^p(\omega_{q+1}) \ldots \sigma^p(\omega_{q+r})t$.

Let $n \geq n_0$ and $j \geq 0$ be arbitrary integers. Since

$$N(v, \omega_{[k_j, k_{j+n}]}) \leq |s| + |t| + \sum_{\substack{q \leq l \leq q+r \\ \omega_l \in A_i \setminus A_{i-1}}} N(v, \sigma^p(\omega_l))$$
$$+ |v|\sharp\{q-1 \leq l \leq q+r+1; \omega_l \in A_i \setminus A_{i-1}\},$$

we obtain

$$\Big|N(v, \omega_{[k_j, k_{j+n}]}) - \sum_{\substack{q \leq l \leq q+r \\ \omega_l \in A_i \setminus A_{i-1}}} N(v, \sigma^p(\omega_l))\Big|$$
$$\leq 2\max_{a \in A} |\sigma^p(a)| + |v|\sharp\{q-1 \leq l \leq q+r+1; \omega_l \in A_i \setminus A_{i-1}\}.$$

However, since

$$
\begin{aligned}
&\frac{\sharp\{q-1 \le l \le q+r+1; \omega_l \in A_i \setminus A_{i-1}\}}{\sum\limits_{\substack{q \le l \le q+r \\ \omega_l \in A_i \setminus A_{i-1}}} \sum\limits_{a \in A_i \setminus A_{i-1}} N(a, \sigma^p(\omega_l))} \\
&\quad \le \frac{\sharp\{q-1 \le l \le q+r+1; \omega_l \in A_i \setminus A_{i-1}\}}{\sharp\{q \le l \le q+r; \omega_l \in A_i \setminus A_{i-1}\} \min\limits_{a,b \in A_i \setminus A_{i-1}} N(a, \sigma^p(b))} \\
&\quad \le 3\Big\{\min_{a,b \in A_i \setminus A_{i-1}} N(a, \sigma^p(b))\Big\}^{-1} \le \frac{1}{4}|v|^{-1}\epsilon,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\left|\frac{1}{n} N(v, \omega_{[k_j, k_{j+n}]}) - \frac{1}{n} \sum_{\substack{q \le l \le q+r \\ \omega_l \in A_i \setminus A_{i-1}}} N(v, \sigma^p(\omega_l))\right| \\
&\quad < \frac{1}{4}\epsilon + |v| \cdot \frac{1}{4}|v|^{-1}\epsilon \cdot \frac{1}{n} \sum_{\substack{q \le l \le q+r \\ \omega_l \in A_i \setminus A_{i-1}}} \sum_{a \in A_i \setminus A_{i-1}} N(a, \sigma^p(\omega_l)) \le \frac{1}{2}\epsilon.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
&\left|\frac{1}{n} \sum_{\substack{q \le l \le q+r \\ \omega_l \in A_i \setminus A_{i-1}}} N(v, \sigma^p(\omega_l)) - \Delta_v\right| \\
&\quad \le \frac{1}{n} \sum_{\substack{q \le l \le q+r \\ \omega_l \in A_i \setminus A_{i-1}}} \left|N(v, \sigma^p(\omega_l)) - \Delta_v \sum_{a \in A_i \setminus A_{i-1}} N(a, \sigma^p(\omega_l))\right| \\
&\qquad + 2\Delta_v n^{-1} \max_{a \in A} |\sigma^p(a)| \\
&\quad < \frac{1}{4}\epsilon \cdot \frac{1}{n} \sum_{\substack{q \le l \le q+r \\ \omega_l \in A_i \setminus A_{i-1}}} \sum_{a \in A_i \setminus A_{i-1}} N(a, \sigma^p(\omega_l)) + \frac{1}{4}\epsilon \le \frac{1}{2}\epsilon.
\end{aligned}
$$

Finally,

$$
\left|\frac{1}{n} N(v, \omega_{[k_j, k_{j+n}]}) - \Delta_v\right| < \epsilon.
$$

This completes the proof. □

We are now in a position to see the unique ergodicity left to be proved in Proposition 3.2. Assume the hypothesis of the proposition, so that $\theta_1 = 1$ and $\theta_2 > 1$. Let ω be the fixed point of σ as in the proposition. For this ω, the proof of Theorem 5.5 may reach the same conclusion, that is, the first return map of X_{σ_2} induced on $X_{\sigma_2} \setminus [a]$ is uniquely ergodic. This means the unique ergodicity of X_{σ_2}.

Corollary 5.6 *The subshift X_σ is uniquely ergodic if and only if one of the following holds*:

(i) $\lambda = \theta_1 > 1$;
(ii) $\theta_1 = 1$, $\lambda = \theta_2 > 1$, *and* $s^\infty \notin X_\sigma$, *where* $A_1 = \{s\}$.
(iii) $\lambda = 1$;

Proof. Assume that X_σ is uniquely ergodic. In view of Theorem 3.15, we first consider the case where $\lim_{n\to\infty} |\sigma^n(a)| = \infty$ for any $a \in A_1$. Since X_{σ_1} is the unique minimal set and $\theta_1 > 1$, Theorem 5.5 implies that $\theta_i < \theta_1$ for any integer i with $1 < i \le n_\sigma$. This corresponds to (i). We then consider the case where A_1 is a singleton, say $\{s\}$, and $s^\infty \notin X_\sigma$. Then, σ_2 must satisfy the hypothesis of Proposition 3.2, and hence $\theta_2 > 1$. Theorem 5.5 implies $\theta_2 > \theta_i$ for any integer i with $2 < i \le n_\sigma$. This corresponds to (ii). We then consider the case where A_1 is a singleton and σ_2 is almost primitive. In this case, $\{s^\infty\}$ is the unique minimal set, where $A_1 = \{s\}$. It follows from Theorem 5.5 again that $\theta_2 = \theta_3 = \cdots = \theta_{n_\sigma} = 1$. This corresponds to (iii). The converse implication is straightforward in view of Lemma 4.1 (i), Theorems 3.15 and 5.5. □

Among the examples studied above, uniquely ergodic systems are exactly (i), (ii) of Example 3.6, (i), (ii) of Example 3.10, and (i) of Example 4.5.

Remark 5.7 In [2], θ_i is said to be distinguished if $\theta_i > \lambda_{i-1}$. It follows from Lemma 4.1 (i) and Theorem 5.5 that the case $\theta_i = 1$ corresponds to the counting measure on $\mathrm{Orb}_{T_\sigma}(y_i)$ or $\mathrm{Orb}_{T_\sigma}(x_{ij})$. This kind of result is not obtained by [2]. Compare Theorem 5.5 and Corollary 5.6 with [2, Corollary 5.5].

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