

## An inverse scattering problem for the Klein-Gordon equation with a classical source in quantum field theory

Hironobu SASAKI and Akito SUZUKI

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**Abstract.** An inverse scattering problem for a quantized scalar field  $\phi$  obeying a linear Klein-Gordon equation

$$(\square + m^2 + V)\phi = J \quad \text{in } \mathbb{R} \times \mathbb{R}^3$$

is considered, where  $V$  is a repulsive external potential and  $J$  an external source. We prove that the scattering operator  $\mathcal{S} = \mathcal{S}(V, J)$  associated with  $\phi$  uniquely determines  $V$ . Assuming that  $J$  is of the form  $J(t, x) = j(t)\rho(x)$ ,  $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ , we represent  $\rho$  (resp.  $j$ ) in terms of  $j$  (resp.  $\rho$ ) and  $\mathcal{S}$ .

*Key words:* Quantum field theory, scattering theory, inverse scattering problem, external field problem.

### 1. Introduction

We consider an inverse scattering problem for a quantized scalar field  $\phi$  interacting with an external potential  $V$  and an external source  $J$  (see, e.g., [3], [5]) which obeys the Klein-Gordon equation

$$(\square + m^2 + V(x))\phi(t, x) = J(t, x) \quad \text{in } \mathbb{R} \times \mathbb{R}^3. \quad (1.1)$$

Here  $\square = \frac{\partial^2}{\partial t^2} - \Delta$ ,  $\Delta$  is the Laplacian in  $\mathbb{R}^3$ ,  $m > 0$  and  $J, V$  are real functions.  $\phi(t, x)$  and its conjugate field  $\pi(t, x) = \frac{\partial}{\partial t}\phi(t, x)$  are operator valued distributions (see, e.g., [8]). A typical example of (1.1) is the nucleon-pion interaction, that is,  $\phi$  describes the pion field and  $J$  the distribution function of the nucleons (see, e.g., [3]).

Under suitable conditions, one can show that the asymptotic fields

$$\phi_{\text{out/in}}(t, x) = \text{s-} \lim_{s \rightarrow \pm\infty} \phi_s(t, x), \quad (1.2)$$

$$\pi_{\text{out/in}}(t, x) = \text{s-} \lim_{s \rightarrow \pm\infty} \pi_s(t, x) \quad (1.3)$$

exist. Here  $\pi_s(t, x) = \frac{\partial}{\partial t} \phi_s(t, x)$  and  $\phi_s(t, x)$  is the solution of the free Klein-Gordon equation with the initial condition:  $\phi_s(s, x) = \phi(s, x)$  and  $\pi_s(s, x) = \pi(s, x)$ . Suppose that  $\phi_{\text{in}}(x) = \phi_{\text{in}}(0, x)$  and  $\pi_{\text{in}}(x) = \pi_{\text{in}}(0, x)$  give the Fock representation of the canonical commutation relations (CCR):

$$[\phi_{\text{in}}(x), \pi_{\text{in}}(y)] = i\delta(x - y), \quad (1.4)$$

$$[\phi_{\text{in}}(x), \phi_{\text{in}}(y)] = [\pi_{\text{in}}(x), \pi_{\text{in}}(y)] = 0. \quad (1.5)$$

See Section 3 for the detail. The scattering operator  $\mathcal{S} = \mathcal{S}(V, J)$  is defined by the following relations (up to a constant factor):

$$\mathcal{S}^{-1} \phi_{\text{in}}(x) \mathcal{S} = \phi_{\text{out}}(x), \quad \mathcal{S}^{-1} \pi_{\text{in}}(x) \mathcal{S} = \pi_{\text{out}}(x). \quad (1.6)$$

We prove that  $\mathcal{S}$  uniquely determines  $V$ . Suppose that  $J(t, x) = j(t)\rho(x)$ . Then we show that  $\rho$  (resp.  $j$ ) is uniquely determined by  $\mathcal{S}$  and  $j$  (resp.  $\rho$ ).

To state our results precisely, we introduce several assumptions. We set  $\mathfrak{h} = L^2(\mathbb{R}^3; dx)$  and assume the following:

**Assumption 1.1** *The potential function  $V : \mathbb{R}^3 \rightarrow \mathbb{R}$  is non negative and satisfies  $V \in H^2(\mathbb{R}^3)$ .*

Then the multiplication operator  $V$  acting in  $\mathfrak{h}$  is infinitesimally small with respect to  $h_0 = -\Delta$  since  $V \in L^2(\mathbb{R}^3)$ . Hence the operator

$$h = h_0 + V$$

is self-adjoint with the domain  $D(h) = D(h_0) = H^2(\mathbb{R}^3)$ . Since  $V$  is relative compact with respect to  $h_0$ , i.e.,  $V(h_0 + 1)^{-1}$  is compact, and  $V$  is positive, the spectrum of  $h$  is  $\sigma(h) = \sigma_{\text{ess}}(h) = [0, \infty)$ . The condition  $V \in H^2(\mathbb{R}^3)$  allows us to construct the solution of (1.1) by a Bogoliubov transformation (see Lemma 2.2 and Proposition 2.4).

We set

$$\omega = \varphi(h), \quad \omega_0 = \varphi(h_0),$$

where  $\varphi(s) = \sqrt{s + m^2}$ .

**Assumption 1.2** *We assume that  $h$  has no positive eigenvalue and*

$$\int_0^\infty dR \|V(x)(-\Delta + 1)^{-1}F(|x| \geq R)\| < \infty, \quad (1.7)$$

where

$$F(|x| \geq R) = \begin{cases} 1 & \text{if } |x| \geq R, \\ 0 & \text{if } |x| < R. \end{cases}$$

We make some comments on Assumption 1.2:

- By (1.7), the following limits exist

$$w_\pm := s\text{-}\lim_{t \rightarrow \pm\infty} e^{ith} e^{-ith_0}$$

and the intertwining property  $hw_\pm = w_\pm h_0$  holds. By Enss and Weder [2], we see that the scattering map for the Schrödinger operator defined by

$$\mathcal{V}_{\text{SR}} \ni V \mapsto S(V) = w_+^* w_-$$

is injective, where

$$\mathcal{V}_{\text{SR}} = \{V : \mathbb{R}^3 \rightarrow \mathbb{R} \mid V \text{ is Kato-small in } \mathfrak{h} \text{ and satisfies (1.7)}\}.$$

- Since  $h$  has no positive eigenvalue, (1.7) implies that  $h$  has purely absolutely continuous spectrum. In particular, we have  $w_\pm^* w_\pm = w_\pm w_\pm^* = I$  on  $\mathfrak{h}$ .
- We see that the following limits exist:

$$s\text{-}\lim_{t \rightarrow \pm\infty} e^{it\omega} e^{-it\omega_0}. \quad (1.8)$$

For a proof of the existence, see Appendix A.2. By [9, Theorem 1], above limits (1.8) are equal to  $w_\pm$ , respectively, i.e., the invariance principal holds for  $\varphi$ .

**Assumption 1.3** *The function  $J : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{R}$  satisfies*

- For each  $t \in \mathbb{R}$ , the function  $J_t(x) := J(t, x)$  satisfies  $J_t \in H^{-1/2}(\mathbb{R}^3)$ .
- The vector valued function  $\mathbb{R} \ni t \mapsto e^{-it\omega} \omega^{-1/2} J_t \in \mathfrak{h}$  satisfies

$$\int_{-\infty}^{\infty} dt \|\omega^{-1/2} J_t\|_{\mathfrak{h}} < \infty.$$

We say that  $V \in \mathcal{V}$  if  $V$  satisfies Assumptions 1.1 and 1.2 and that  $J \in \mathcal{J}$  if  $J$  satisfies Assumption 1.3.

**Theorem 1.1** *Suppose that  $V, V' \in \mathcal{V}$  and  $J, J' \in \mathcal{J}$ . If  $\mathcal{S}(V, J) = \mathcal{S}(V', J')$ , then:*

- (i)  $S(V) = S(V')$ ,
- (ii)  $V = V'$ ,
- (iii)  $\int_{-\infty}^{+\infty} ds e^{-is\omega} J_t = \int_{-\infty}^{+\infty} ds e^{-is\omega} J'_t$ .

By the above theorem, we immediately see the following:

**Corollary 1.2** *Let  $J \in \mathcal{J}$  be given. Then the map  $\mathcal{V} \ni V \mapsto \mathcal{S}(V, J)$  is injective.*

*Proof of Theorem 1.1.* (i) will be proved in Theorem 4.1. Since  $\mathcal{V} \subset \mathcal{V}_{\text{SR}}$ , (ii) follows from the injectivity of the map  $V \mapsto S(V)$ . We will prove (iii) in Proposition 4.5.  $\square$

In order to recover the external source  $J$ , we henceforth suppose that  $J \in \mathcal{J}$  is expressed by

$$J(t, x) = j(t) \times \rho(x), \quad (1.9)$$

where  $j \in L^1(\mathbb{R})$  and  $\rho \in H^{-1/2}(\mathbb{R}^3)$ . Let

$$F(t, f) = (\Omega_{\text{in}}, \phi_{\text{out}}(t, f) \Omega_{\text{in}}), \quad f \in \mathcal{S}(\mathbb{R}^d),$$

where  $\Omega_{\text{in}}$  is the Fock vacuum. From (1.6), we see that  $F(t, f)$  is uniquely determined by  $\mathcal{S}$ . One can show the following:

- (1) For a given function  $j$  such that the Fourier transform  $\hat{j}$  of  $j$  is real analytic, we represent  $\rho$  in terms of  $j$  and  $F(t, f)$ . See Theorem 4.6 for the detail.
- (2) Let  $\rho$  be a given function and assume that  $j$  satisfies the following:

$$\text{For some } \delta > 0, \quad e^{\delta|t|} j(t) \in L^1(\mathbb{R}_t). \quad (1.10)$$

Then we express  $j$  by means of  $\rho$  and  $F(t, f)$ . See Theorem 4.8 for the detail.

From the reconstruction formulas above, we observe that the scattering operator  $\mathcal{S}(V, j \times \rho)$  uniquely determines  $j$  and  $\rho$ :

**Theorem 1.3** *Let  $V \in \mathcal{V}$  and  $j \times \rho, j' \times \rho' \in \mathcal{J}$ . Suppose that  $j, j' \in L^1(\mathbb{R})$  and  $\rho, \rho' \in H^{-1/2}(\mathbb{R})$ . Then:*

- (i) *Assume that  $\hat{j}$  is real analytic. Then  $\rho = \rho'$  if  $\mathcal{S}(V, j \times \rho) = \mathcal{S}(V, j \times \rho')$ .*
- (ii) *Assume that (1.10) holds. Then  $j = j'$  if  $\mathcal{S}(V, j \times \rho) = \mathcal{S}(V, j' \times \rho)$ .*

*Proof.* See Theorem 4.6.  $\rho$  is uniquely determined by  $j$  and a function  $z$  defined in (4.6).  $z$  is uniquely determined by  $F(t, f)$ . Since, as was noted above,  $\mathcal{S}$  determines  $F(t, f)$  uniquely, we see that  $\rho \mapsto \mathcal{S}(V, j \times \rho)$  is injective. Hence (i) holds. (ii) is proved similarly.  $\square$

This paper is organized as follows. Section 2 is devoted to some mathematical preliminaries. In Subsections 2.1 and 2.2, we review well-known facts. The quantized Klein-Gordon field is constructed in Subsection 2.3 and the wave operator in Subsection 2.4. In Section 3, we discuss the scattering theory and define the scattering operator  $\mathcal{S}$ . Section 4 deals with the inverse scattering problem. In Subsection 4.1, we show the uniqueness of  $V$ . The reconstruction formulas of  $\rho$  and  $j$  are given in Subsections 4.2 and 4.3, respectively. In Appendix, we prove Lemmas 2.2 and 2.6.

## 2. Preliminary

In general we denote the inner product and the associated norm of a Hilbert space  $\mathcal{L}$  by  $(*, \cdot)_{\mathcal{L}}$  and  $\|\cdot\|_{\mathcal{L}}$ , respectively. The inner product is linear in  $\cdot$  and antilinear in  $*$ . If there is no danger of confusion, we omit the subscript  $\mathcal{L}$  in  $(\cdot, \cdot)_{\mathcal{L}}$  and  $\|\cdot\|_{\mathcal{L}}$ . For a linear operator  $T$  on  $\mathcal{L}$ , we denote the domain of  $T$  by  $D(T)$  and, if  $D(T)$  is dense in  $\mathcal{H}$ , the adjoint of  $T$  by  $T^*$ .

### 2.1. Boson Fock space

We first recall the abstract Boson Fock space and operators therein. The Boson Fock space over a Hilbert space  $\mathfrak{h}$  is defined by

$$\begin{aligned} \Gamma(\mathfrak{h}) &= \bigoplus_{n=0}^{\infty} \bigotimes_s^n \mathfrak{h} \\ &= \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \mid \Psi^{(n)} \in \bigotimes_s^n \mathfrak{h}, \sum_{n=0}^{\infty} \|\Psi^{(n)}\|_{\bigotimes_s^n \mathfrak{h}}^2 < \infty \right\}, \end{aligned}$$

where  $\bigotimes_s^n \mathfrak{h}$  denotes the symmetric tensor product of  $\mathfrak{h}$  with the convention  $\bigotimes_s^0 \mathfrak{h} = \mathbb{C}$ .

The creation operator  $c^*(f)$  ( $f \in \mathfrak{h}$ ) acting in  $\Gamma(\mathfrak{h})$  is defined by

$$(c^*(f)\Psi)^{(n)} = \sqrt{n} S_n(f \otimes \Psi^{(n-1)})$$

with the domain

$$D(c^*(f)) = \left\{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \mid \sum_{n=0}^{\infty} n \|S_n(f \otimes \Psi^{(n-1)})\|_{\bigotimes_s^n \mathfrak{h}}^2 < \infty \right\},$$

where  $S_n$  denotes the symmetrization operator on  $\bigotimes_s^n \mathfrak{h}$  satisfying  $S_n = S_n^* = S_n^2$  and  $S_n(\bigotimes_s^n \mathfrak{h}) = \bigotimes_s^n \mathfrak{h}$ .

The annihilation operator  $c(f)$  ( $f \in \mathfrak{h}$ ) is defined by the adjoint of  $c^*(f)$ , i.e.,  $c(f) := c^*(f)^*$ . By definition,  $c^*(f)$  (resp.  $c(f)$ ) is linear (resp. anti-linear) in  $f \in \mathfrak{h}$ . As is well known, the creation and annihilation operators leave the finite particle subspace

$$D_f = \bigcup_{m=1}^{\infty} \{ \Psi = \{\Psi^{(n)}\}_{n=0}^{\infty} \mid \Psi^{(n)} = 0, \quad n \geq m \}$$

invariant. The canonical commutation relations (CCR) hold on  $D_f$ :

$$[c(f), c^*(g)] = (f, g)_{\mathfrak{h}}, \quad [c(f), c(g)] = [c^*(f), c^*(g)] = 0. \quad (2.1)$$

It follows from (2.1) that

$$\|c^*(f)\Psi\|^2 = \|f\|^2 \|\Psi\|^2 + \|c(f)\Psi\|^2, \quad \Psi \in D_f. \quad (2.2)$$

The Segal field operator  $\tau(f) = \frac{1}{\sqrt{2}}(c(f) + c^*(f))$  ( $f \in \mathfrak{h}$ ) is essentially self-adjoint on  $D_f$ . We denote its closure by the same symbol. By (2.1), the following equation holds

$$\|c^*(f)\Psi\|^2 = \frac{1}{2}(\|\tau(f)\Psi\|^2 + \|\tau(if)\Psi\|^2 + \|f\|^2\|\Psi\|^2), \quad \Psi \in D_f. \quad (2.3)$$

Since  $D_f$  is a core for  $c(f)$ ,  $c^*(f)$  and  $\tau(f)$  ( $f \in \mathfrak{h}$ ), we observe from (2.2) and (2.3) that

$$D(\tau(f)) \cap D(\tau(if)) = D(c(f)) = D(c^*(f)). \quad (2.4)$$

Hence the following operator equalities hold true:

$$\begin{aligned} c(f) &= \frac{1}{\sqrt{2}}(\tau(f) + i\tau(if)), \\ c^*(f) &= \frac{1}{\sqrt{2}}(\tau(f) - i\tau(if)). \end{aligned}$$

Let

$$\tilde{D} := \bigcap_{f \in \mathfrak{h}} D(c(f)).$$

Since  $\tilde{D} \supset D_f$ ,  $\tilde{D}$  is dense in  $\Gamma(\mathfrak{h})$ . From (2.4), we observe that

$$\tilde{D} = \bigcap_{f \in \mathfrak{h}} D(c^*(f)) = \bigcap_{f \in \mathfrak{h}} D(\tau(f)). \quad (2.5)$$

The Fock vacuum  $\Omega = \{\Omega^{(n)}\}_{n=0}^\infty \in \Gamma(\mathfrak{h})$  is defined by  $\Omega^{(0)} = 1$  and  $\Omega^{(n)} = 0$  ( $n \geq 1$ ), which satisfies

$$c(f)\Omega = 0, \quad f \in \mathfrak{h}. \quad (2.6)$$

$\Omega$  is a unique vector satisfying (2.6) up to a constant factor.

Let  $A$  be a contraction operator on  $\mathfrak{h}$ , i.e.,  $\|A\| \leq 1$ . We define a contraction operator  $\Gamma(A)$  on  $\Gamma(\mathfrak{h})$  by

$$(\Gamma(A)\Psi)^{(n)} = (\otimes^n A)\Psi^{(n)}, \quad \Psi = \{\Psi^{(n)}\}_{n=0}^\infty$$

with the convention  $\otimes^0 A = 1$ . If  $U$  is unitary, i.e.  $U^{-1} = U^*$ , then  $\Gamma(U)$  is also unitary and satisfies  $\Gamma(U)^* = \Gamma(U^*)$  and

$$\Gamma(U)c(f)\Gamma(U)^* = c(Uf), \quad \Gamma(U)c^*(f)\Gamma(U)^* = c^*(Uf).$$

For a self-adjoint operator  $T$  on  $\mathfrak{h}$ , i.e.,  $T = T^*$ ,  $\{\Gamma(e^{itT})\}_{t \in \mathbb{R}}$  is a strongly continuous one-parameter unitary group on  $\Gamma(\mathfrak{h})$ . Then, by the Stone theorem, there exists a unique self-adjoint operator  $d\Gamma(T)$  such that

$$\Gamma(e^{itT}) = e^{itd\Gamma(T)}.$$

The number operator  $N_f$  is defined by  $d\Gamma(1)$ .

## 2.2. Bogoliubov transformations

Let  $\mathcal{H}$  be the direct sum of two Hilbert spaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$ , where  $\mathcal{H}_+ = \mathfrak{h}$  and  $\mathcal{H}_-$  is a copy of it:

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- = \left\{ v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \mid v_+, v_- \in \mathfrak{h} \right\}.$$

We denote by  $P_+$  (resp.  $P_-$ ) the projection from  $\mathcal{H}$  onto  $\mathcal{H}_+$  (resp.  $\mathcal{H}_-$ ):

$$P_+ \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} v_+ \\ 0 \end{bmatrix}, \quad P_- \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} 0 \\ v_- \end{bmatrix}.$$

A vector  $v \in \text{Ran}P_+ = \mathcal{H}_+$  is identified with one in  $\mathfrak{h}$ :  $\begin{bmatrix} v_+ \\ 0 \end{bmatrix} = v_+ \in \mathfrak{h}$ . We define an involution  $Q$  on  $\mathcal{H}$  by

$$Q = P_+ - P_-$$

and a conjugation  $C$  on  $\mathcal{H}$  by

$$C \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} \bar{v}_- \\ \bar{v}_+ \end{bmatrix},$$

where  $\bar{f}$  stands for the complex conjugation of  $f$  in  $\mathfrak{h}$ , i.e.,  $\bar{f}(x) = \overline{f(x)}$ , a.e.  $x \in \mathbb{R}^3$ .

A bounded operator  $A$  on  $\mathcal{H}$  is written as

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix},$$

where  $A_{\epsilon\epsilon'} = P_\epsilon A P_{\epsilon'}$  ( $\epsilon, \epsilon' = +$  or  $-$ ). Then we observe that

$$A \begin{bmatrix} v_+ \\ v_- \end{bmatrix} = \begin{bmatrix} A_{++}v_+ + A_{+-}v_- \\ A_{-+}v_+ + A_{--}v_- \end{bmatrix}$$

and  $A_{\epsilon\epsilon'}^* = (A_{\epsilon\epsilon'})^*$ . We introduce field operators defined on  $\tilde{D}$  by

$$\begin{aligned} \psi(v) &= c(P_+v) + c^*(CP_-v) \\ &= c(v_+) + c^*(\bar{v}_-), \quad v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \in \mathcal{H}. \end{aligned}$$

One observes that  $\psi(v)^* = \psi(Cv)$  on  $\tilde{D}$  and hence that  $\psi(v)$  is closable. We denote its closure by the same symbol. Let  $\mathcal{H}_C$  be the set of vectors satisfying  $Cv = v$ :

$$\mathcal{H}_C = \left\{ v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \mid v_+ = \bar{v}_- \in \mathfrak{h} \right\}.$$

Clearly, for any  $v = \begin{bmatrix} f \\ \bar{f} \end{bmatrix} \in \mathcal{H}_C$  ( $f \in \mathfrak{h}$ ), the operator  $\psi(v)$  is essentially self-adjoint on  $D_f$  and is equal to  $\sqrt{2}\tau(f)$ . By (2.5), we see that

$$\tilde{D} = \bigcap_{v \in \mathcal{H}_C} D(\psi(v)).$$

Note that, for any  $v = \begin{bmatrix} v_+ \\ v_- \end{bmatrix} \in \mathcal{H}$ , the vectors

$$v + Cv = \begin{bmatrix} v_+ + \bar{v}_- \\ v_- + \bar{v}_+ \end{bmatrix}, \quad i(v - Cv) = \begin{bmatrix} iv_+ - i\bar{v}_- \\ iv_- - i\bar{v}_+ \end{bmatrix}$$

belong to  $\mathcal{H}_C$  and the following holds on  $\tilde{D}$ :

$$\psi(v) = \frac{1}{2}\psi(v + Cv) + \frac{i}{2}\psi(i(v - Cv)).$$

It is straightforward from (2.1) that

$$[\psi(u), \psi(v)^*] = (u, Qv)$$

holds on  $D_f$ . The following is well known (see, e.g., [7]):

**Lemma 2.1**

(1) Let  $U$  be a bounded operator on  $\mathcal{H}$  satisfying

$$CU = UC, \quad UQU^* = U^*QU = Q. \quad (2.7)$$

Then there exists a unitary operator  $\mathcal{U}$  such that

$$e^{i\psi(U^*v)} = \mathcal{U}^* e^{i\psi(v)} \mathcal{U}, \quad \mathcal{U} e^{i\psi(v)} \mathcal{U}^* = e^{i\psi(QUQv)}, \quad v \in \mathcal{H}_C$$

if and only if  $U_{-+}$  is Hilbert-Schmidt. In this case,  $\mathcal{U}$  leaves  $\tilde{D}$  invariant.

(2) Let  $l$  be a linear functional from  $\mathcal{H}$  to  $\mathbb{C}$ . Then there exists a unitary operator  $\mathcal{U}_l$  such that

$$e^{i(\psi(u)+l(u))} = \mathcal{U}_l^* e^{i\psi(u)} \mathcal{U}_l, \quad u \in \mathcal{H}_C$$

if and only if there exists a  $u_l \in \mathcal{H}_C$  such that

$$l(u) = i(v_l, Qu), \quad u \in \mathcal{H}_C.$$

In this case,  $\mathcal{U}_l$  leaves  $\tilde{D}$  invariant and  $\mathcal{U}_l = e^{-i\psi(v_l)}$ .

**2.3. Quantized Klein Gordon equation**

Let  $\mathfrak{h} = L^2(\mathbb{R}^3; dx)$  and  $h_0 = -\Delta$  with the domain  $D(h_0) = D(-\Delta) = H^2(\mathbb{R}^3)$ . Then we define the Schrödinger operator  $h$  by

$$h = -\Delta + V(x) \quad (2.8)$$

with the potential  $V : \mathbb{R}^3 \mapsto \mathbb{R}$  satisfying Assumption 1.1.  $h$  is self-adjoint with the domain  $D(h) = D(h_0)$ . Let

$$\omega_0 := (h_0 + m^2)^{1/2} \quad \text{and} \quad \omega := (h + m^2)^{1/2}.$$

The free field Hamiltonian  $H_f$  is defined by

$$H_f = d\Gamma(\omega_0).$$

Since  $\omega_0$  and  $\omega$  are strictly positive,  $\omega_0^{-1}$  and  $\omega^{-1}$  are bounded. Note that

$\omega_0^\theta \omega^{-\theta}$  and  $\omega^\theta \omega_0^{-\theta}$  are bounded operators for any  $0 \leq \theta \leq 1$ . Indeed, since  $h_0 \leq h$ , we observe that  $\|\omega_0^\theta f\| \leq \|\omega^\theta f\|$ . Hence  $\omega_0^\theta \omega^{-\theta}$  is bounded. On the other hand, since  $D(h) = D(h_0)$ , it follows from the closed graph theorem that  $\|hf\| \leq C\|(h_0 + 1)f\|$  with some  $C > 0$ . Hence we observe that  $\omega^\theta \omega_0^{-\theta}$  is bounded. From this fact, we see that  $\omega^{-\theta} \omega_0^\theta$  and  $\omega_0^{-\theta} \omega^\theta$  can be extended to bounded operators on  $\mathfrak{h}$ . We denote the extended operators by the same symbols. The following holds.

**Lemma 2.2** *Suppose that Assumption 1.1 holds. Then the operators  $\omega_0^{1/2} \omega^{-1/2} - 1$ ,  $\omega_0^{-1/2} \omega^{1/2} - 1$  and  $\omega_0^{-1/2} (\omega_0 - \omega) \omega_0^{-1/2}$  are Hilbert-Schmidt.*

*Proof.* See Appendix A.1. □

For real  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ , we set

$$\phi_0(f) = \psi(u_0) \quad \text{and} \quad \pi_0(g) = \psi(v_0),$$

where

$$u_0 = \begin{bmatrix} \omega_0^{-1/2} f / \sqrt{2} \\ \omega_0^{-1/2} f / \sqrt{2} \end{bmatrix} \quad \text{and} \quad v_0 = \begin{bmatrix} i\omega_0^{+1/2} g / \sqrt{2} \\ -i\omega_0^{+1/2} g / \sqrt{2} \end{bmatrix}.$$

For non real  $f \in H^{-1/2}(\mathbb{R}^3)$  (resp.  $g \in H^{1/2}(\mathbb{R}^3)$ ),  $\phi_0(f)$  (resp.  $\pi_0(g)$ ) is defined by  $\phi_0(\operatorname{Re}f) + i\phi_0(\operatorname{Im}f)$  (resp.  $\pi_0(\operatorname{Re}g) + i\pi_0(\operatorname{Im}g)$ ). Note that for non real  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ ,  $\phi_0(f)$  and  $\pi_0(g)$  are non self-adjoint and the following equations hold on  $\tilde{D}$ :

$$\begin{aligned} \phi_0(f) &= \frac{1}{\sqrt{2}} (c^*(\omega_0^{-1/2} f) + c(\omega_0^{-1/2} \bar{f})), \\ \pi_0(g) &= \frac{i}{\sqrt{2}} (c^*(\omega_0^{+1/2} g) - c(\omega_0^{+1/2} \bar{g})). \end{aligned}$$

By (2.1), one can show that the CCR holds on  $D_{\mathfrak{f}}$ :

$$[\phi_0(f), \pi_0(g)] = i(\bar{f}, g), \quad [\phi_0(f), \phi_0(\tilde{f})] = [\pi_0(g), \pi_0(\tilde{g})] = 0, \quad (2.9)$$

for any  $f, \tilde{f} \in H^{-1/2}(\mathbb{R}^3)$  and  $g, \tilde{g} \in H^{1/2}(\mathbb{R}^3)$ . It holds from (2.9) that

$$\begin{aligned}\|\phi_0(f)\Psi\|^2 &= \|\phi_0(\operatorname{Re}f)\Psi\|^2 + \|\phi_0(\operatorname{Re}f)\|^2, \\ \|\pi_0(g)\Psi\|^2 &= \|\pi_0(\operatorname{Re}g)\Psi\|^2 + \|\pi_0(\operatorname{Re}g)\|^2, \quad \Psi \in D_f.\end{aligned}$$

Hence we observe that, for non real  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ ,  $\phi_0(f)$  and  $\pi_0(g)$  are closed on the natural domain.

We introduce the bounded operator on  $\mathcal{H}$  by

$$U(t) = \begin{bmatrix} U_{++}(t) & U_{+-}(t) \\ U_{-+}(t) & U_{--}(t) \end{bmatrix}, \quad t \in \mathbb{R},$$

where

$$\begin{aligned}U_{++}(t) &= \frac{1}{2}(\omega_0^{-1/2} \cos(t\omega)\omega_0^{1/2} + \omega_0^{1/2} \cos(t\omega)\omega_0^{-1/2}) \\ &\quad - \frac{i}{2}(\omega_0^{1/2} \omega^{-1/2} \sin(t\omega)\omega^{-1/2}\omega_0^{1/2} + \omega_0^{-1/2} \omega^{1/2} \sin(t\omega)\omega^{1/2}\omega_0^{-1/2})\end{aligned}$$

and

$$\begin{aligned}U_{-+}(t) &= \frac{1}{2}(\omega_0^{1/2} \cos(t\omega)\omega_0^{-1/2} - \omega_0^{-1/2} \cos(t\omega)\omega_0^{1/2}) \\ &\quad - \frac{i}{2}(\omega_0^{1/2} \omega^{-1/2} \sin(t\omega)\omega^{-1/2}\omega_0^{1/2} - \omega_0^{-1/2} \omega^{1/2} \sin(t\omega)\omega^{1/2}\omega_0^{-1/2})\end{aligned}$$

with  $U_{--}(t) = \overline{U_{++}(t)}$  and  $U_{+-}(t) = \overline{U_{-+}(t)}$ . Here, for a linear operator  $A$ , we define  $\bar{A}$  by  $\bar{A}f = \overline{Af}$ .

**Lemma 2.3** *Suppose that Assumption 1.1 holds. Then there exists a family of unitary operators  $\mathcal{U}_t$  on  $\Gamma(\mathfrak{h})$  such that  $\mathcal{U}_t$  maps  $\tilde{D}$  to  $\tilde{D}$  and for  $v \in \mathcal{H}_C$*

$$e^{i\psi(U(t)^*v)} = \mathcal{U}_t^* e^{i\psi(v)} \mathcal{U}_t, \quad \mathcal{U}_t e^{i\psi(v)} \mathcal{U}_t^* = e^{i\psi(QU(t)Qv)}.$$

*Proof.* By direct calculation, we observe that  $U(t)$  satisfies (2.7) with  $U = U(t)$ . We note that  $2U_{-+}(t)$  is equal to

$$\begin{aligned}(\omega_0^{1/2} \omega^{-1/2} - 1) \cos(t\omega)\omega^{1/2}\omega_0^{-1/2} \\ + \cos(t\omega) \cdot \omega^{-1/2}\omega_0^{1/2} \cdot \omega_0^{-1/2}(\omega - \omega_0)\omega_0^{-1/2}\end{aligned}$$

$$\begin{aligned}
& + (1 - \omega_0^{-1/2} \omega^{1/2}) \cos(t\omega) \omega^{-1/2} \omega_0^{1/2} \\
& - i(\omega_0^{1/2} \omega^{-1/2} - 1) \sin(t\omega) \cdot \omega^{-1/2} \omega_0^{1/2} \\
& - i \sin(t\omega) \cdot \omega^{-1/2} \omega_0^{1/2} \cdot \omega_0^{-1/2} (\omega_0 - \omega) \omega_0^{-1/2} \\
& - i(1 - \omega_0^{-1/2} \omega^{1/2}) \sin(t\omega) \cdot \omega^{1/2} \omega_0^{-1/2}.
\end{aligned}$$

By Lemma 2.2,  $\omega_0^{1/2} \omega^{-1/2} - 1$ ,  $\omega_0^{-1/2} \omega^{1/2} - 1$  and  $\omega_0^{-1/2} (\omega_0 - \omega) \omega_0^{-1/2}$  are Hilbert-Schmidt, so is  $U_{-+}(t)$ .  $\square$

Let

$$\begin{aligned}
g_t &= -\frac{1}{\sqrt{2}} \int_0^t ds \omega_0^{-1/2} \cos[(t-s)\omega] J_s \\
&+ \frac{i}{\sqrt{2}} \int_0^t ds \omega_0^{1/2} \frac{\sin[(t-s)\omega]}{\omega} J_s
\end{aligned}$$

and

$$j_t = \begin{bmatrix} g_t \\ \bar{g}_t \end{bmatrix}.$$

For  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ , we define field operators

$$\phi(t, f) = \mathcal{U}(t)^* \phi_0(f) \mathcal{U}(t) \quad \text{and} \quad \pi(t, g) = \mathcal{U}(t)^* \pi_0(g) \mathcal{U}(t),$$

where

$$\mathcal{U}(t) = e^{-i\psi(j_t)} \mathcal{U}_t.$$

The following propositions are standard:

**Proposition 2.4** *Suppose that Assumptions 1.1 and 1.3 hold. Then it holds that,  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ ,*

$$\begin{aligned}
\phi(t, f) &= \psi(u(t)) + \left( \int_0^t ds \frac{\sin[(t-s)\omega]}{\omega} J_s, f \right), \\
\pi(t, f) &= \psi(v(t)) - \left( \int_0^t ds \cos[(t-s)\omega] J_s, f \right),
\end{aligned}$$

where

$$u(t) = \begin{bmatrix} \omega_0^{-1/2}(\cos(t\omega) + i\omega_0 \sin(t\omega)\omega^{-1})\bar{f}/\sqrt{2} \\ \omega_0^{-1/2}(\cos(t\omega) - i\omega_0 \sin(t\omega)\omega^{-1})\bar{f}/\sqrt{2} \end{bmatrix},$$

$$v(t) = \begin{bmatrix} i\omega_0^{1/2}(\cos(t\omega) + i\omega_0^{-1} \sin(t\omega)\omega)\bar{f}/\sqrt{2} \\ -i\omega_0^{1/2}(\cos(t\omega) - i\omega_0^{-1} \sin(t\omega)\omega)\bar{f}/\sqrt{2} \end{bmatrix}.$$

**Proposition 2.5** *Suppose that Assumptions 1.1 and 1.3 hold. Let  $\Psi \in D(N_f^{1/2})$  and  $f \in H^{-1/2}(\mathbb{R}^3) \cap H^{3/2}(\mathbb{R}^3)$ . Then  $\phi(0, f) = \phi_0(f)$ ,  $\pi(0, f) = \pi_0(f)$  and*

$$\frac{d}{dt}\phi(t, f)\Psi = \pi(t, f)\Psi,$$

$$\frac{d^2}{dt^2}\phi(t, f)\Psi + \phi(t, (m^2 - \Delta)f)\Psi = (J_t, f).$$

**Remark 2.1** Let  $\Psi \in D(N_f^{1/2})$ . Then  $\mathcal{S}(\mathbb{R}^d) \ni f \mapsto (\Psi, \phi(t, f)\Psi)$  and  $\mathcal{S}(\mathbb{R}^d) \ni g \mapsto (\Psi, \pi(t, g)\Psi)$  are tempered distributions and symbols  $\varphi_\Psi(t, x)$  and  $\varpi_\Psi(t, x)$ , defined formally as

$$\int \varphi_\Psi(t, x)f(x)dx = (\Psi, \pi(t, f)\Psi) \quad \text{and}$$

$$\int \varpi_\Psi(t, x)f(x)dx = (\Psi, \phi(t, f)\Psi),$$

satisfy

$$\varphi_\Psi(t, x) \in H^{1/2}(\mathbb{R}^3), \quad \varpi_\Psi(t, x) \in H^{-1/2}(\mathbb{R}^3).$$

We denote by  $\Gamma_0$  the linear hull of

$$\{\Omega\} \cup \{c^*(f_1) \cdots c^*(f_n)\Omega \mid f_j \in D(\omega_0), j = 1, \dots, n, n \geq 1\}.$$

Note that  $\Gamma_0$  is dense in  $\Gamma(\mathfrak{h})$  and that  $\phi(t, f)$  and  $\pi(t, f)$  ( $f \in \mathcal{S}(\mathbb{R}^3)$ ) leave  $\Gamma_0$  invariant. Suppose that  $J_t \in H^{1/2}(\mathbb{R}^3)$  holds. If  $\Psi$  is a vector belonging to  $\Gamma_0$ , then  $\varphi_\Psi(t) \in H^{3/2}(\mathbb{R}^3)$ ,  $\varpi_\Psi(t) \in H^{1/2}(\mathbb{R}^3)$  and

$$\begin{aligned} \begin{bmatrix} \varphi_{\Psi}(t) \\ \varpi_{\Psi}(t) \end{bmatrix} &= \begin{bmatrix} \cos(t\omega) & \omega^{-1} \sin(t\omega) \\ -\omega \sin(t\omega) & \cos(t\omega) \end{bmatrix} \begin{bmatrix} \varphi_{\Psi}(0) \\ \varpi_{\Psi}(0) \end{bmatrix} \\ &+ \begin{bmatrix} \int_0^t ds \sin[(t-s)\omega] \omega^{-1} J_s \\ -\int_0^t ds \cos[(t-s)\omega] J_s \end{bmatrix}, \end{aligned}$$

which gives a solution of

$$i \frac{d}{dt} \begin{bmatrix} \varphi(t) \\ \varpi(t) \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i\omega^2 & 0 \end{bmatrix} \begin{bmatrix} \varphi(t) \\ \varpi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ J_t \end{bmatrix} \quad (2.10)$$

with the initial value

$$\begin{bmatrix} \varphi(0) \\ \varpi(0) \end{bmatrix} = \begin{bmatrix} \varphi_{\Psi}(0) \\ \varpi_{\Psi}(0) \end{bmatrix}. \quad (2.11)$$

We note that  $\Psi \in \Gamma_{\infty}$  belongs to  $D(N^{1/2})$  and is an analytic vector of  $\phi(t, f)$  and  $\pi(t, f)$ , i.e., for any  $0 < t \leq t_0$ ,

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \|\phi(t, f)^n \Psi\| < \infty, \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} \|\pi(t, f)^n \Psi\| < \infty$$

with some  $t_0 > 0$ .

**Remark 2.2** Suppose that  $J_t \in H^{1/2}(\mathbb{R}^3)$  holds. Then the pair of field operators  $\phi(t, f)$  and  $\pi(t, f)$  is unique in the following sense: If there exist a pair of field operators  $\phi'(t, f) = \mathcal{U}'(t)^* \phi_0(f) \mathcal{U}'(t)$  and  $\pi'(t, f) = \mathcal{U}'(t)^* \pi_0(f) \mathcal{U}'(t)$  with a family of unitary operators  $\mathcal{U}'(t)$  satisfying the following conditions (1)–(4), then  $\phi(t, f) = \phi'(t, f)$  and  $\pi(t, f) = \pi'(t, f)$ :

- (1)  $\mathcal{U}'(0) = I$ .
- (2)  $\phi'(t, f)$  and  $\pi'(t, f)$  leave  $\Gamma_0$  invariant.
- (3) The vector  $\Psi \in \Gamma_0$  is an analytic vector of  $\phi'(t, f)$  and  $\pi'(t, f)$ .
- (4) The distributional kernels  $\varphi'_{\Psi}(t)$  and  $\varpi'_{\Psi}(t)$  of

$$\begin{aligned} \varphi'_{\Psi}(t, f) &= (\Psi, \phi'(t, f) \Psi), \\ \varpi'_{\Psi}(t, f) &= (\Psi, \pi'(t, f) \Psi) \end{aligned}$$

with  $\Psi \in \Gamma_0$  satisfy the equation (2.10) with the initial value (2.11).

The uniqueness can be proved as follows. Using the conditions (1), (4) and the uniqueness of the solution of (2.10), we infer

$$\phi(t, f) = \phi'(t, f), \quad \pi(t, g) = \pi'(t, g)$$

on  $\Gamma_0$ . By the conditions (2) and (3), we have

$$e^{it\phi(t,f)}\Psi = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \phi(t, f)^n \Psi = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \phi'(t, f)^n \Psi = e^{i\phi'(t,f)}\Psi$$

for  $\Psi \in \Gamma_0$  and sufficiently small  $t > 0$ . Since  $\Gamma_0$  is dense, we observe that  $e^{i\psi(t,f)} = e^{i\phi'(t,f)}$  for real  $f \in H^{-1/2}(\mathbb{R}^3)$ . By the uniqueness of the generator, we have the operator equality  $\psi(t, f) = \phi'(t, f)$  for real  $f \in H^{-1/2}(\mathbb{R}^3)$ . Note that  $\phi(t, f)$  and  $\phi'(t, f)$  are unitary equivalent to  $\phi_0(f)$ . By the similar argument as in the proof of the closedness of  $\phi_0(f)$ , one can prove that the following operator equation holds for non real  $f \in H^{-1/2}(\mathbb{R}^3)$ :

$$\begin{aligned} \phi(t, f) &= \phi(t, \operatorname{Re}f) + i\phi(t, \operatorname{Im}f) \\ &= \phi'(t, \operatorname{Re}f) + i\phi'(t, \operatorname{Im}f) = \phi'(t, f). \end{aligned}$$

$\pi(t, f) = \pi'(t, f)$  is proved similarly.

## 2.4. Wave operators

Let

$$\mathcal{U}_0(t) = e^{-itH_t}$$

and

$$U_0(t) = \begin{bmatrix} e^{-it\omega_0} & 0 \\ 0 & e^{it\omega_0} \end{bmatrix}.$$

One observe that

$$\mathcal{W}(t) = \mathcal{U}(t)^* \mathcal{U}_0(t)$$

satisfies

$$\mathcal{W}(t)e^{i\psi(v)}\mathcal{W}(t)^* = \mathcal{W}_t^* e^{i(\psi(U_0(t)v) + i(j_t, QU_0(t)v))} \mathcal{W}_t \quad (2.12)$$

$$= e^{i\psi((U(t)^*U_0(t)v) + i(U_0(t)^*j_t, Qv))} \quad (2.13)$$

for  $v \in \mathcal{H}_C$ . By the Stone theorem, we have

$$\mathscr{W}(t)\psi(v)\mathscr{W}(t)^* = \psi(U(t)^*U_0(t)v) + i(U_0(t)^*j_t, Qv). \quad (2.14)$$

**Lemma 2.6** *Suppose that Assumptions 1.1 and 1.2 hold. Then:*

$$s\text{-}\lim_{t \rightarrow \pm\infty} U(t)^*U_0(t) = W_{\pm},$$

where

$$W_{\pm} = \begin{bmatrix} (W_{\pm})_{++} & (W_{\pm})_{+-} \\ (W_{\pm})_{-+} & (W_{\pm})_{--} \end{bmatrix}$$

with

$$\begin{aligned} (W_{\pm})_{++} &= \overline{(W_{\pm})_{--}} \\ &= \frac{1}{2}(\omega_0^{-1/2}w_{\pm}\omega_0^{+1/2} + \omega_0^{+1/2}w_{\pm}\omega_0^{-1/2}), \\ (W_{\pm})_{-+} &= \overline{(W_{\pm})_{+-}} \\ &= \frac{1}{2}(\omega_0^{-1/2}w_{\pm}\omega_0^{+1/2} - \omega_0^{+1/2}w_{\pm}\omega_0^{-1/2}). \end{aligned}$$

*Proof.* See Appendix A.3. □

**Lemma 2.7** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then:*

$$\lim_{t \rightarrow \pm\infty} U_0(t)^*j_t = j_{\pm},$$

where

$$j_{\pm} = \begin{bmatrix} g_{\pm} \\ \bar{g}_{\pm} \end{bmatrix} \quad \text{and} \quad g_{\pm} = -\frac{1}{\sqrt{2}} \int_0^{\pm\infty} ds \omega_0^{-1/2} e^{is\omega_0} w_{\pm}^* J_s.$$

*Proof.* It suffices to prove that

$$\lim_{t \rightarrow \pm\infty} e^{it\omega_0} g_t = -\frac{1}{\sqrt{2}} \int_0^{\pm\infty} ds \omega_0^{-1/2} e^{is\omega_0} w_{\pm}^* J_s. \quad (2.15)$$

By a direct calculation, we have

$$\begin{aligned}
e^{it\omega_0} g_t &= -\frac{1}{2\sqrt{2}} \int_0^t ds e^{it\omega_0} \omega_0^{-1/2} (\omega - \omega_0) \omega_0^{-1/2} \cdot \omega_0^{1/2} \omega^{-1/2} \cdot e^{i(t-s)\omega} \omega^{-1/2} J_s \\
&\quad - \frac{1}{2\sqrt{2}} \int_0^t ds e^{it\omega_0} (\omega_0^{-1/2} + \omega_0^{1/2} \omega^{-1}) e^{-i(t-s)\omega} J_s. \tag{2.16}
\end{aligned}$$

Since  $\omega_0^{-1/2}(\omega - \omega_0)\omega_0^{-1/2}$  is Hilbert-Schmidt by Lemma 2.2 and since  $\int_0^t ds \|\omega^{-1/2} J_s\| < \infty$  by (b) of Assumption 1.3, the first term of the r.h.s in (2.16) tends to zero as  $t$  goes to  $\pm\infty$ . We show that the limit of the second term in (2.16) equals (2.15). It holds that

$$\begin{aligned}
&e^{it\omega_0} (\omega_0^{-1/2} + \omega_0^{1/2} \omega^{-1}) e^{-it\omega} \\
&= (\omega_0^{-1/2} e^{it\omega_0} e^{-it\omega} + e^{it\omega_0} e^{-it\omega} \omega^{-1/2}) \\
&\quad + e^{it\omega_0} (\omega_0^{-1/2} - \omega^{-1/2}) \omega^{-1} e^{-it\omega}. \tag{2.17}
\end{aligned}$$

The first term of the r.h.s in (2.17) tends to  $2\omega_0^{-1/2} w_{\pm}^*$ . The second term tends to zero since

$$(\omega_0^{-1/2} - \omega^{-1/2}) \omega^{-1} e^{-it\omega} = \omega_0^{-1/2} (1 - \omega_0^{1/2} \omega^{-1/2}) \omega^{-1} e^{-it\omega}$$

and since, by Lemma 2.2,  $(1 - \omega_0^{1/2} \omega^{-1/2})$  is Hilbert-Schmidt. Using these facts and Assumption 1.3 (b) again, we infer the limit of the second term in (2.16) equals (2.15).  $\square$

By Lemmas 2.6 and 2.7, we have

**Lemma 2.8** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then it holds that, for  $v \in \mathcal{H}_C$ ,*

$$s\text{-}\lim_{t \rightarrow \pm} \mathscr{W}(t) e^{i\psi(v)} \mathscr{W}(t)^* = e^{i(\psi(W_{\pm}v) + i(j_{\pm}, Qv))}. \tag{2.18}$$

*In particular, the following properties hold: for real  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ ,*

$$s\text{-}\lim_{t \rightarrow \pm} \mathscr{W}(t) e^{i\phi_0(f)} \mathscr{W}(t)^* = e^{i\phi_{\pm}(f)},$$

$$s\text{-}\lim_{t \rightarrow \pm} \mathscr{W}(t) e^{i\pi_0(g)} \mathscr{W}(t)^* = e^{i\pi_{\pm}(g)},$$

where

$$\begin{aligned}\phi_{\pm}(f) &= \psi(W_{\pm}u_0) + i(j_{\pm}, Qu_0), \\ \pi_{\pm}(f) &= \psi(W_{\pm}v_0) + i(j_{\pm}, Qv_0).\end{aligned}$$

*Proof.* Since  $e^{i\psi(v)}$  ( $v \in \mathcal{H}_C$ ) is unitary, it suffices to prove (2.18) on  $D_f$ , which is an easy exercise.  $\square$

**Lemma 2.9** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then there exists a unitary operator  $\mathcal{W}_{\pm}$  such that*

$$\mathcal{W}_{\pm}e^{i\psi(v)}\mathcal{W}_{\pm}^* = e^{i(\psi(W_{\pm}v) + i(j_{\pm}, Qv))}, \quad v \in \mathcal{H}_C.$$

*In particular, it holds that: for  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ ,*

$$\begin{aligned}\mathcal{W}_{\pm}\phi_0(f)\mathcal{W}_{\pm}^* &= \phi_{\pm}(f), \\ \mathcal{W}_{\pm}\pi_0(f)\mathcal{W}_{\pm}^* &= \pi_{\pm}(f).\end{aligned}$$

*Proof.* Note that  $(W_{\pm}^*)_{-+} = W_{+-}^*$  is equal to

$$\begin{aligned}& \frac{1}{2}\omega_0^{1/2}w_{\mp}^*\omega_0^{-1/2} - \omega_0^{-1/2}w_{\mp}^*\omega_0^{1/2}) \\ &= \frac{1}{2}(w_{\mp}^*\omega^{-1/2}\omega_0^{1/2}) \cdot (\omega_0^{-1/2}(\omega - \omega_0)\omega_0^{-1/2})\end{aligned}$$

and hence is Hilbert-Schmidt. By Lemma 2.1, it holds that there exists a unitary operator  $\mathcal{U}_{\pm}$  such that

$$\mathcal{U}_{\pm}e^{i\psi(v)}\mathcal{U}_{\pm}^* = e^{i\psi(W_{\pm}v)}, \quad v \in \mathcal{H}_C.$$

Setting  $\mathcal{W}_{\pm} = \mathcal{U}_{\pm}e^{i\psi(j_{\pm})}$ , one obtains the desired result.  $\square$

### 3. Scattering theory

Throughout this section, we suppose that Assumptions 1.1–1.3 hold. Since any two quantum systems which are transformed from one to the other by a unitary transformation are physically equivalent, one can redefine the solution of the Klein-Gordon field as

$$\phi(t, f) = \mathcal{W}_-^* \phi(t, f) \mathcal{W}_-, \quad \pi(t, g) = \mathcal{W}_-^* \pi(t, g) \mathcal{W}_-$$

for  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ . Then it follows from Proposition 2.4 (see also Remarks 2.1–2.2) that  $\phi(t, f)$  and  $\pi(t, g)$  satisfy (1.1) in the operator valued distributional sense. The field operators  $\phi_s(t, f)$  and  $\pi_s(t, g)$  defined by

$$\begin{aligned} \phi_s(t, f) &= \mathcal{W}_-^* \mathcal{U}(s)^* \mathcal{U}(s-t) \phi_0(f) \mathcal{U}(s-t)^* \mathcal{U}(s) \mathcal{W}_-, \\ \pi_s(t, g) &= \mathcal{W}_-^* \mathcal{U}(s)^* \mathcal{U}(s-t) \pi_0(g) \mathcal{U}(s-t)^* \mathcal{U}(s) \mathcal{W}_- \end{aligned}$$

satisfy the free Klein-Gordon equation with the initial condition:

$$\phi_s(s, f) = \phi(s, f), \quad \pi_s(t, g) = \pi(s, g).$$

The asymptotic fields  $\phi_{\sharp}(t, f)$  and  $\pi_{\sharp}(t, g)$  ( $\sharp$  stands for out or in) are defined as

$$e^{i\phi_{\text{out/in}}(t, f)} = \lim_{t \rightarrow \pm\infty} e^{i\phi_s(t, f)}, \quad e^{i\pi_{\text{out/in}}(t, g)} = \lim_{t \rightarrow \pm\infty} e^{i\pi_s(t, g)}$$

for any real  $f \in H^{-1/2}(\mathbb{R}^3)$  and  $g \in H^{1/2}(\mathbb{R}^3)$ . Then, by Lemmas 2.8 and 2.9, the incoming fields  $\phi_{\text{in}}(f) = \phi_{\text{in}}(0, f)$  and  $\pi_{\text{in}}(g) = \pi_{\text{in}}(0, g)$  are

$$\phi_{\text{in}}(f) = \phi_0(f), \quad \pi_{\text{in}}(g) = \pi_0(g) \tag{3.1}$$

and

$$\{\phi_{\text{in}}(f), \pi_{\text{in}}(g) \mid f \in H^{-1/2}(\mathbb{R}^3), g \in H^{1/2}(\mathbb{R}^3)\}$$

gives the Fock representation of the CCR (see, e.g., [1]). The out going fields  $\phi_{\text{out}}(f) = \phi_{\text{out}}(0, f)$  and  $\pi_{\text{out}}(g) = \pi_{\text{out}}(0, g)$  are

$$\begin{aligned} \phi_{\text{out}}(f) &= (\mathcal{W}_+^* \mathcal{W}_-)^* \phi_0(f) (\mathcal{W}_+^* \mathcal{W}_-), \\ \pi_{\text{in}}(g) &= (\mathcal{W}_+^* \mathcal{W}_-)^* \pi_0(g) (\mathcal{W}_+^* \mathcal{W}_-). \end{aligned} \tag{3.2}$$

The scattering matrix  $\mathcal{S} = \mathcal{S}(V, J)$  is defined by

$$\mathcal{S} = \mathcal{W}_+^* \mathcal{W}_-.$$

**Proposition 3.1** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then:*

$$\mathcal{S}^{-1}\phi_{\text{in}}(f)\mathcal{S} = \phi_{\text{out}}(f) \quad \text{and} \quad \mathcal{S}^{-1}\pi_{\text{in}}(f)\mathcal{S} = \pi_{\text{out}}(f).$$

*Proof.* By (3.1) and (3.2), we see that

$$\mathcal{S}^{-1}\psi_{\text{in}}(f)\mathcal{S} = \mathcal{W}_-^*\mathcal{W}_+\psi_0(f)\mathcal{W}_+^*\mathcal{W}_- = \psi_{\text{out}}(f),$$

where  $\psi_{\text{in/out},0}$  stands for  $\phi_{\text{in/out},0}$  or  $\pi_{\text{in/out},0}$ . □

Let us define the associated annihilation and creation operators by

$$c_{\text{in}}(f) = c(f) \quad \text{and} \quad c_{\text{out}}(f) = \mathcal{S}^*c(f)\mathcal{S}$$

and  $c_{\#}^*(f) = c_{\#}(f)^*$ . The free Hamiltonians of the incoming field and outgoing field are defined by

$$H_{\text{in}} = d\Gamma(\omega_0) \quad \text{and} \quad H_{\text{out}} = \mathcal{S}^*d\Gamma(\omega_0)\mathcal{S}.$$

The asymptotic vacua

$$\Omega_{\text{in}} = \Omega \quad \text{and} \quad \Omega_{\text{out}} = \mathcal{S}^*\Omega$$

satisfy

$$H_{\#}\Omega_{\#} = 0 \quad \text{and} \quad c_{\#}(f)\Omega_{\#} = 0.$$

The asymptotic fields satisfy

$$\phi_{\#}(t, f) = e^{itH_{\#}}\phi_{\#}(f)e^{-itH_{\#}}, \quad \text{and} \quad \pi_{\#}(t, f) = e^{itH_{\#}}\pi_{\#}(f)e^{-itH_{\#}},$$

and

$$\begin{aligned} \phi_{\#}(t, f) &= \frac{1}{\sqrt{2}}(c_{\#}^*(e^{it\omega_0}\omega_0^{-1/2}f) + c_{\#}(e^{it\omega_0}\omega_0^{-1/2}\bar{f})), \\ \pi_{\#}(t, f) &= \frac{i}{\sqrt{2}}(c_{\#}^*(e^{it\omega_0}\omega_0^{+1/2}f) - c_{\#}(e^{it\omega_0}\omega_0^{+1/2}\bar{f})) \end{aligned}$$

hold on  $\tilde{D}$ .

## 4. Inverse scattering

### 4.1. Uniqueness of the potential $V$

Let

$$S = w_+^* w_-.$$

When we want to emphasize the dependence of  $V$ , we write  $S = S(V)$ . We will prove the following theorem:

**Theorem 4.1** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. If  $\mathcal{S}(V, J) = \mathcal{S}(V', J')$ , then  $S(V) = S(V')$ .*

We need the following:

**Lemma 4.2** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then:*

$$\lim_{t \rightarrow \pm\infty} (c^*(e^{it\omega_0} f)\Omega, \mathcal{S}c^*(e^{it\omega_0} g)\Omega) = (\Omega, \mathcal{S}\Omega)(f, Sg). \quad (4.1)$$

*Proof of Theorem 4.1.* For notational simplicity, we write  $\mathcal{S}(V', J') = \mathcal{S}'$  and  $S(V') = S'$ . Note that  $(\Omega, \mathcal{S}\Omega) \neq 0$  since  $\mathcal{S}$  is unitary. By Lemma 4.2, we have

$$\begin{aligned} (f, Sg) &= \lim_{t \rightarrow \pm\infty} (c^*(e^{it\omega_0} f)\Omega, \mathcal{S}c^*(e^{it\omega_0} g)\Omega) / (\Omega, \mathcal{S}\Omega) \\ &= \lim_{t \rightarrow \pm\infty} (c^*(e^{it\omega_0} f)\Omega, \mathcal{S}'c^*(e^{it\omega_0} g)\Omega) / (\Omega, \mathcal{S}'\Omega) = (f, S'g). \quad \square \end{aligned}$$

In the remainder of this subsection, we will prove Lemma 4.2. Let

$$\mathcal{I}_t(f, g) = (c^*(e^{it\omega_0} f)\Omega, \mathcal{S}c^*(e^{it\omega_0} g)\Omega).$$

We see from Lemma 2.9 that  $\mathcal{W}_\pm = \mathcal{U}_\pm e^{i\psi(j_\pm)}$ . By a direct calculation, one obtains the following:

$$\begin{aligned} \mathcal{U}_\pm c(f)\mathcal{U}_\pm^* &= c((W_\pm)_{++}f) + c^*((W_\pm)_{+-}\bar{f}), \\ \mathcal{U}_\pm c^*(f)\mathcal{U}_\pm^* &= c^*((W_\pm)_{++}f) + c((W_\pm)_{+-}\bar{f}), \\ e^{i\psi(j_\pm)}c(f)e^{-i\psi(j_\pm)} &= c(f) - i(f, j_\pm), \\ e^{i\psi(j_\pm)}c^*(f)e^{-i\psi(j_\pm)} &= c^*(f) + i(j_\pm, f). \end{aligned}$$

It holds from the above that

$$\begin{aligned} & \mathcal{U}_\pm e^{i\psi(j_\pm)} c^*(e^{it\omega_0} f) \Omega \\ &= (c^*((W_\pm)_{++} e^{it\omega_0} f) + c((W_\pm)_{+-} e^{-it\omega_0} \bar{f}) + i(g_\pm, e^{it\omega_0} f)) \mathcal{U}_\pm e^{i\psi(j_\pm)} \Omega \\ &= c^*((W_\pm)_{++} e^{it\omega_0} f) \mathcal{U}_\pm e^{i\psi(j_\pm)} \Omega + o(1) \end{aligned}$$

as  $t \rightarrow \pm\infty$  in the strong topology since  $(W_\pm)_{+-}$  is Hilbert-Schmidt and since  $\lim_{t \rightarrow \pm\infty} (g_\pm, e^{it\omega_0} f) = 0$  by the Riemann-Lebesgue Lemma. Hence we have

$$\begin{aligned} & \mathcal{I}_t(f, g) \\ &= (c^*((W_+)_{++} e^{it\omega_0} f) \mathcal{U}_+ e^{i\psi(j_+)} \Omega, c^*((W_-)_{++} e^{it\omega_0} g) \mathcal{U}_- e^{i\psi(j_-)} \Omega) + o(1) \\ &= ((W_+)_{++} e^{it\omega_0} f, (W_-)_{++} e^{it\omega_0} g)(\Omega, \mathcal{S}\Omega) \\ &\quad + (c((W_-)_{++} e^{it\omega_0} g) \mathcal{U}_+ e^{i\psi(j_+)} \Omega, c((W_+)_{++} e^{it\omega_0} f) \mathcal{U}_- e^{i\psi(j_-)} \Omega) + o(1) \end{aligned} \tag{4.2}$$

as  $t \rightarrow \pm\infty$ . It is straightforward that

$$\begin{aligned} & (W_+)_{++}^* (W_-)_{++} \\ &= \frac{1}{4} (\omega_0^{-1/2} w_+^* \omega_0^{1/2} + \omega_0^{+1/2} w_+^* \omega_0^{-1/2}) (\omega_0^{-1/2} w_- \omega_0^{1/2} + \omega_0^{+1/2} w_- \omega_0^{-1/2}) \\ &= S + \frac{1}{4} w_+^* [(\omega_0^{-1/2} \omega_0 \omega_0^{-1/2} - 1) + (\omega_0^{1/2} \omega_0^{-1} \omega_0^{1/2} - 1)] w_-. \end{aligned} \tag{4.3}$$

Note that  $S e^{it\omega_0} = e^{it\omega_0} S$  and that the second term in the r.h.s of (4.3) is Hilbert-Schmidt by Lemma 2.2. Hence, by (4.2), we have (4.1) if the following holds true:

$$(c((W_-)_{++} e^{it\omega_0} g) \mathcal{U}_+ e^{i\psi(j_+)} \Omega, c((W_+)_{++} e^{it\omega_0} f) \mathcal{U}_- e^{i\psi(j_-)} \Omega) = o(1). \tag{4.4}$$

To prove (4.4), we use the relation

$$\mathcal{U}_\pm^* c(f) \mathcal{U}_\pm = c((W_\pm)_{++}^*) - c^*((W_\pm)_{-+}^* \bar{f}),$$

which is obtained from  $\mathcal{U}_\pm^* e^{i\psi(v)} \mathcal{U}_\pm = e^{i\psi(QW_\pm^* Qv)}$ . By the above and the

fact that  $(W_{\pm})_{-+}^*$  is Hilbert-Schmidt, we observe that

$$c((W_-)_{++}e^{it\omega_0}g)\mathcal{U}_+e^{i\psi(j_+)}\Omega = \mathcal{U}_+e^{i\psi(j_+)}c((W_-)_{++}e^{it\omega_0}g)\Omega + o(1).$$

Since the first term of the above equals zero, the proof of the lemma is completed.

#### 4.2. Characterization of the external source $J$

Our aim of this subsection is to represent the classical source  $J$  in terms of the functional

$$F(t, f) = (\Omega_{\text{in}}, \phi_{\text{out}}(t, f)\Omega_{\text{in}}), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

We set

$$Z_{\pm}[f] = X[f] \mp iY[f],$$

where

$$X[f] = \left. \frac{d}{dt}F(t, \omega_0^{-1/2}f) \right|_{t=0} \quad \text{and} \quad Y[f] = F(0, \omega_0^{+1/2}f).$$

Note that

$$\phi_0(t, f) = \psi(v_t),$$

where  $v_t$  is

$$v_t = \begin{bmatrix} e^{it\omega_0}\omega_0^{-1/2}\bar{f}/\sqrt{2} \\ e^{-it\omega_0}\omega_0^{-1/2}\bar{f}/\sqrt{2} \end{bmatrix}.$$

We see that

$$Cv_t = \begin{bmatrix} e^{it\omega_0}\omega_0^{-1/2}f/\sqrt{2} \\ e^{-it\omega_0}\omega_0^{-1/2}f/\sqrt{2} \end{bmatrix} \neq v_t$$

and  $v_t \notin \mathcal{H}_C$ . By a direct calculation, we have for  $v \in \mathcal{H}_C$ :

$$\mathcal{S}^{-1}\psi(v)\mathcal{S} = \psi(QW_-^*QW_+v) + i(Qj_+ - W_+^*QW_-j_-, v).$$

Since  $(\Omega, \psi(v)\Omega) = 0$  and

$$\begin{aligned} (Qj_+ - W_+^*QW_-j_-, v) &= (W_+^*Q(W_+Qj_+ - W_-j_-), v) \\ &= (W_+j_+ - W_-j_-, QW_+v), \end{aligned}$$

one obtains

$$(\Omega, \mathcal{S}^{-1}\psi(v)\mathcal{S}\Omega) = i(W_+j_+ - W_-j_-, QW_+v), \quad v \in \mathcal{H}_C.$$

It holds that

$$\begin{aligned} F(t, f) &= \frac{1}{2}(\Omega, \mathcal{S}^{-1}\psi(v_t + Cv_t)\mathcal{S}\Omega) \\ &\quad + \frac{i}{2}(\Omega, \mathcal{S}^{-1}\psi(i(v_t - Cv_t))\mathcal{S}\Omega) \\ &= \frac{i}{2}(W_+j_+ - W_-j_-, QW_+(v_t + Cv_t)) \\ &\quad - \frac{i}{2}(W_+j_+ - W_-j_-, QW_+(v_t - Cv_t)) \\ &= i(W_+j_+ - W_-j_-, QW_+Cv_t). \end{aligned}$$

Thus we infer

$$\begin{aligned} X[f] &= \frac{1}{\sqrt{2}}\left(W_+j_+ - W_-j_-, QW_+\begin{bmatrix} f \\ -f \end{bmatrix}\right), \\ Y[f] &= \frac{i}{\sqrt{2}}\left(W_+j_+ - W_-j_-, QW_+\begin{bmatrix} f \\ f \end{bmatrix}\right) \end{aligned}$$

and

$$\begin{aligned} Z_+[f] &= \sqrt{2}\left(W_+j_+ - W_-j_-, QW_+\begin{bmatrix} f \\ 0 \end{bmatrix}\right), \\ Z_-[f] &= -\sqrt{2}\left(W_+j_+ - W_-j_-, QW_+\begin{bmatrix} 0 \\ f \end{bmatrix}\right). \end{aligned}$$

**Lemma 4.3** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Let*

$$Z[f, g] := Z_+[f] - Z_-[g].$$

Then

$$\begin{aligned} Z[Sf, f] &= -2\sqrt{2}i(\operatorname{Im}(g_\infty), w_- f), \\ Z[Sf, -f] &= 2\sqrt{2}(\operatorname{Re}(g_\infty), w_- f), \end{aligned}$$

where

$$g_\infty := w_+ g_+ - w_- g_- = -\frac{1}{\sqrt{2}} \int_{-\infty}^{+\infty} ds \omega^{-1/2} e^{is\omega} J_s.$$

*Proof.* By definition, we see that

$$Z[w_+^* f, \pm w_-^* f] = \sqrt{2} \left( W_{+j_+} - W_{-j_-}, QW_+ \begin{bmatrix} w_+^* f \\ \pm w_-^* f \end{bmatrix} \right).$$

By direct calculation, we have

$$W_{+j_+} - QW_{-j_-} = \begin{bmatrix} \omega_0^{-1/2} \omega^{1/2} \operatorname{Re}(g_\infty) + i\omega_0^{1/2} \omega^{-1/2} \operatorname{Im}(g_\infty) \\ \omega_0^{-1/2} \omega^{1/2} \operatorname{Re}(g_\infty) - i\omega_0^{1/2} \omega^{-1/2} \operatorname{Im}(g_\infty) \end{bmatrix}$$

and

$$QW_+ \begin{bmatrix} w_+^* f \\ \pm w_-^* f \end{bmatrix} = \begin{bmatrix} \omega_0^{\mp 1/2} \omega^{\pm 1/2} f \\ \mp \omega_0^{\mp 1/2} \omega^{\pm 1/2} f \end{bmatrix},$$

which completes the proof.  $\square$

We introduce the functional  $Z : L^2(\mathbb{R}^3) \rightarrow \mathbb{C}$  by

$$Z[f] := \frac{1}{2\sqrt{2}} \left( Z[Sf, f] + Z[Sf, -f] \right), \quad f \in L^2(\mathbb{R}^3).$$

For  $f \in L^2(\mathbb{R}^3)$ ,  $\lambda > 0$  and  $k, x \in \mathbb{R}^3$ , we denote  $e^{-ik \cdot x} f(\lambda x)$  by  $f_k^\lambda(x)$ .  $\mathcal{F}_0$  stands for the Fourier transform:  $\mathfrak{h} \ni f \mapsto (\mathcal{F}_0 f) = \hat{f}$  and  $\hat{f}(k) = (2\pi)^{-3/2} \int dk e^{-ik \cdot x} f(x)$ . The generalized Fourier transform  $\mathcal{F}_\pm$  is defined by  $\mathcal{F}_0 w_\pm^*$ . Let  $\chi \in \mathfrak{h}$  such that  $0 \leq \chi \leq 1$  and  $\chi(x) = 1$  if  $|x| \leq 1$  and  $\chi(x) = 0$  if  $|x| \geq 2$ . We introduce a function  $z_\lambda$  by

$$z_\lambda(k) := -\sqrt{2}(2\pi)^{-3/2} Z[\chi_k^\lambda].$$

**Lemma 4.4** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Then  $z_\lambda \in \mathfrak{h}$  and*

$$\lim_{\lambda \rightarrow 0} z_\lambda = \int_{-\infty}^{\infty} ds \mathcal{F}_+(e^{-is\omega} \omega^{-1/2} J_s) \quad \text{in } \mathfrak{h}.$$

*Proof.* Since, by the above lemma, we have

$$Z[f] = (w^* g_\infty, f), \tag{4.5}$$

it follows that

$$\begin{aligned} z_\lambda(k) &= \left( \int_{-\infty}^{\infty} ds w_-^* e^{is\omega} \omega^{-1/2} J_s, \chi_k^\lambda \right) \\ &= (2\pi)^{-3/2} \int dx e^{-ik \cdot x} \chi(\lambda x) \times \left[ \int_{-\infty}^{\infty} ds w_+^* e^{is\omega} \omega^{-1/2} J_s \right](k). \end{aligned}$$

Hence  $z_\lambda$  converges in  $\mathfrak{h}$  to

$$\mathcal{F}_0 \left[ \int_{-\infty}^{\infty} ds w_+^* e^{-is\omega} \omega^{-1/2} J_s \right] = \int_{-\infty}^{\infty} ds \mathcal{F}_+ e^{-is\omega} \omega^{-1/2} J_s. \quad \square$$

Let

$$z := \lim_{\lambda \rightarrow 0} z_\lambda. \tag{4.6}$$

**Proposition 4.5** *Suppose that Assumptions 1.1–1.2 and 1.3 hold. If  $\mathcal{S}(V, J) = \mathcal{S}(V', J')$ , then*

$$\int_{-\infty}^{\infty} ds e^{-is\omega} J_s = \int_{-\infty}^{\infty} ds e^{-is\omega} J'_s.$$

*Proof.* Let  $z'(k)$  be defined as  $z(k)$  with replacing  $\mathcal{S}(V, J)$  by  $\mathcal{S}(V', J')$ . By Theorem 4.1, we have  $S(V) = S(V')$ ,  $V = V'$  and hence  $z = z'$ . By Lemma 4.4, we obtain

$$\int_{-\infty}^{\infty} ds e^{-is\omega} \omega^{-1/2} J_s = \int_{-\infty}^{\infty} ds e^{-is\omega} \omega^{-1/2} J'_s$$

by the unitarity of  $\mathcal{F}_+$ . □

Henceforth, we suppose that  $J$  is expressed by

$$J(t, x) = j(t)\rho(x), \quad (4.7)$$

where  $j \in L^1(\mathbb{R})$  and  $\rho \in H^{-1/2}(\mathbb{R}^3)$ . In Subsection 4.3 below, assuming that  $j$  is a given function and that  $\hat{j}$  is analytic, we will represent  $\rho$  in terms of  $z$  and  $j$ . In Subsection 4.4 below, we next assume that  $\rho$  is a given function. We will show that  $j$  is determined by  $z$  and  $\rho$  if  $j$  satisfies the following:

$$\text{For some } \delta > 0, \quad e^{\delta|t|}j(t) \in L^1(\mathbb{R}_t). \quad (4.8)$$

### 4.3. Reconstruction of $\rho$

Let  $j$  be a given function belonging to  $L^1(\mathbb{R}_t)$ . Then we immediately obtain the reconstruction formula for determining  $\rho$ :

**Theorem 4.6** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Assume that  $J \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^3)$  satisfies (4.7) and that  $\hat{j}$  is a nonzero, real analytic given function. If  $\omega(k) \notin (\hat{j})^{-1}(0)$ , then  $(\mathcal{F}_+\rho)(k)$  is uniquely determined by*

$$(\mathcal{F}_+\rho)(k) = \frac{z(k)}{(\hat{j})(\hat{\omega}_0(k))}. \quad (4.9)$$

**Remark 4.1** Since  $\hat{j}$  is real analytic function,  $(\hat{j})^{-1}(0)$  is a discrete set and hence countable. Thus, (4.9) holds for almost all  $k \in \mathbb{R}^3$  and we have  $\rho = \mathcal{F}_+(z/(\hat{j}(\hat{\omega}_0)))$ . If the generalized eigenfunction  $\psi_+(k, x)$  of  $-\Delta + V$  exists, then the inverse of the generalized Fourier transform  $\mathcal{F}_+$  is given by

$$(\mathcal{F}_+^{-1}f)(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} dk \psi_+(k, x) f(k),$$

where we denote  $\lim_{R \rightarrow +\infty} \int_{|k| \leq R} dk$  by  $\int_{\mathbb{R}^3} dk$ . In this case, we have

$$\rho(x) = (2\pi)^{-3/2} \int_{\mathbb{R}^3} dk \frac{\psi_+(k, x) z(k)}{(\hat{j})(\hat{\omega}_0(k))}.$$

### 4.4. Reconstruction of $j$

In this subsection we suppose that  $\rho$  is a nonzero, given function belonging to  $H^{-1/2}(\mathbb{R}^3)$ . The following lemma will help us to identify  $j$ :

**Lemma 4.7** *Suppose that  $j$  satisfies (4.8). Then  $\hat{j}$  is real analytic. Furthermore, the radius of convergence of the Taylor series of  $\hat{j}$  is larger than  $\delta$ .*

*Proof.* By the assumption (4.8), it follows that for any non negative integer  $m$ ,

$$|t|^m j(t) \in L^1(\mathbb{R}_t).$$

Therefore, we have  $\hat{j} \in C^\infty(\mathbb{R})$  and

$$(\hat{j})^{(m)}(\tau) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} (-it)^m e^{-i\tau t} j(t) dt, \quad \tau \in \mathbb{R},$$

where  $(\hat{j})^{(m)}$  is the  $m$ -th order derivative of  $\hat{j}$ .

Since

$$\sup_{t>0} t^m e^{-\delta t} = m^m \delta^{-m} e^{-m},$$

we obtain for any  $\tau \in \mathbb{R}$ ,

$$\begin{aligned} |(\hat{j})^{(m)}(\tau)| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |t|^m |j(t)| dt \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |j(t)| e^{\delta|t|} |t|^m e^{-\delta|t|} dt \\ &\leq \frac{1}{\sqrt{2\pi}} m^m \delta^{-m} e^{-m} \int_{\mathbb{R}} |j(t)| e^{\delta|t|} dt. \end{aligned}$$

Thus, we have

$$\left| \frac{(\hat{j})^{(m)}(\tau)}{m!} \right| \leq \frac{1}{\sqrt{2\pi}} \left( \int_{\mathbb{R}} |j(t)| e^{\delta|t|} dt \right) \frac{m^m e^{-m}}{m!} \delta^{-m}.$$

Using Stirling's formula

$$m! = \sqrt{2\pi m} m^{m+1/2} e^{-m} e^{\theta(m)/12m}, \quad 0 < \theta(m) < 1,$$

we see that

$$\limsup_{m \rightarrow \infty} \left| \frac{(\hat{j})^{(m)}(\tau)}{m!} \right|^{1/m} \leq \delta^{-1},$$

which completes the lemma.  $\square$

Applying Lemmas 4.4 and 4.7, we have the following result:

**Theorem 4.8** *Suppose that Assumptions 1.1, 1.2 and 1.3 hold. Assume that  $J$  satisfies (4.7) and  $\rho \in H^{-1/2}(\mathbb{R}^3)$  is a nonzero, given function. If  $j$  satisfies (4.8), then  $j$  is uniquely reconstructed by the following steps:*

(Step I) *Fix a point  $k_0 \notin (\mathcal{F}_+\rho)^{-1}(0)$ . Let  $U_0$  be a  $k_0$ -neighborhood such that  $0 \notin (\hat{J}_X)(U_0)$ . Then we have*

$$(\hat{j})(\omega(k)) = \frac{z(k)}{(\mathcal{F}_+\rho)(k)}, \quad k \in U_0. \quad (4.10)$$

*Therefore, we see exact values of  $(\hat{j})^{(m)}(\tau_0)$ ,  $m = 0, 1, 2, \dots$ , where  $\tau_0 = \omega(k_0)$ .*

(Step II) *For any  $\tau \in (\tau_0 - \delta, \tau_0 + \delta)$ , we have*

$$(\hat{j})(\tau) = \sum_{m=0}^{\infty} \frac{(\hat{j})^{(m)}(\tau_0)}{m!} (\tau - \tau_0)^m.$$

(Step III) *Let  $l$  be a positive integer. Suppose that we have already determined  $(\hat{j})(\tau)$  with  $\tau \in (\tau_0 - \frac{(l+1)\delta}{2}, \tau_0 + \frac{(l+1)\delta}{2})$ . For any  $\tau \in [\tau_0 + \frac{(l+1)\delta}{2}, \tau_0 + \frac{(l+2)\delta}{2})$ , we see the value of  $(\hat{J}_T)(\tau)$  by*

$$(\hat{j})(\tau) = \sum_{m=0}^{\infty} \frac{(\hat{j})^{(m)}(\tau_0 + l\delta/2)}{m!} (\tau - \tau_0 - l\delta/2)^m.$$

*On the other hand, For any  $\tau \in (\tau_0 - \frac{(l+2)\delta}{2}, \tau_0 - \frac{(l+1)\delta}{2}]$ , we see the value of  $(\hat{J}_T)(\tau)$  by*

$$(\hat{j})(\tau) = \sum_{m=0}^{\infty} \frac{(\hat{j})^{(m)}(\tau_0 - l\delta/2)}{m!} (\tau - \tau_0 + l\delta/2)^m.$$

(Step VI) *From (Step III),  $\hat{j}$  is reconstructed completely. Hence we can determine  $j$  by the inverse Fourier transform.*

## A. Appendix

### A.1. Hilbert-Schmidt operators

We prove Lemma 2.2. We first show that  $\omega^{1/2}\omega_0^{-1/2} - 1$ . To this end, we use the formula

$$A^\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty d\lambda \lambda^{\alpha-1} (A + \lambda)^{-1} A \quad (\text{A.1})$$

on  $D(A)$  ( $0 < \alpha < 1$ ). With the aid of (A.1) for  $\alpha = 1/4$ , we have

$$\begin{aligned} \omega^{1/2}\omega_0^{-1/2} - 1 &= [(\omega^2)^{1/4} - (\omega_0^2)^{1/4}] \omega_0^{-1/2} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\lambda \lambda^{-3/4} [(\omega^2 + \lambda)^{-1} \omega^2 - (\omega_0^2 + \lambda)^{-1} \omega_0^2] \omega_0^{-1/2} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\lambda \lambda^{1/4} [ -(\omega^2 + \lambda)^{-1} + (\omega_0^2 + \lambda)^{-1} ] \omega_0^{-1/2} \\ &= \frac{1}{\sqrt{2\pi}} \int_0^\infty d\lambda \lambda^{1/4} (\omega^2 + \lambda)^{-1} V (\omega_0^2 + \lambda)^{-1} \omega_0^{-1/2}, \end{aligned}$$

where the above equation holds true on some dense domain, e.g.  $D(\omega_0^{3/2})$ . It suffices to prove that the operator  $(\omega^2 + \lambda)^{-1} V (\omega_0^2 + \lambda)^{-1} \omega_0^{-1/2}$  is Hilbert-Schmidt and satisfies

$$\int_0^\infty d\lambda \lambda^{1/4} \|(\omega^2 + \lambda)^{-1} V (\omega_0^2 + \lambda)^{-1} \omega_0^{-1/2}\|_2 < \infty, \quad (\text{A.2})$$

where  $\|\cdot\|_2$  is the Hilbert-Schmidt norm. Since  $V \in L^2(\mathbb{R}^3)$  and  $(\sqrt{|k|^2 + m^2})^{-3/2-\epsilon} \in L^2(\mathbb{R}^3)$ ,  $V\omega_0^{-3/2-\epsilon}$  is Hilbert-Schmidt for  $\epsilon > 0$  (see [6, Theorem XI.20] for details). Since  $\|(\omega^2 + \lambda)^{-1}\| \leq (m^2 + \lambda)^{-1}$ , we have

$$\begin{aligned} &\|(\omega^2 + \lambda)^{-1} V (\omega_0^2 + \lambda)^{-1} \omega_0^{-1/2}\|_2 \\ &\leq (m^2 + \lambda)^{-1} \|V\omega_0^{-3/2-\epsilon}\|_2 \cdot \|\omega_0^{1+\epsilon} (\omega_0^2 + \lambda)^{-1}\| \\ &\leq (m^2 + \lambda)^{-1} \|V\omega_0^{-3/2-\epsilon}\|_2 \cdot \|\omega_0^{1+\epsilon} (\omega_0^2 + \lambda)^{-(1+\epsilon)/2}\| \\ &\quad \times \|(\omega_0^2 + \lambda)^{-1+(1+\epsilon)/2}\| \\ &\leq (m^2 + \lambda)^{-3/2+\epsilon/2}. \end{aligned}$$

Thus (A.2) holds if  $0 < \epsilon < 1/2$  and hence  $\omega^{1/2}\omega_0^{-1/2} - 1$  is Hilbert-Schmidt.

We shall prove that  $\omega^{-1/2}\omega_0^{1/2} - 1$  is Hilbert-Schmidt. From a similar argument as above, we infer that it suffices to show that

$$\int_0^\infty \lambda^{1/4} \left\| \omega^{-1/2}(\omega^2 + \lambda)^{-1} V(\omega_0^2 + \lambda)^{-1} \right\|_2 < \infty. \quad (\text{A.3})$$

Since, as will be seen later,  $\omega_0^{-1/2} V \omega_0^{-1-\epsilon}$  ( $\epsilon > 0$ ) is Hilbert-Schmidt, we have

$$\left\| \omega^{-1/2}(\omega^2 + \lambda)^{-1} V(\omega_0^2 + \lambda)^{-1} \right\|_2 \leq C(m^2 + \lambda)^{-3/2+\epsilon/2} \left\| \omega_0^{-1/2} V \omega_0^{-1-\epsilon} \right\|_2$$

with some positive  $C$ . Thus (A.4) holds true if  $0 < \epsilon < 1/2$ .

To prove that  $\omega_0^{-1/2}(\omega_0 - \omega)\omega_0^{-1/2}$  is Hilbert-Schmidt, we use the formula (A.1) for  $\alpha = 1/2$  and write

$$\omega_0^{-1/2}(\omega_0 - \omega)\omega_0^{-1/2} = -\frac{1}{\pi} \int_0^\infty d\lambda \lambda^{1/2} \omega_0^{-1/2}(\omega^2 + \lambda)^{-1} V(\omega_0^2 + \lambda)^{-1} \omega_0^{-1/2}.$$

where the above equation hold true on some dense domain (for instance  $D(\omega_0^{3/2})$ ). It suffices to show

$$\int_0^\infty d\lambda \lambda^{1/2} \left\| \omega_0^{-1/2}(\omega^2 + \lambda)^{-1} V(\omega_0^2 + \lambda)^{-1} \omega_0^{-1/2} \right\|_2 < \infty. \quad (\text{A.4})$$

If  $\omega_0^{-1/2} V \omega_0^{-1-\epsilon}$  ( $\epsilon > 0$ ) is Hilbert-Schmidt, then

$$\begin{aligned} & \left\| \omega_0^{-1/2}(\omega^2 + \lambda)^{-1} V(\omega_0^2 + \lambda)^{-1} \omega_0^{-1/2} \right\|_2 \\ & \leq \left\| \omega_0^{-1/2}(\omega^2 + \lambda)^{-1} \omega_0^{1/2} \right\| \cdot \left\| \omega_0^{-1/2} V \omega_0^{-1-\epsilon} \right\|_2 \cdot \left\| \omega_0^{1/2+\epsilon}(\omega_0^2 + \lambda)^{-1} \right\| \\ & \leq \left\| \omega_0^{-1/2} \omega^{1/2} \right\| \cdot \left\| (\omega^2 + \lambda)^{-1} \right\| \cdot \left\| \omega^{-1/2} \omega_0^{1/2} \right\| \cdot \left\| \omega_0^{-1/2} V \omega_0^{-1-\epsilon} \right\|_2 \\ & \quad \times \left\| \omega^{1/2+\epsilon}(\omega_0^2 + \lambda)^{-(1/2+\epsilon)/2} \right\| \cdot \left\| (\omega_0^2 + \lambda)^{-3/4+\epsilon/2} \right\| \\ & \leq C(m^2 + \lambda)^{\epsilon/2-7/4} \end{aligned}$$

with some  $C > 0$  and  $0 < \epsilon < 3/2$ . This implies that (A.3) holds true for  $0 < \epsilon < 1/2$  and we obtain the desired results.

It remains to show the following.

**Lemma A.1** *For any  $\epsilon > 0$ ,  $\omega_0^{-1/2}V\omega_0^{-1-\epsilon}$  is Hilbert-Schmidt.*

*Proof.* Note that

$$\omega_0^{-1/2}V\omega_0^{-1-\epsilon} = V\omega_0^{-3/2-\epsilon} + [\omega_0^{-1/2}, V]\omega_0^{-1-\epsilon}.$$

As seen above, the first term of the r.h.s. is Hilbert-Schmidt. It suffices to show that  $[\omega_0^{-1/2}, V]$  is Hilbert-Schmidt. To this end, we write it as

$$[\omega_0^{-1/2}, V] = \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{-1/2} [(\omega_0^2 + \lambda)^{-1}, V].$$

By direct calculation, we see that

$$\begin{aligned} [(\omega_0^2 + \lambda)^{-1}, V] &= (\omega_0^2 + \lambda)^{-1} [V, \omega_0^2] (\omega_0^2 + \lambda)^{-1} \\ &= V_1 + V_2, \end{aligned}$$

where

$$\begin{aligned} V_1 &= (\omega_0^2 + \lambda)^{-1} (\Delta V) (\omega_0^2 + \lambda)^{-1}, \\ V_2 &= 2(\omega_0^2 + \lambda)^{-1} (\nabla V) \cdot \nabla (\omega_0^2 + \lambda)^{-1}. \end{aligned}$$

One observes that the operators  $(\Delta V)(\omega_0^2 + \lambda)^{-1}$  and  $(\omega_0^2 + \lambda)^{-1}(\nabla_j V)$  are integral operators with the kernels  $(4\pi|x-y|)^{-1}(\Delta V)(x)e^{\sqrt{m^2+\lambda}|x-y|}$  and  $(4\pi|x-y|)^{-1}e^{\sqrt{m^2+\lambda}|x-y|}(\nabla_j V)(y)$ , respectively. Hence we have

$$\begin{aligned} \|(\Delta V)(\omega_0^2 + \lambda)^{-1}\|_2^2 &= (4\pi)^{-2} \int dx dy \frac{|(\Delta V)(x)|^2 e^{-2\sqrt{m^2+\lambda}|x-y|}}{|x-y|^2} \\ &= (4\pi)^{-2} \|(\Delta V)\|_{L^2}^2 \int dx \frac{e^{-2\sqrt{m^2+\lambda}|x|}}{|x|^2} \\ &= \frac{1}{\sqrt{m^2+\lambda}} \left( \frac{\|(\Delta V)\|_{L^2}^2}{(4\pi)^2} \int dx \frac{e^{-2|x|}}{|x|^2} \right) \end{aligned}$$

and

$$\|(\omega_0^2 + \lambda)^{-1}(\nabla_j V)\|_2^2 = \frac{1}{\sqrt{m^2 + \lambda}} \left( \frac{\|(\nabla V)\|_{L^2}^2}{(4\pi)^2} \int dx \frac{e^{-2|x|}}{|x|^2} \right).$$

Thus we obtain

$$\begin{aligned} \|V_1\|_2 &\leq \|(\omega_0^2 + \lambda)^{-1}\| \cdot \|V(\omega_0^2 + \lambda)^{-1}\|_2 \\ &\leq C(m^2 + \lambda)^{-5/4} \end{aligned}$$

and

$$\begin{aligned} \|V_2\|_2 &\leq \sum_{j=1}^3 \|(\omega_0^2 + \lambda)^{-1}(\nabla_j V)\|_2 \cdot \|\nabla_j(\omega_0^2 + \lambda)^{-1/2}\| \cdot \|(\omega_0^2 + \lambda)^{-1/2}\| \\ &\leq C(m^2 + \lambda)^{-3/4} \end{aligned}$$

with some  $C$  independent of  $\lambda$ . Hence  $[\omega_0^{-1/2}, V]$  is Hilbert-Schmidt since

$$\|[\omega_0^{-1/2}, V]\|_2 \leq C \int_0^\infty d\lambda \lambda^{-1/2} ((m^2 + \lambda)^{-5/4} + (m^2 + \lambda)^{-3/4}) < \infty. \quad \square$$

### A.2. Existence of the limits (1.8)

Under Assumptions 1.1 and 1.2, we prove the existence of

$$s\text{-}\lim_{t \rightarrow \pm\infty} e^{it\omega} e^{-it\omega_0}.$$

Since  $\omega - \omega_0$  is  $\omega_0$ -bounded and  $\mathcal{S}(\mathbb{R}^3)$  is dense in  $L^2(\mathbb{R}^3) = \mathcal{H}_{\text{ac}}(\omega_0)$ , it suffices to show that

$$\int_1^\infty dt \|(\omega - \omega_0)e^{-it\omega_0} f\|_{\mathfrak{h}} < \infty \quad (\text{A.5})$$

for any  $f \in \mathcal{S}(\mathbb{R}^3)$ . From (A.1), we see that

$$\begin{aligned} \omega - \omega_0 &= \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{-1/2} [(\omega^2 + \lambda)^{-1} \omega^2 - (\omega_0^2 + \lambda)^{-1} \omega_0^2] \\ &= \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{-1/2} [1 - \lambda(\omega^2 + \lambda)^{-1} - 1 + \lambda(\omega_0^2 + \lambda)^{-1}] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{1/2} [(\omega^2 + \lambda)^{-1} - (\omega_0^2 + \lambda)^{-1}] \\
&= \frac{1}{\pi} \int_0^\infty d\lambda \lambda^{1/2} (\omega^2 + \lambda)^{-1} [(\omega_0^2 + \lambda) - (\omega^2 + \lambda)] (\omega_0^2 + \lambda)^{-1} \\
&= -\frac{1}{\pi} \int_0^\infty d\lambda \lambda^{1/2} (\omega^2 + \lambda)^{-1} V (\omega_0^2 + \lambda)^{-1}.
\end{aligned}$$

Therefore, we have for any  $f \in \mathcal{S}(\mathbb{R}^3)$ ,

$$\begin{aligned}
&\|(\omega - \omega_0)e^{-it\omega_0} f\|_{\mathfrak{h}} \\
&\leq C \int_0^\infty d\lambda \lambda^{1/2} \|(\omega^2 + \lambda)^{-1} V (\omega_0^2 + \lambda)^{-1} e^{-it\omega_0} f\|_{\mathfrak{h}} \\
&\leq C \int_0^\infty d\lambda \lambda^{1/2} (m^2 + \lambda)^{-1} \|V (\omega_0^2 + \lambda)^{-1} e^{-it\omega_0} f\|_{\mathfrak{h}} \\
&\leq C \|V\|_{L^2(\mathbb{R}^3)} \int_0^\infty d\lambda \lambda^{1/2} (m^2 + \lambda)^{-1} \|(\omega_0^2 + \lambda)^{-1} e^{-it\omega_0} f\|_{L^\infty(\mathbb{R}^3)}.
\end{aligned}$$

Here, we have used the Hölder inequality in the last inequality. It follows from [4] that

$$\|(\omega_0^2 + \lambda)^{-1} e^{-it\omega_0} f\|_{L^\infty(\mathbb{R}^3)} \leq C |t|^{-3/2} \|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\|_{L^1(\mathbb{R}^3)}$$

for any  $t \in \mathbb{R}$ . Since

$$\begin{aligned}
&\|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\|_{L^1(\mathbb{R}^3)} \\
&\leq \|(1 + |x|)^{-2} (1 + |x|)^2 (\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\|_{L^1(\mathbb{R}^3)} \\
&\leq \|(1 + |x|)^{-2}\|_{L^2(\mathbb{R}^3)} \|(1 + |x|)^2 (\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\|_{L^2(\mathbb{R}^3)} \\
&\leq C \|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\|_{L^2(\mathbb{R}^3)} + C \| |x|^2 (\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f \|_{L^2(\mathbb{R}^3)}
\end{aligned}$$

and

$$|x|^2 (\omega_0^2 + \lambda)^{-1} = (\omega_0^2 + \lambda)^{-1} |x|^2 + 8(\omega_0^2 + \lambda)^{-3} \Delta - 6(\omega_0^2 + \lambda)^{-2},$$

we see that

$$\begin{aligned} & \|(\omega_0^2 + \lambda)^{-1} \omega_0^{5/2} f\|_{L^1(\mathbb{R}^3)} \\ & \leq C(m^2 + \lambda)^{-1} \{ \|\omega_0^{5/2} f\|_{\mathfrak{h}} + \| |x|^2 \omega_0^{5/2} f \|_{\mathfrak{h}} + \|\Delta \omega_0^{5/2} f\|_{\mathfrak{h}} \}. \end{aligned}$$

Thus, we obtain for any  $t \in \mathbb{R}$ ,

$$\|(\omega - \omega_0) e^{-it\omega_0} f\|_{\mathfrak{h}} \leq C|t|^{-3/2} \int_0^\infty d\lambda \lambda^{1/2} (m^2 + \lambda)^{-2}$$

and hence (A.5) holds for any  $f \in \mathcal{S}(\mathbb{R}^3)$ .

### A.3. Classical wave operator

We prove Lemma 2.6. It holds that

$$U(t)^* U_0(t) = \begin{bmatrix} U_{++}(t)^* e^{-it\omega_0} & U_{-+}(t)^* e^{it\omega_0} \\ U_{+-}(t)^* e^{-it\omega_0} & U_{--}(t)^* e^{it\omega_0} \end{bmatrix}.$$

For  $a, b, c, t$  and  $s \in \mathbb{R}$ , we set

$$I_{(a,b,c)}(s, t) = \omega_0^a e^{is\omega} \omega^b e^{it\omega_0} \omega_0^c.$$

If  $a + b + c = 0$ , then  $I_{(a,b,c)}(s, t)$  is bounded and

$$\overline{I_{(a,b,c)}(s, t)} = I_{(a,b,c)}(-s, -t).$$

By direct calculation, we have

$$\begin{aligned} U_{++}(t)^* e^{-it\omega_0} &= \overline{U_{--}(t)^* e^{it\omega_0}} \\ &= \frac{1}{4} [I_{(1/2,0,-1/2)}(t, -t) + I_{(-1/2,0,1/2)}(t, -t) \\ &\quad + I_{(1/2,-1,1/2)}(t, -t) + I_{(-1/2,1,-1/2)}(t, -t) \\ &\quad + I_{(1/2,0,-1/2)}(-t, -t) + I_{(-1/2,0,1/2)}(-t, -t) \\ &\quad - I_{(1/2,-1,1/2)}(-t, -t) - I_{(-1/2,1,-1/2)}(-t, -t)] \end{aligned}$$

and

$$\begin{aligned}
U_{-+}(t)^* e^{it\omega_0} &= \overline{U_{+-}(t)^* e^{-it\omega_0}} \\
&= \frac{1}{4} \left[ I_{(-1/2,0,1/2)}(t,t) - I_{(1/2,0,-1/2)}(t,t) \right. \\
&\quad + I_{(1/2,-1,1/2)}(t,t) - I_{(-1/2,1,-1/2)}(t,t) \\
&\quad + I_{(-1/2,0,1/2)}(-t,t) - I_{(1/2,0,-1/2)}(-t,t) \\
&\quad \left. - I_{(1/2,-1,1/2)}(-t,t) + I_{(-1/2,1,-1/2)}(-t,t) \right].
\end{aligned}$$

By the similar argument as in the proof of Lemma 2.7 with the aid of Lemma 2.2, one can prove the following lemma:

**Lemma A.2** *Suppose that Assumptions 1.1 and 1.2. Then:*

$$\begin{aligned}
\lim_{t \rightarrow \pm\infty} I_{(1/2,0,-1/2)}(t,-t) &= \lim_{t \rightarrow \pm\infty} I_{(1/2,-1,1/2)}(t,-t) = \omega_0^{1/2} w_{\pm} \omega_0^{-1/2}, \\
\lim_{t \rightarrow \pm\infty} I_{(-1/2,0,1/2)}(t,-t) &= \lim_{t \rightarrow \pm\infty} I_{(-1/2,1,-1/2)}(t,-t) = \omega_0^{-1/2} w_{\pm} \omega_0^{1/2}, \\
\lim_{t \rightarrow \pm\infty} I_{(1/2,0,-1/2)}(t,t) &= \lim_{t \rightarrow \pm\infty} I_{(1/2,-1,1/2)}(t,t) = 0, \\
\lim_{t \rightarrow \pm\infty} I_{(-1/2,0,1/2)}(t,t) &= \lim_{t \rightarrow \pm\infty} I_{(-1/2,1,-1/2)}(t,t) = 0.
\end{aligned}$$

By the above lemma, we have

$$\begin{aligned}
\lim_{t \rightarrow \pm\infty} U_{++}(t)^* e^{-it\omega_0} &= \frac{1}{2} [\omega_0^{-1/2} w_{\pm} \omega_0^{+1/2} + \omega_0^{+1/2} w_{\pm} \omega_0^{-1/2}], \\
\lim_{t \rightarrow \pm\infty} U_{-+}(t)^* e^{it\omega_0} &= \frac{1}{2} [\omega_0^{-1/2} w_{\mp} \omega_0^{+1/2} - \omega_0^{+1/2} w_{\mp} \omega_0^{-1/2}].
\end{aligned}$$

This completes the proof.

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Hironobu Sasaki  
Department of Mathematics and Informatics  
Chiba 263-8522, Japan  
E-mail: sasaki@math.s.chiba-u.ac.jp

Akito Suzuki  
Department of Engineering  
Shinshu University  
Nagano, 380-8553, Japan  
E-mail: akito@shinshu-u.ac.jp