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A remark on the commensurability for inclusions of ergodic measured equivalence relations

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Abstract. It is shown that, for each inclusion of ergodic discrete measured equivalence relations, the commensurability can be characterized in terms of measure theoretical arguments. As an application, we also include a measure theoretical proof concerning a property of the commensurability groupoid which determines the commensurability in terms of operator algebras. It is proven that a family of typical elements in the commensurability groupoid is closed under the product operation. This proof supplements a gap in the proof of [2, Lemma 7.5].

Key words: measured equivalence relation, commensurablility subrelation, choice functions, Cartan subalgebra.

1. Introduction

Let $\mathcal{S} \subseteq \mathcal{R}$ be an inclusion of ergodic discrete measured equivalence relations on a standard probability space (X, μ). As applications of the group theory, we can consider two notions for this inclusion–the normality and the commensurability.

The normality of equivalence relations is defined by the index cocycles ([3, Definition 2.1]). It is known that the normality of $\mathcal{S}$ in $\mathcal{R}$ is equivalent to the existence of a 1-cocycle on $\mathcal{R}$ to a countable group whose kernel coincides with $\mathcal{S}$ ([3, Theorem 2.2]). In [2], the author and T. Yamanouchi succeeded in characterizing this property in terms of operator algebras. They showed that the subrelation $\mathcal{S}$ is normal in $\mathcal{R}$ if and only if the corresponding factor $W^*(\mathcal{R})$ is generated by the normalizing groupoid of the subfactor $W^*(\mathcal{S})$ in $W^*(\mathcal{R})$ ([2, Theorem 5.11]).

On the other hand, to determine the commensurability, they further defined the commensurability groupoid for each inclusion of factors. They say that the subrelation $\mathcal{S}$ is commensurable in $\mathcal{R}$ if $W^*(\mathcal{R})$ is generated by the commensurability groupoid of $W^*(\mathcal{S})$ in $W^*(\mathcal{R})$ ([2, Theorem 7.11]). This means that their definition of the commensurability depends on the

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theory of operator algebras.

So it is natural to seek a characterization of the commensurability in terms of measure theoretical arguments. Our aim of this paper is to give such a characterization (Theorem 3.7).

The idea for our characterization is to count the number of orbits for each inclusion of equivalent classes $\mathcal{S}(x) := \{y \in X : (x, y) \in \mathcal{S}\} \subseteq \mathcal{R}(x) := \{y \in X : (x, y) \in \mathcal{R}\}$ of $x \in X$. In fact, we define the index $\Phi(\rho)$ for each measurable nonsingular map ρ whose graph is contained in $\mathcal{R}$. Roughly, the index $\Phi(\rho)$ is the number of $\mathcal{S}$-equivalent classes in $\mathcal{S}(\rho(\mathcal{S}(x))) := \bigcup_{y \in \mathcal{S}(x) \cap \mathrm{Dom}(\rho)} \mathcal{S}(\rho(y))$ (Lemma 3.3). We will show that the index $\Phi(\rho)$ is determined by the corresponding projection with an operator valued weight (Proposition 3.4). It follows that the commensurability subrelation is constructed by countable elements in $\Phi^{-1}(\mathbf{N})$.

Moreover, as an application of our arguments, we shall give a measure theoretical proof of [2, Lemma 7.5] (Corollary 3.6). We note that although their claim is valid, there exists a gap in their proof. So our complete proof will justify all the rest of arguments in [2].

We also develop the theory of choice functions. We will show that the commensurability coincides with the existence of choice functions which have remarkable properties (Theorem 3.8). This result is a generalization of a characterization of normality by choice functions in [3, Theorem 2.2].

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2. Preparation

In this section, we summarize basic facts about the theory of measured equivalence relations with the corresponding von Neumann algebras. Further details for these matters can be found in [1], [2], [3] and [4].

Let $\mathcal{R}$ be a discrete measured equivalence relation on a standard probability space (X, μ) with a normalized 2-cocycle ω, the left-hand projection π_l, the left counting measure ν and the Radon-Nikodym derivative δ. A measurable map φ on X is called nonsingular if $\varphi^{-1}(E)$ is a null set for each null set E in X. We denote by $[\mathcal{R}]_*$ the groupoid of $\mathcal{R}$. Namely, each $\varphi \in [\mathcal{R}]_*$ is a bimeasurable nonsingular map from a measurable subset $\mathrm{Dom}(\varphi)$ of X onto a measurable subset $\mathrm{Im}(\varphi)$ of X such that the graph $\Gamma(\varphi) := \{(x, \varphi(x)) : x \in \mathrm{Dom}(\rho)\}$ is contained in $\mathcal{R}$ up to a null set. We denote the equivalent class $\{y \in X : (x, y) \in \mathcal{R}\}$ by $\mathcal{R}(x)$ for each $x \in X$. Since $\mathcal{R}$ is discrete, each $\mathcal{R}(x)$ is a countable set.

For each suitable function f on $\mathcal{R}$, we can consider a bounded operator $L^\omega(f)$ on $L^2(\mathcal{R}, \nu)$ which is defined by the following:

$$L^\omega(f)\xi(x, z) := \sum_{y \in \mathcal{R}(x)} f(x, y)\xi(y, z)\omega(x, y, z), \quad (\xi \in L^2(\mathcal{R}, \nu)).$$

We denote the set of all such bounded operators by $W^*(\mathcal{R}, \omega)$. It is known that $\xi_0 := \chi_{\mathcal{D}}$ is a cyclic and separating vector for $W^*(\mathcal{R}, \omega)$ in $L^2(\mathcal{R}, \nu)$, where $\mathcal{D} := \{(x, x) : x \in X\}$ and χ_S is in general the characteristic function of a set S. For each discrete measured equivalence subrelation $\mathcal{S}$ of $\mathcal{R}$, the subset $\{L^\omega(f) \in W^*(\mathcal{R}, \omega) : \operatorname{supp}(f) \subseteq \mathcal{S}\}$ of $W^*(\mathcal{R}, \omega)$ is denoted by $W^*(\mathcal{S}, \omega)$. It is known that $W^*(\mathcal{S}, \omega)$ is a von Neumann subalgebra of $W^*(\mathcal{R}, \omega)$. In particular, the diagonal algebra $W^*(X) := \{L(g) : g \in L^\infty(X)\}$ is called a Cartan subalgebra of $W^*(\mathcal{R}, \omega)$, where $L(g) \in W^*(\mathcal{R}, \omega)$ acts on $L^2(\mathcal{R}, \nu)$ by the following:

$$L(g)\xi(x, z) := g(x)\xi(x, z), \quad (\xi \in L^2(\mathcal{R}, \nu)).$$

For each $\rho \in [\mathcal{R}]_*$, we define an element v_ρ in $W^*(\mathcal{R}, \omega)$ by the following:

$$v_\rho := L^\omega(f_\rho),\ f_\rho(x, y) := \delta(x, y)^{-1/2}\chi_{\Gamma(\rho)}(x, y).$$

A direct calculation shows that each v_ρ belongs to the normalizing groupoid $\mathcal{GN}(W^*(X))$ of $W^*(X)$ in $W^*(\mathcal{R}, \omega)$. We recall that, for an inclusion of von Neumann algebras $B \subseteq A$, the normalizing groupoid $\mathcal{GN}(B)$ of B in A is defined by the following:

$$\begin{aligned}\mathcal{GN}(B) := \{v \in A :\ & v \text{ is a partial isometry}, \\ & vv^*,\ v^*v \in B,\ vBv^* = vv^*Bvv^*\}.\end{aligned}$$

It is known that $W^*(\mathcal{R}, \omega)$ is generated by $\mathcal{GN}(W^*(X))$. Moreover, for each $v \in W^*(\mathcal{R}, \omega)$, v is in $\mathcal{GN}(W^*(X))$ if and only if v is of the form $L(g)v_\rho$, where $\rho \in [\mathcal{R}]_*$ and g is a measurable function on $\operatorname{Dom}(\rho)$ of absolute value one.

Suppose that $\mathcal{S}$ is an equivalence subrelation of an ergodic discrete measured equivalence relation $\mathcal{R}$. A countable functions $\{\varphi_i\}_{i \in I}$ are called choice functions if $\mathcal{R}(x)$ is a disjoint union of $\mathcal{S}$-equivalent classes $\{\mathcal{S}(\varphi_i(x))\}_{i \in I}$ for a.e. $x \in X$. For each choice functions $\{\varphi_i\}_{i \in I}$, we define a 1-cocycle σ

from $\mathcal{R}$ to the permutation group Σ_I of I by the following rule:

$$\sigma(x, y)(i) = j \text{ if } (\varphi_j(x), \varphi_i(y)) \in \mathcal{S}.$$

We call σ the index cocycle determined by $\{\varphi_i\}_{i\in I}$. The equivalence subrelation $\mathcal{S}$ is called normal in $\mathcal{R}$ if the restriction of the index cocycle σ to $\mathcal{S}$ cobounds. It is known that $\mathcal{S}$ is normal in $\mathcal{R}$ if and only if there exist choice functions $\{\varphi_i\}_{i\in I}$ with the index cocycle σ such that $\mathcal{S}$ coincides with $\mathrm{Ker}(\sigma)$, i.e., $(\varphi_i(x), \varphi_i(y))$ is in $\mathcal{S}$ for each $i \in I$ and a.e. $(x, y) \in \mathcal{S}$ ([3, Theorem 2.2]).

In the rest of this section, we assume that $\mathcal{S}$ is ergodic. Set $(D \subseteq B \subseteq A) := (W^*(X) \subseteq W^*(\mathcal{S}, \omega) \subseteq W^*(\mathcal{R}, \omega))$. By [2, Theorem 5.16], there exists the largest intermediate subrelation $N_{\mathcal{R}}(\mathcal{S})$ of $\mathcal{S} \subseteq \mathcal{R}$ such that $\mathcal{S}$ is normal in $N_{\mathcal{R}}(\mathcal{S})$. We call $N_{\mathcal{R}}(\mathcal{S})$ the normalizer of $\mathcal{S}$ in $\mathcal{R}$. It is known that the normalizing groupoid $\mathcal{GN}(B)$ of B in A is contained in the intermediate subfactor $W^*(N_{\mathcal{R}}(\mathcal{S}), \omega)$, and $W^*(N_{\mathcal{R}}(\mathcal{S}), \omega)$ is generated by $\mathcal{GN}(B) \cap \mathcal{GN}(D)$.

Furthermore, we denote the unique faithful normal conditional expectation from A onto B by E_B with the Jones projection e_B and the basic extension $A_1 := A \vee \{e_B\}$. The inclusion $B \subseteq A$ is called discrete if the map $\widehat{E_B}|_{A_1\cap B'}$ is semifinite, where $\widehat{E_B}$ is the operator valued weight dual to E_B. It is known that the inclusion $B \subseteq A$ is discrete if and only if there exist minimal projections $\{e_n\}_{n=1}^{\infty}$ in $A_1 \cap B'$ which satisfies $\sum_{n=1}^{\infty} e_n = 1$ and $\widehat{E_B}(e_n) < \infty$ for all $n \in \mathbf{N}$ (see [4]).

For each $L^\omega(f)\in A$, the projection $z_{L^\omega(f)}$ from $L^2(\mathcal{R},\nu)$ onto $[BL^\omega(f)B\xi_0]$ belongs to $A_1 \cap B'$, where $[S]$ in general stands for the closed subspace spanned by a set S. For each measurable nonsingular map ρ satisfying $\Gamma(\rho) \subseteq \mathcal{R}$ up to a null set, we define a projection z_ρ in $A_1 \cap B'$ by the following:

$$z_\rho := \bigvee\{z_{v_\theta} \colon \theta \in [\mathcal{R}]_*, \Gamma(\theta) \subseteq \Gamma(\rho)\}.$$

We note that z_ρ coincides with z_{v_ρ} if ρ is in $[\mathcal{R}]_*$. Moreover, we get the following

Proposition 2.1 *Let the notations be as above. For each $L^\omega(f) \in A$, the projection $z_{L^\omega(f)}$ is equal to $\bigvee\{z_\rho \colon \rho \in [\mathcal{R}]_*, \Gamma(\rho) \subseteq \mathrm{supp}(f)\}$. In particular, $z_{L^\omega(f)}$ is equal to χ_{E_f}, where*

$$E_f := \{(x, y) \colon \exists \rho \in [\mathcal{R}]_* \textit{ s.t. } \Gamma(\rho) \subseteq \mathrm{supp}(f) \textit{ and } y \in \mathcal{S}(\rho(\mathcal{S}(x)))\}.$$

Proof. Put $p := \bigvee\{z_\rho : \rho \in [\mathcal{R}]_*, \Gamma(\rho) \subseteq \mathrm{supp}(f)\}$. A direct computation shows that $z_{L^\omega(f)}$ contains $\chi_{\mathrm{supp}(f_1 * f * f_2)}$ if both $L^\omega(f_1)$ and $L^\omega(f_2)$ belong to B. In particular, $z_{L^\omega(f)}$ contains $\chi_{\Gamma(\rho)}$ if $\Gamma(\rho) \subseteq \mathrm{supp}(f)$. So we obtain $z_\rho \leq z_{L^\omega(f)}$, and $p \leq z_{L^\omega(f)}$.

If $z_{L^\omega(f)} - p > 0$, then there exists $\psi \in [\mathcal{R}]_*$ such that $0 < \chi_{\Gamma(\psi)} \leq z_{L^\omega(f)} - p$. Since $\chi_{\Gamma(\psi)}$ is contained in $z_{L^\omega(f)}$, there exist $L^\omega(f_1)$, $L^\omega(f_2) \in B$ such that $\mathrm{supp}(f_1 * f * f_2) \cap \Gamma(\psi)$ is not a null set. Since A is generated by $\mathcal{GN}(D)$, there exists a sequence $\{f_{0,n}\}_{n=1}^\infty$ in the linear span of $\mathcal{GN}(D)$ such that $L^\omega(f_1 * f_{0,n} * f_2)$ converges to $L^\omega(f_1 * f * f_2)$ in the sense of the strong operator topology. This means that there exists $\rho \in [\mathcal{R}]_*$ such that $\mathrm{supp}(f_1 * f_\rho * f_2) \cap \Gamma(\psi)$ is not a null set. Moreover, since $\mathcal{GN}(D) \cap B$ generates B, by using the same arguments, there exist θ_1, $\theta_2 \in [\mathcal{S}]_*$ such that $\mathrm{supp}(f_{\theta_1} * f_\rho * f_{\theta_2}) \cap \Gamma(\psi)$ is not a null set. Hence there exists a measurable non-null subset F of $\mathrm{Dom}(\psi)$ such that $\Gamma(\psi|_F)$ is contained in $\mathrm{supp}(f_{\theta_1} * f_\rho * f_{\theta_2}) = \Gamma(\theta_2 \circ \rho \circ \theta_1)$. It follows that $z_{\rho|_{\mathrm{Im}(\theta_1)}}$ contains $\chi_{\Gamma(\psi|_F)}$. This contradicts the assumption $p\chi_{\Gamma(\psi)} = 0$. So we conclude that $z_{L^\omega(f)}$ is equal to p.

The last assertion follows from the fact that z_ρ is equal to χ_{E_ρ} for each $\rho \in [\mathcal{R}]_*$, where $E_\rho := \{(x, y) \colon x \in X,\ y \in \mathcal{S}(\rho(\mathcal{S}(x)))\}$. Indeed, there exists a countable subset $\{\theta_n\}_{n=1}^\infty$ of $[\mathcal{S}]_*$ such that $\bigcup_{n=1}^\infty \Gamma(\theta_n)$ is equal to $\mathcal{S}$ up to a null set. Since the orbit $\mathcal{S}(\rho(\mathcal{S}(x)))$ is equal to $\bigcup_{n_1,n_2=1}^\infty \theta_{n_2}(\rho(\theta_{n_1}(x)))$ for a.e. $x \in X$, we have that E_ρ coincides with a union of $\{\Gamma(\theta_{n_2} \circ \rho \circ \theta_{n_1})\}_{n_1,n_2=1}^\infty$ up to a null set. In particular, E_ρ is measurable. Moreover, for each θ_{n_1}, $\theta_{n_2} \in [\mathcal{S}]_*$, we have

$$\Gamma(\theta_{n_2} \circ \rho \circ \theta_{n_1}) = \mathrm{supp}(f_{\theta_{n_1}} * f_\rho * f_{\theta_{n_2}}) \subseteq [B v_\rho B \xi_0].$$

This yields $\chi_{E_\rho} \leq z_\rho$. The converse inequality holds because $\Gamma(\rho)$ is contained in E_ρ and χ_{E_ρ} belongs to $A_1 \cap (\mathcal{GN}(D) \cap B)' = A_1 \cap B'$.

Therefore we get the conclusion. □

We define the commensurability groupoid $\mathcal{CG}(B)$ of B in A by the following:

$$\mathcal{CG}(B) := \{v \in A \colon v \text{ is a partial isometry}, \quad vv^*,\ v^*v \in B,\ \widehat{E_B}(z_v),\ \widehat{E_B}(z_{v^*}) < \infty\}.$$

By [1, Corollary 3.5], there exists an intermediate subrelation $\mathrm{Comm}_{\mathcal{R}}(\mathcal{S})$ of $\mathcal{S} \subseteq \mathcal{R}$ such that $\mathcal{CG}(B)$ generates $W^*(\mathrm{Comm}_{\mathcal{R}}(\mathcal{S}), \omega)$. The subrelation

$\mathrm{Comm}_{\mathcal{R}}(\mathcal{S})$ is called the commensurability subrelation of $\mathcal{S}$ in $\mathcal{R}$. The subrelation $\mathcal{S}$ is called commensurable in $\mathcal{R}$ if $\mathrm{Comm}_{\mathcal{R}}(\mathcal{S})$ is equal to $\mathcal{R}$.

By [2, Theorem 7.9] and [2, Theorem 7.11], we have that $W^*(\mathrm{Comm}_{\mathcal{R}}(\mathcal{S}), \omega)$ is generated by $\mathcal{CG}(B) \cap \mathcal{GN}(D)$, and the inclusion $B \subseteq A$ is discrete if and only if $\mathcal{S}$ is commensurable in $\mathcal{R}$.

3. Characterizations of the commensurability

For our purpose, we first introduce a notion of index for each measurable nonsingular map whose graph is contained in the equivalence relation.

Let $\mathcal{R}$ be an ergodic discrete measured equivalence relation on (X, μ) and $\mathcal{S}$ be an equivalence subrelation of $\mathcal{R}$ with choice functions $\{\varphi_i\}_{i\in I}$. For each measurable nonsingular map ρ satisfying $\mu(\mathrm{Dom}(\rho)) > 0$ and $\Gamma(\rho) \subseteq \mathcal{R}$ up to a null set, we define a function $\varPhi_\rho$ from $\mathrm{Dom}(\rho)$ to $\mathbf{N} \cup \{\infty\}$ by the following:

$$\varPhi_\rho(x) := |\{i \in I \colon \rho(\mathcal{S}(x)) \cap \mathcal{S}(\varphi_i(\rho(x))) \neq \emptyset\}|,$$

where $|S|$ in general stands for the cardinality of a set S.

We note that the function $\varPhi_\rho$ does not depend on the choice of $\{\varphi_i\}_{i\in I}$. Moreover, we get the following

Lemma 3.1 *Under the above setting, $\varPhi_\rho$ is a measurable $\mathcal{S}_{\mathrm{Dom}(\rho)}$-invariant function, where $\mathcal{S}_{\mathrm{Dom}(\rho)} := \mathcal{S} \cap \mathrm{Dom}(\rho) \times \mathrm{Dom}(\rho)$.*

Proof. By [1, Lemma 2.3], there exists a countable subset $\{\theta_n\}_{n=1}^\infty$ of $[\mathcal{S}]_*$ such that $\mathcal{S}$ is equal to a disjoint union of $\{\Gamma(\theta_n)\}_{n=1}^\infty$ up to a null set. For each $n \in \mathbf{N}$, we have

$$\varPhi_\rho^{-1}(\{k \in \mathbf{N} \colon k \geq n\}) = \bigcup_{\substack{i_1, \ldots, i_n \in I \\ i_l \neq i_m (l \neq m)}} \bigcap_{k=1}^{n} \bigcup_{p,q=1}^{\infty} \pi_l\bigl(\Gamma(\rho \circ \theta_p) \cap \Gamma(\theta_q \circ \varphi_{i_k} \circ \rho)\bigr).$$

Since each $\Gamma(\rho \circ \theta_p) \cap \Gamma(\theta_q \circ \varphi_{i_k} \circ \rho)$ is a measurable subset of $\mathcal{R}$, $\varPhi_\rho^{-1}(\{k \in \mathbf{N} \colon k \geq n\})$ is also measurable, i.e., $\varPhi_\rho$ is a measurable function.

Moreover, for a.e. $(x, y) \in \mathcal{S}_{\mathrm{Dom}(\rho)}$, we have $\rho(\mathcal{S}(x)) = \rho(\mathcal{S}(y))$ and $\mathcal{S}(\varphi_i(\rho(x))) = \mathcal{S}(\varphi_{\sigma(\rho(y), \rho(x))(i)}(\rho(y)))$ for each $i \in I$. This yields $\varPhi_\rho(x) = \varPhi_\rho(y)$.

So we complete the proof. ∎

Hence, under the condition that the subrelation $\mathcal{S}$ is ergodic, for each such measurable map ρ, the function Φ_ρ is constant on $\mathrm{Dom}(\rho)$ up to a null set. Namely, we have the following

Definition 3.2 Suppose that $\mathcal{S}$ is an ergodic equivalence subrelation of $\mathcal{R}$. For each measurable nonsingular map ρ satisfying $\mu(\mathrm{Dom}(\rho)) > 0$ and $\Gamma(\rho) \subseteq \mathcal{R}$ up to a null set, there exists a unique number n in $\mathbf{N} \cup \{\infty\}$ such that $\Phi_\rho^{-1}(\{n\})$ is conull in $\mathrm{Dom}(\rho)$. The number n is called the index of ρ in $\mathcal{S}$ and denoted by $\Phi(\rho)$.

We note that, by using Lemma 3.1, $\Phi(\rho \circ \theta)$ is equal to $\Phi(\rho)$ for each measurable nonsingular map θ satisfying $\mathrm{Im}(\theta) = \mathrm{Dom}(\rho)$ and $\Gamma(\theta) \subseteq \mathcal{S}$ up to null sets. Moreover, the index $\Phi(\rho)$ is, in a sense, equal to the number of $\mathcal{S}$-equivalent classes. Namely, we have the following

Lemma 3.3 *Under the above situation, the following are equivalent*:

(1) *The index* $\Phi(\rho)$ *is equal to* $n \in \mathbf{N}$.

(2) *There exist a measurable non-null subset* E *of* $\mathrm{Dom}(\rho)$ *and* n *elements* $\{i_k\}_{k=1}^n$ *in* I *which satisfy the following for all* $x \in E$:

$$\rho(\mathcal{S}(x)) \cap \mathcal{S}((\varphi_{i_k} \circ \rho)(x)) \neq \emptyset \quad (k = 1, \ldots, n),$$

$$\rho(\mathcal{S}(x)) \subseteq \bigcup_{k=1}^{n} \mathcal{S}((\varphi_{i_k} \circ \rho)(x)).$$

(3) *There exist a measurable non-null subset* E *of* $\mathrm{Dom}(\rho)$ *and measurable nonsingular maps* $\{\rho_k\}_{k=1}^n$ *on* E *which satisfy the following for all* $x \in E$:

$$\Gamma(\rho_k) \subseteq \mathcal{R} \quad (k = 1, \ldots, n),$$

$$(\rho_{k_1}(x), \rho_{k_2}(x)) \notin \mathcal{S} \quad (k_1 \neq k_2),$$

$$\mathcal{S}(\rho(\mathcal{S}(x))) = \bigcup_{k=1}^{n} \mathcal{S}(\rho_k(x)).$$

Proof. (1) $\Rightarrow$ (2): Since the set of finite subsets of I is countable, there exists a measurable non-null subset E_0 of $\mathrm{Dom}(\rho)$ and n elements $\{i_k\}_{k=1}^n$ in I such that $\rho(\mathcal{S}(x)) \cap \mathcal{S}(\varphi_{i_k}(\rho(x)))$ is not empty for each $x \in E_0$ and $k = 1, \ldots, n$. Put $E_1 := \{x \in E_0 \colon \rho(\mathcal{S}(x)) \not\subseteq \bigcup_{k=1}^{n} \mathcal{S}(\varphi_{i_k}(\rho(x))\}$. It is easy to check that E_1 is measurable. We claim that E_1 is a null set. Indeed, if E_1 were non-null, then, by the definition of the choice functions $\{\varphi_i\}_{i \in I}$,

there exist $i_0 \in I \setminus \{i_k\}_{k=1}^n$ and a measurable non-null subset F of E_1 such that $\rho(\mathcal{S}(x)) \cap \mathcal{S}(\varphi_{i_0}(\rho(x)))$ is not empty for each $x \in F$. This yields $\Phi(\rho) \geq n+1$, a contradiction. Hence $E := E_0 \setminus E_1$ and $\{i_k\}_{k=1}^n$ satisfy the desired properties.
(2) $\Rightarrow$ (3): By the definition of the choice functions, there exist a measurable non-null subset E' of E and a subset $\{i'_k\}_{k=1}^n$ of I such that $((\varphi_{i_k} \circ \rho)(x),\, \varphi_{i'_k}(x))$ belongs to $\mathcal{S}$ for each $k = 1, \ldots, n$ and $x \in E'$. It follows that $\rho(\mathcal{S}(x))$ is contained in $\bigcup_{k=1}^n \mathcal{S}(\varphi_{i'_k}(x))$, and $\rho(\mathcal{S}(x)) \cap \mathcal{S}(\varphi_{i'_k}(x))$ is not empty for each $x \in E'$. So we conclude that $\mathcal{S}(\rho(\mathcal{S}(x)))$ is equal to a disjoint union of $\{\mathcal{S}(\varphi_{i'_k}(x))\}_{k=1}^n$ for each $x \in E'$. Hence E' and $\{\varphi_{i'_k}\}_{k=1}^n$ satisfy the desired properties.
(3) $\Rightarrow$ (1): By using the same arguments, there exist a measurable non-null subset E' of E and a subset $\{i_k\}_{k=1}^n$ of I such that $((\varphi_{i_k} \circ \rho)(x),\, \rho_k(x))$ belongs to $\mathcal{S}$ for each $k = 1, \ldots, n$ and $x \in E'$. It follows that $\rho(\mathcal{S}(x))$ is contained in $\bigcup_{k=1}^n \mathcal{S}((\varphi_{i_k} \circ \rho)(x))$ and $\rho(\mathcal{S}(x)) \cap \mathcal{S}((\varphi_{i_k} \circ \rho)(x))$ is not empty for each $x \in E'$. This means that the index $\Phi(\rho)$ is equal to n.

So we get the conclusion. □

In what follows, we assume that $\mathcal{S}$ is an ergodic equivalence subrelation of $\mathcal{R}$ and ω is a 2-cocycle on $\mathcal{R}$. Set $(D \subseteq B \subseteq A) := (W^*(X) \subseteq W^*(\mathcal{S}, \omega) \subseteq W^*(\mathcal{R}, \omega))$. We will characterize the index $\Phi(\rho)$ in terms of operator algebras.

Proposition 3.4 *For each measurable nonsingular map ρ satisfying $\mu(\mathrm{Dom}(\rho)) > 0$ and $\Gamma(\rho) \subseteq \mathcal{R}$ up to a null set, the index $\Phi(\rho)$ of ρ coincides with $\widehat{E_B}(z_\rho)$. Moreover, if $\Phi(\rho)$ is finite, then there exists a finite number of measurable non-null subsets $\{F_k\}_{k=1}^m$ of $\mathrm{Dom}(\rho)$ which satisfy $\rho|_{F_k} \in [\mathcal{R}]_*$ $(k = 1, \ldots, m)$, $z_\rho = \sum_{k=1}^m z_{\rho|_{F_k}}$ and $\Phi(\rho) = \sum_{k=1}^m \Phi(\rho|_{F_k})$.*

Proof. We first prove when $\Phi(\rho)$ is finite. Put $n := \Phi(\rho)$. Since $\mathcal{S}$ is ergodic, by a standard maximal argument, there exists a measurable nonsingular map θ_0 on $X \setminus \mathrm{Dom}(\rho)$ such that $\Gamma(\theta_0) \subseteq \mathcal{S}$ and $\mathrm{Im}(\theta_0) \subseteq \mathrm{Dom}(\rho)$ up to a null set. We define a measurable nonsingular map θ on X by the following:

$$\theta(x) = \begin{cases} x & x \in \mathrm{Dom}(\rho), \\ \theta_0(x) & x \in X \setminus \mathrm{Dom}(\rho). \end{cases}$$

We can easily check that θ satisfies $\Gamma(\theta) \subseteq \mathcal{S}$ and $\mathrm{Im}(\theta) = \mathrm{Dom}(\rho)$ up to null sets. This yields $\Phi(\rho \circ \theta) = \Phi(\rho) = n$.

We claim that there exists a measurable function g from $X \times \{1, \ldots, n\}$ to I which satisfies $\rho(\mathcal{S}(x)) \subseteq \bigcup_{k=1}^{n} \mathcal{S}(\varphi_{g(x,k)}((\rho\circ\theta)(x)))$ and $\rho(\mathcal{S}(x)) \cap \mathcal{S}(\varphi_{g(x,k)}((\rho\circ\theta)(x))) \neq \emptyset$ for a.e. $x \in X$. Indeed, by renumbering, we may and do assume that I is equal to $\{m \in \mathbf{N} \colon m < M\}$ for some $M \in \mathbf{N} \cup \{\infty\}$. For each $x \in X$ and $k = 1, \ldots, n$, we define an element $g(x, k)$ in I by the following:

$$\begin{aligned} &g(x, k) \\ &:= \min\bigl\{m \in I \colon |\{i \leq m \colon \rho(\mathcal{S}(x)) \cap \mathcal{S}(\varphi_i((\rho\circ\theta)(x))) \neq \emptyset\}| = k\bigr\}. \end{aligned}$$

A direct computation shows that the function g has the desired properties.

Put $\rho_k(x) := \varphi_{g(x,k)}(\rho\circ\theta(x))$ for each $k = 1, \ldots, n$ and $x \in X$. It is easy to check that $\{\rho_k\}_{k=1}^{n}$ are measurable and satisfy the following for a.e. $x \in X$:

$$(x, \rho_k(x)) \in E_\rho, \tag{3.1}$$

$$(\rho_{k_1}(x), \rho_{k_2}(x)) \notin \mathcal{S} \quad (k_1 \neq k_2). \tag{3.2}$$

On the other hand, for each $k = 1, \ldots, n$, there exists a measurable partition $\{F_{k,m}\}_{m=1}^{\infty}$ of X such that each $\rho_k|_{F_{k,m}}$ belongs to $[\mathcal{R}]_*$. Put $v_{k,m} := v_{\rho_k|_{F_{k,m}}}$. By (3.1), we have that $v_{k,m} e_B v_{k,m}^*$ is a projection onto $[v_{k,m} B \xi_0]$ and majorized by $z_\rho = \chi_{E_\rho}$ for each $k = 1, \ldots, n$ and $m \in \mathbf{N}$. We claim that $\{v_{k,m} e_B v_{k,m}^*\}_{k,m}$ are mutually orthogonal projections. Indeed, by using (3.2), we have

$$\begin{aligned} v_{k_1,m_1} e_B v_{k_1,m_1}^* v_{k_2,m_2} e_B v_{k_2,m_2}^* &= v_{k_1,m_1} E_B(v_{k_1,m_1}^* v_{k_2,m_2}) e_B v_{k_2,m_2}^* \\ &= 0, \end{aligned}$$

for $k_1 \neq k_2$. On the other hand, for each $k = 1, \ldots, n$, we have $v_{k,m_1}^* v_{k,m_2} = 0$ if $m_1 \neq m_2$. Thus our claim has been proven. Put

$$p := \sum_{k=1}^{n} \sum_{m=1}^{\infty} v_{k,m} e_B v_{k,m}.$$

It is easy to check that p is a projection which satisfies the following:

$$p \leq z_\rho, \quad \widehat{E_B}(p) = \sum_{k=1}^{n} \sum_{m=1}^{\infty} v_{k,m} v_{k,m}^* = n. \tag{3.3}$$

We claim that p is equal to z_ρ. Indeed, if $z_\rho - p$ is not equal to 0, then there exists $\psi \in [\mathcal{R}]_*$ such that $0 < \chi_{\Gamma(\psi)} \leq z_\rho - p$. It follows that there exists a subset $\{i_k\}_{k=1}^n$ of I such that $F' := \{x \in X : g(x, k) = i_k \ (k = 1, \ldots, n)\} \cap \mathrm{Dom}(\psi)$ is not a null set. This means that the equation $\rho_k(x) = \varphi_{i_k}((\rho \circ \theta)(x))$ holds for each $x \in F'$ and $k = 1, \ldots, n$. On the other hand, since $\psi(x)$ is in $\mathcal{S}((\rho \circ \theta)(\mathcal{S}(x)))$ for each $x \in F'$, we have that there exists a measurable non-null subset F of F' and $k \in \{1, \ldots, n\}$ such that $\psi(x)$ belongs to $\mathcal{S}(\rho_k(x))$ for each $x \in F$. So we get $p\chi_{\Gamma(\psi)} \geq z_{\rho_k|_F}\chi_{\Gamma(\psi)} > 0$, a contradiction. Hence we conclude that $\widehat{E_B}(z_\rho)$ is equal to n.

Moreover, by using the same arguments as in the proof of Proposition 2.1, for each measurable non-null subsets F_1, F_2 of $\mathrm{Dom}(\rho)$, the following equation holds up to a null set:

$$\begin{aligned} E_{\rho|_{F_1 \cup F_2}} &= \bigcup_{n_1, n_2 = 1}^{\infty} \Gamma(\theta_{n_2} \circ \rho|_{F_1 \cup F_2} \circ \theta_{n_1}) \\ &= \bigcup_{j=1,2} \bigcup_{n_1, n_2 = 1}^{\infty} \Gamma(\theta_{n_2} \circ \rho|_{F_j} \circ \theta_{n_1}) = E_{\rho|_{F_1}} \cup E_{\rho|_{F_2}}, \end{aligned}$$

where $\{\theta_n\}_{n=1}^{\infty} \subseteq [\mathcal{S}]_*$ satisfies $\mathcal{S} = \bigcup_{n=1}^{\infty} \Gamma(\theta_n)$ up to a null set. It follows that, for each measurable non-null subset F of $\mathrm{Dom}(\rho)$ such that $E_\rho \setminus E_{\rho|_F}$ is a non-null set, there exists a measurable non-null subset F' of $\mathrm{Dom}(\rho)$ such that $E_{\rho|_{F'}}$ does not intersect $E_{\rho|_F}$ up to a null set. Since $\widehat{E_B}(\chi_{E_\rho}) = n < \infty$ and $\widehat{E_B}(\chi_{E_{\rho|_F}}) \in \mathbf{N}$ for each measurable non-null subset F of $\mathrm{Dom}(\rho)$, we conclude that there exist a finite number of measurable non-null subsets $\{F_k\}_{k=1}^m$ of $\mathrm{Dom}(\rho)$ which satisfy $\rho|_{F_k} \in [\mathcal{R}]_*$ and E_ρ is a disjoint union of $\{E_{\rho|_{F_k}}\}_{k=1}^m$ up to a null set. It follows that z_ρ is equal to $\sum_{k=1}^m z_{\rho|_{E_k}}$. This yields

$$\Phi(\rho) = \widehat{E_B}(\chi_{E_\rho}) = \sum_{k=1}^{m} \widehat{E_B}(z_{\rho|_{E_k}}) = \sum_{k=1}^{m} \Phi(\rho|_{E_k}).$$

So the second half assertion follows.

Secondly, suppose that $\Phi(\rho)$ is equal to ∞. By using the same arguments, we have that there exists a projection p which satisfies (3.3) for each $n \in \mathbf{N}$. This means that $\widehat{E_B}(z_\rho)$ is also equal to ∞.

So we complete the proof. $\square$

We shall next show that the set $\Phi^{-1}(\mathbf{N})$ is closed under the composition. We note that this claim is equivalent to [2, Lemma 7.5]. But their proof contains a gap. So we will give a complete proof for the claim. We emphasize that the gap in the proof of [2, Lemma 7.5] do not influence any other arguments in [2].

Proposition 3.5 *Suppose that ρ_1 and ρ_2 are measurable nonsingular maps satisfying $\Gamma(\rho_k) \subseteq \mathcal{R}$ up to a null set for $k = 1, 2$, and $\mu(\mathrm{Im}(\rho_2) \cap \mathrm{Dom}(\rho_1)) > 0$. Then the inequality $\varPhi(\rho_1 \circ \rho_2) \leq \varPhi(\rho_1)\varPhi(\rho_2)$ holds.*

Proof. It suffices to show when both $\varPhi(\rho_1)$ and $\varPhi(\rho_2)$ are finite and $\mathrm{Im}(\rho_2)$ is contained in $\mathrm{Dom}(\rho_1)$. Put $n_k := \varPhi(\rho_k)$ for $k = 1, 2$. By using Lemma 3.3, for $k = 1, 2$, there exist a measurable non-null subset E_k of $\mathrm{Dom}(\rho_k)$ and a subset $\{i_{k,p}\}_{p=1}^{n_k}$ of I such that the following inclusion holds for each $x \in E_k$:

$$\rho_k\bigl(\mathcal{S}(x)\bigr) \subseteq \bigcup_{p=1}^{n_k} \mathcal{S}\bigl((\varphi_{i_{k,p}} \circ \rho_k)(x)\bigr).$$

On the other hand, since $\mathcal{S}$ is ergodic, for each $q = 1, \ldots, n_2$, there exists a measurable nonsingular map θ_q which satisfies $\mathrm{Dom}(\theta_q) \supseteq \mathrm{Im}(\varphi_{i_{2,q}})$, $\mathrm{Im}(\theta_q) \subseteq E_1$ and $\Gamma(\theta_q) \subseteq \mathcal{S}$ up to null sets. Hence we have the following inclusion for a.e. $x \in E_2$:

$$\begin{aligned}(\rho_1 \circ \rho_2)\bigl(\mathcal{S}(x)\bigr) &\subseteq \rho_1\Bigl(\bigcup_{q=1}^{n_2} \mathcal{S}(\theta_q(\varphi_{i_{2,q}}(\rho_2(x))))\Bigr)\\ &\subseteq \bigcup_{p=1}^{n_1}\bigcup_{q=1}^{n_2} \mathcal{S}\bigl((\varphi_{i_{1,p}} \circ \rho_1 \circ \theta_q \circ \varphi_{i_{2,q}} \circ \rho_2)(x)\bigr).\end{aligned}$$

Moreover, by the definition of the choice functions, for each $i_{1,p}$, $i_{2,q} \in I$ and a.e. $x \in E_2$, there exists a unique element i in I such that $\bigl((\varphi_{i_{1,p}} \circ \rho_1 \circ \theta_q \circ \varphi_{i_{2,q}} \circ \rho_2)(x), (\varphi_i \circ \rho_1 \circ \rho_2)(x)\bigr)$ is in $\mathcal{S}$. Since both $\varphi_{i_{1,p}} \circ \rho_1 \circ \theta_q \circ \varphi_{i_{2,q}} \circ \rho_2$ and $\varphi_i \circ \rho_1 \circ \rho_2$ are measurable and I is countable, there exist a measurable non-null subset E_3 of E_2 with a subset $\{i_k\}_{k=1}^{n_1 n_2}$ of I such that $(\rho_1 \circ \rho_2)(\mathcal{S}(x))$ is contained in $\bigcup_{k=1}^{n_1 n_2} \mathcal{S}\bigl((\varphi_{i_k} \circ \rho_1 \circ \rho_2)(x)\bigr)$ for each $x \in E_3$. Now, we define a natural number n_3 by the following:

$$\begin{aligned}n_3 := |\{k\colon (\rho_1 \circ \rho_2)(\mathcal{S}(x)) \cap \mathcal{S}((\varphi_{i_k} \circ \rho_1 \circ \rho_2)(x)) \neq \emptyset \\ \text{(for a.e. } x \in E_3)\}|.\end{aligned}$$

By using Lemma 3.3 again, we obtain $\varPhi(\rho_1 \circ \rho_2) = n_3 \leq n_1 n_2 = \varPhi(\rho_1)\varPhi(\rho_2)$. Thus we are done. □

Corollary 3.6 ([2, Lemma 7.5]) *The subset $\mathcal{CG}(B) \cap \mathcal{GN}(D)$ is closed under the product operation.*

Proof. Suppose that v_i is in $\mathcal{CG}(B) \cap \mathcal{GN}(D)$ for $i = 1, 2$. We may and do assume that $v_1 v_2$ is not equal to 0. By assumption, for $i = 1, 2$, there exists $\rho_i \in [\mathcal{R}]_*$ such that v_i is equal to $L(g_i)v_{\rho_i}$, where g_i is a measurable function on $\mathrm{Dom}(\rho_i)$ of absolute value one. This means that the equations $z_{v_i} = z_{v_{\rho_i}}$ $(i = 1, 2)$ and $z_{v_1 v_2} = z_{\rho_2 \circ \rho_1}$ hold. By Proposition 3.4 and Proposition 3.5, we get

$$\begin{aligned}\widehat{E_B}(z_{v_1 v_2}) = \widehat{E_B}(z_{\rho_2 \circ \rho_1}) &= \Phi(\rho_2 \circ \rho_1) \leq \Phi(\rho_2)\Phi(\rho_1) \\ &= \widehat{E_B}(z_{v_2})\widehat{E_B}(z_{v_1}) < \infty.\end{aligned}$$

By using the same argument, we also get $\widehat{E_B}(z_{(v_1 v_2)^*}) \leq \widehat{E_B}(z_{v_1^*})\widehat{E_B}(z_{v_2^*}) < \infty$. So we conclude that $v_1 v_2$ also belongs to $\mathcal{CG}(B) \cap \mathcal{GN}(D)$. $\square$

We are now in a position to prove our main theorem.

Theorem 3.7 *Suppose that $\mathcal{S} \subseteq \mathcal{R}$ is an inclusion of ergodic discrete measured equivalence relations. Fix a countable subset $\{\rho_n\}_{n=1}^{\infty}$ of $[\mathcal{R}]_*$ satisfying $\mathcal{R} = \bigcup_{n=1}^{\infty} \Gamma(\rho_n)$ up to a null set. Then there exists countable measurable subsets $\{E_{n,m}\}_{n,m=1}^{\infty}$ and $\{F_{n,m}\}_{n,m=1}^{\infty}$ of X such that the two intermediate subrelations–the normalizer and the commensurability subrelation–are expressed as follows up to null sets*:

$$N_{\mathcal{R}}(\mathcal{S}) = \bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{E_{n,m}}),\ \Phi(\rho_n|_{E_{n,m}}) = \Phi((\rho_n|_{E_{n,m}})^{-1}) = 1,$$

$$\mathrm{Comm}_{\mathcal{R}}(\mathcal{S}) = \bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{F_{n,m}}),\ \Phi(\rho_n|_{F_{n,m}}),\ \Phi((\rho_n|_{F_{n,m}})^{-1}) < \infty.$$

Proof. By using a standard maximal argument, for each $n \in \mathbf{N}$, there exist countable disjoint measurable subsets $\{F'_{n,m}\}_{m=1}^{\infty}$ of $\mathrm{Dom}(\rho_n)$ which satisfy the following:

- $\Phi(\rho_n|_{F'_{n,m}}) < \infty$ for all $m \in \mathbf{N}$.
- If a measurable subset F of $\mathrm{Dom}(\rho_n)$ satisfies $\Phi(\rho_n|_F) < \infty$, then F is contained in $\bigcup_{m=1}^{\infty} F'_{n,m}$ up to a null set.

By the construction of $\{F'_{n,m}\}_{n,m=1}^{\infty}$, we have that, if $\rho \in [\mathcal{R}]_*$ satisfies $\Phi(\rho) < \infty$, then $\Gamma(\rho)$ is contained in $\bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{F'_{n,m}})$ up to a null set. In particular, $\bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{F'_{n,m}})$ contains the diagonal set $\mathcal{D}$ up to a null set. By using the same argument, for each n, $m \in \mathbf{N}$, there exist countable disjoint measurable subsets $\{F_{n,m,l}\}_{l=1}^{\infty}$ of $F'_{n,m}$ which satisfy the following:

- $\Phi((\rho_n|_{F_{n,m,l}})^{-1}) < \infty$ for all $l \in \mathbf{N}$.
- If a measurable subset F of $F'_{n,m}$ satisfies $\Phi((\rho_n|_F)^{-1}) < \infty$, then F is contained in $\bigcup_{l=1}^{\infty} F_{n,m,l}$ up to a null set.

By renumbering, we may and do assume that $\{F_{n,m,l}\}_{m,l=1}^{\infty}$ is equal to $\{F_{n,m}\}_{m=1}^{\infty}$ for each $n \in \mathbf{N}$. Set $\mathcal{P} := \bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{F_{n,m}})$. By the construction of $\{F_{n,m}\}_{n,m=1}^{\infty}$, we have that both $\Phi(\rho_n|_{F_{n,m}})$ and $\Phi((\rho_n|_{F_{n,m}})^{-1})$ are finite for each n, $m \in \mathbf{N}$, and the following equation holds up to a null set:

$$\bigcup_{m=1}^{\infty} \Gamma(\rho_n|_{F_{n,m}}) = \bigcup_{m=1}^{\infty} \Gamma(\rho_n|_{F'_{n,m}}) \cap \bigcup_{n,m=1}^{\infty} \Gamma((\rho_n|_{F'_{n,m}})^{-1}).$$

It follows that the equation

$$\mathcal{P} = \bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{F'_{n,m}}) \cap \bigcup_{n,m=1}^{\infty} \Gamma((\rho_n|_{F'_{n,m}})^{-1})$$

holds up to a null set. On the other hand, by Proposition 3.5,

$$\bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{F'_{n,m}})$$

satisfies the transitivity up to a null set. Hence $\mathcal{P}$ is an equivalence subrelation of $\mathcal{R}$. Moreover, by Proposition 3.4, we have that each $v_{\rho_n|_{F_{n,m}}}$ is in $\mathcal{CG}(B)$. So $\mathcal{P}$ is contained in $\mathrm{Comm}_{\mathcal{S}}(\mathcal{R})$ up to a null set. Conversely, by the definition of $\{F'_{n,m}\}_{n,m=1}^{\infty}$ we have that, for each $\rho \in [\mathcal{R}]_*$ satisfying $\Phi(\rho)$, $\Phi(\rho^{-1}) < \infty$, $\Gamma(\rho)$ is contained in $\mathcal{P}$ up to a null set. By Proposition 3.4 again, it follows that $W^*(\mathcal{P}, \omega)$ contains $\mathcal{CG}(B) \cap \mathcal{GN}(D)$. So we conclude that $W^*(\mathcal{P}, \omega)$ contains $W^*(\mathrm{Comm}_{\mathcal{R}}(\mathcal{S}), \omega)$ and $\mathcal{P}$ is equal to $\mathrm{Comm}_{\mathcal{R}}(\mathcal{S})$ up to a null set.

By using the same arguments, for each n, $m \in \mathbf{N}$, there exist a measurable subsets $\{E'_{n,m}\}_{n,m=1}^{\infty}$ and $\{E_{n,m}\}_{n,m=1}^{\infty}$ which satisfy the following:

- $\Phi(\rho_n|_{E'_{n,m}}) = 1$ for all n, $m \in \mathbf{N}$.
- If $\rho \in [\mathcal{R}]_*$ satisfies $\Phi(\rho) = 1$, then $\Gamma(\rho)$ is contained in $\bigcup_{n,m=1}^{\infty} \Gamma(\rho_n|_{E'_{n,m}})$ up to a null set.
- $\Phi(\rho_n|_{E_{n,m}}) = \Phi((\rho_n|_{E_{n,m}})^{-1}) = 1$ for all n, $m \in \mathbf{N}$.

- The equation
$$\bigcup_{n,m=1}^{\infty}\Gamma(\rho_n|_{E_{n,m}})=\bigcup_{n,m=1}^{\infty}\Gamma(\rho_n|_{E'_{n,m}})\cap\bigcup_{n,m=1}^{\infty}\Gamma((\rho_n|_{E'_{n,m}})^{-1})$$
holds up to a null set.

By Proposition 3.5 again, the subset
$$\mathcal{T}:=\bigcup_{n,m=1}^{\infty}\Gamma(\rho_n|_{E'_{n,m}})\cap\bigcup_{n,m=1}^{\infty}\Gamma((\rho_n|_{E'_{n,m}})^{-1})$$
is an equivalence subrelation of $\mathcal{R}$. By the definition of Φ, for each $\rho\in[\mathcal{R}]_*$, the equation $\Phi(\rho)=1$ holds if and only if $\rho(\mathcal{S}(x))$ is contained in $\mathcal{S}(\rho(x))$ for a.e. $x\in\mathrm{Dom}(\rho)$. So we have that each $v_{\rho_n|_{E_{n,m}}}$ is in $\mathcal{GN}(B)$ and $W^*(\mathcal{T},\omega)$ contains $\mathcal{GN}(B)\cap\mathcal{GN}(D)$. It follows that the normalizer $N_{\mathcal{R}}(\mathcal{S})$ is equal to $\bigcup_{n,m=1}^{\infty}\Gamma(\rho_n|_{E_{n,m}})$ up to a null set.

Therefore we get the conclusion. □

We conclude this paper with a characterization of commensurability by choice functions.

Theorem 3.8 *Let $\mathcal{S}\subseteq\mathcal{R}$ be an inclusion of ergodic discrete measured equivalence relation–subrelation on (X,μ) with a 2-cocycle ω. Put $(B\subseteq A):=(W^*(\mathcal{S},\omega)\subseteq W^*(\mathcal{R},\omega))$. Then $\mathcal{S}$ is commensurable in $\mathcal{R}$ if and only if there exist choice functions $\{\varphi_i\}_{i\in I}$ for $\mathcal{S}\subseteq\mathcal{R}$ which satisfies the following*:

(1) *There exist a countable set J and natural numbers $\{n_j\}_{j\in J}$ such that the index set I is equal to $\{(j,n)\colon j\in J,\ n=1,\dots,n_j\}$.*

(2) *The index $\Phi(\varphi_{j,n})$ is equal to n_j for each $(j,n)\in I$.*

(3) *For each $(j,n)\in I$ and a measurable non-null subset F of X, $\mathcal{S}(\varphi_{j,n}(\mathcal{S}(x)))$ is equal to $\mathcal{S}(\varphi_{j,n}|_F(\mathcal{S}(x)))$ for a.e. $x\in X$. Namely, each $z_{\varphi_{j,n}}$ is a minimal projection in $A_1\cap B'$.*

(4) *For each $j\in J$ and $n,m\in\{1,\dots,n_j\}$, $\mathcal{S}(\varphi_{j,n}(\mathcal{S}(x)))$ is equal to $\mathcal{S}(\varphi_{j,m}(\mathcal{S}(x)))$ for a.e. $x\in X$. Namely, $z_{\varphi_{j,n}}$ coincides with $z_{\varphi_{j,m}}$.*

(5) *If $j_1\neq j_2$, then $\mathcal{S}(\varphi_{j_1,m_1}(\mathcal{S}(x)))$ and $\mathcal{S}(\varphi_{j_2,m_2}(\mathcal{S}(x)))$ are disjoint for a.e. $x\in X$. Namely, $z_{\varphi_{j_1,m_1}}z_{\varphi_{j_2,m_2}}$ is equal to 0.*

Proof. Suppose that there exist choice functions $\{\varphi_i\}_{i\in I}$ which satisfy the above properties. Since $\mathcal{R}(x)$ is equal to a disjoint union of $\{\mathcal{S}(\varphi_{j,1}(x))\}_{j\in J}$ for a.e. $x\in X$, we have that $\{z_{\varphi_{j,1}}\}_{j\in J}$ are the partition of the unity satisfying $\widehat{E_B}(z_{\varphi_{j,1}}) = n_j < \infty$. It follows that the inclusion $B \subseteq A$ is discrete, and $\mathcal{S}$ is commensurable in $\mathcal{R}$.

Conversely, if $\mathcal{S}$ is commensurable in $\mathcal{R}$, then there exists a countable partition $\{\mathcal{R}_j\}_{j\in J}$ of $\mathcal{R}$ such that $\{\chi_{\mathcal{R}_j}\}_{j\in J}$ are minimal projections in $A_1 \cap B'$ with $\widehat{E_B}(\chi_{\mathcal{R}_j}) < \infty$ for all $j \in J$. Put $n_j := \widehat{E_B}(\chi_{\mathcal{R}_j})$ and $I := \{(j, n)\colon j \in J,\ n = 1, \ldots, n_j\}$. For each $j \in J$, there exists $\rho_j \in [\mathcal{R}]_*$ which satisfies $\mu(\mathrm{Dom}(\rho_j)) > 0$ and $\Gamma(\rho_j) \subseteq \mathcal{R}_j$. Since $\chi_{\mathcal{R}_j}$ is minimal, we have that z_{ρ_j} coincides with $\chi_{\mathcal{R}_j}$. By Proposition 3.4, it follows that the index $\Phi(\rho_j)$ is equal to n_j for each $j \in J$.

Choose choice functions $\{\psi_i\}_{i\in I_0}$ for $\mathcal{S} \subseteq \mathcal{R}$. By Lemma 3.3, for each $j \in J$, there exist a measurable non-null subset E'_j of $\mathrm{Dom}(\rho_j)$ and n_j elements $\{i'_{j,n}\}_{n=1}^{n_j}$ in I_0 which satisfy $\rho_j(\mathcal{S}(x)) \cap \mathcal{S}(\psi_{i'_{j,n}}(\rho_j(x))) \neq \emptyset$ for all $x \in E'_j$ and $n = 1, \ldots, n_j$. By using the property of choice functions, for each $j \in J$, there exists a measurable non-null subset E_j of E'_j and n_j elements $\{i_{j,n}\}_{n=1}^{n_j}$ in I_0 which satisfy $\psi_{i'_{j,n}}(\rho_j(x)) \in \mathcal{S}(\psi_{i_{j,n}}(x))$ for all $x \in E_j$ and $n = 1, \ldots, n_j$. On the other hand, since $\mathcal{S}$ is ergodic, there exist measurable nonsingular maps $\{\theta_j\}_{j\in J}$ on X satisfying $\Gamma(\theta_j) \subseteq \mathcal{S}$ and $\mathrm{Im}(\theta_j) \subseteq E_j$ for each $j \in J$ up to null sets. Put $\varphi_{j,n} := \psi_{i_{j,n}} \circ \theta_j$ for each $(j, n) \in I$. Since $\Gamma(\varphi_{j,n})$ is contained in $\mathcal{R}_j$ and $\chi_{\mathcal{R}_j}$ is minimal, we have that $z_{\varphi_{j,n}}$ is equal to $\chi_{\mathcal{R}_j}$. By Proposition 2.1 and Proposition 3.4, we get $\mathcal{S}(\varphi_{j,n}(\mathcal{S}(x))) = \mathcal{R}_j(x) := \{y \in X\colon (x, y) \in \mathcal{R}_j\}$ and $\Phi(\varphi_{j,n}) = \widehat{E_B}(z_{\varphi_{j,n}}) = \widehat{E_B}(\chi_{\mathcal{R}_j}) = n_j$ for each $(j, n) \in I$ and a.e. $x \in X$. In particular, we have that $\mathcal{S}(\varphi_{j,1}(\mathcal{S}(x)))$ contains $\bigcup_{n=1}^{n_j} \mathcal{S}(\varphi_{j,n}(x))$ for each $j \in J$ and a.e. $x \in X$. On the other hand, by using the property of choice functions $\{\psi_i\}_{i\in I_0}$ for the equivalent class $\mathcal{R}(\theta_j(x))$, $\mathcal{S}(\varphi_{j,1}(\mathcal{S}(x)))$ is contained in $\bigcup_{i\in I_0} \mathcal{S}(\psi_i(\theta_j(x)))$ for a.e. $x \in X$. Since $\Phi(\varphi_{j,1})$ is equal to n_j and $\mathcal{S}(\theta_j(x))$ coincides with $\mathcal{S}(x)$, by using the condition of Lemma 3.3 (3), we have the following inequality:

$$|\{i \in I_0\colon \mathcal{S}(\varphi_{j,1}(\mathcal{S}(x))) \cap \mathcal{S}(\psi_i(\theta_j(x))) \neq \emptyset \quad \text{for a.e. } x \in X\}| \leq n_j.$$

It means that $\mathcal{S}(\psi_i(\theta_j(x)))$ does not intersect $\mathcal{S}(\varphi_{j,1}(\mathcal{S}(x)))$ for each $i \in I_0 \setminus \{i_{j,n}\}_{n=1}^{n_j}$ and a.e. $x \in X$. So $\mathcal{S}(\varphi_{j,1}(\mathcal{S}(x)))$ is actually equal to $\bigcup_{n=1}^{n_j} \mathcal{S}(\varphi_{j,n}(x))$ for each $j \in J$ and a.e. $x \in X$. This means that $\mathcal{R}_j(x)$ is equal to the disjoint union of $\{\mathcal{S}(\varphi_{j,n}(x))\}_{n=1}^{n_j}$ for each $j \in J$ and a.e. $x \in X$. Hence we conclude that $\{\varphi_{j,n}\}_{(j,n)\in I}$ are choice functions for $\mathcal{S} \subseteq \mathcal{R}$ which satisfy the desired properties.

Therefore we get the conclusion. $\square$

Remark Under the above setting, we have the following:

(1) $\varphi_{j,n}(\mathcal{S}(y))$ is contained in a disjoint union of $\{\mathcal{S}(\varphi_{j,m}(y))\}_{m=1}^{n_j}$ for each $(j, n) \in J$ and a.e. $y \in X$. This means that, for a.e. $(x, y) \in \mathcal{S}$ and $(j, n) \in I$, there exists a unique $m \in \{1, \ldots, n_j\}$ which satisfies $(\varphi_{j,n}(x), \varphi_{j,m}(y)) \in \mathcal{S}$.

(2) $\mathcal{S}$ is normal in $\mathcal{R}$ if and only if n_j is equal to 1 for all $j \in J$. Hence our result is a generalization of a characterization of normality in [3, Theorem 2.2] when the subrelation is ergodic.

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