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How to Determine the *Sign* of a Valuation on $\mathbb{C}[x, y]$

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Abstract. Given a divisorial discrete valuation *centered at infinity* on $\mathbb{C}[x, y]$, we show that its sign on $\mathbb{C}[x, y]$ (i.e. whether it is negative or nonpositive on $\mathbb{C}[x, y] \setminus \mathbb{C}$) is completely determined by the sign of its value on the *last key form* (key forms being the avatar of *key polynomials* of valuations [Mac36] in "global coordinates"). We also describe the cone of curves and the nef cone of certain compactifications of $\mathbb{C}^2$ associated with a given valuation centered at infinity and give a characterization of the divisorial valuations centered at infinity whose *skewness* can be interpreted in terms of the *slope* of an extremal ray of these cones, yielding a generalization of a result of [FJ07]. A byproduct of these arguments is a characterization of valuations that "determine" normal compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity in terms of an associated "semigroup of values".

1. Introduction

Notation 1.1. Throughout this section, k is a field, and R is a finitely generated k-algebra.

In algebraic (or analytic) geometry and commutative algebra, valuations are usually treated in the *local setting,* and the values are always positive or nonnegative. Even if it is a priori not known if a given discrete valuation ν is positive or nonnegative on $R \setminus k$, it is evident how to verify this, at least if $\nu(k \setminus \{0\}) = 0$: we have only to check the values of ν on the k-algebra generators of R. For valuations *centered at infinity* however, in general, it is nontrivial to determine if it is negative or nonpositive on $R \setminus k$:

Example 1.2. Let $R := \mathbb{C}[x, y]$, and for every $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < 1$, let ν_ε be the valuation (with values in $\mathbb{R}$) on $\mathbb{C}(x, y)$ defined as follows:

$$\nu_\varepsilon(f(x, y)) := -\deg_x(f(x, y)|_{y=x^{5/2}+x^{-1}+\xi x^{-5/2-\varepsilon}}) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\}, \tag{1}$$

where ξ is a new indeterminate, and $\deg_x$ is the degree in x. A direct computation shows that

$$\nu_\varepsilon(x) = -1, \qquad \nu(y) = -5/2,$$
$$\nu_\varepsilon(y^2 - x^5) = -3/2, \qquad \nu_\varepsilon(y^2 - x^5 - 2x^{-1}y) = \varepsilon.$$

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Is ν_ε negative on $\mathbb{C}[x, y]$? Let $g := y^2 - x^5 - 2x^{-1}y$. The fact that $\nu_\varepsilon(g) > 0$ does not seem to be of much help for the answer (especially if ε is very small) since $g \notin \mathbb{C}[x, y]$ and $\nu_\varepsilon(xg) < 0$. However, g is precisely the *last key form* (Definition 2.4) of ν_ε (see Example 2.8), and therefore Theorem 1.4 implies that ν_ε is *not* nonpositive on $\mathbb{C}[x, y]$, that is, no matter how small ε is, there exists $f_\varepsilon \in \mathbb{C}[x, y]$ such that $\nu_\varepsilon(f_\varepsilon) > 0$.

In this paper, we settle the question of how to determine if a valuation centered at infinity is negative or nonpositive on R for the case that $R = \mathbb{C}[x, y]$. Given a valuation ν centered at infinity on $\mathbb{C}[x, y]$, we explicitly construct a rational function (which is the 'last key form' of the valuation) g such that the sign of $\nu(g)$ determines the answer. Let $\bar{X}$ be a compactification of $\mathbb{C}^2$ such that the center of ν on $\bar{X}$ is a curve C. The construction of the rational function g depends on the Puiseux parametrization of generic 'curvettes' transversal to C. Replacing the computation of this Puiseux parametrization with 'infinitely near points' yields a straightforward generalization of our results in non-zero characteristic. This was accomplished in [GM16] following the arXiv publication of the first version of this article in 2013.

In Section 1.1 below we state the main results and describe the relation of this question to the study of algebraic compactifications of affine varieties. In Section 1.2 we give a geometric interpretation of our results. After introducing some background material in Section 2, we prove the results in Section 3.

1.1. Motivation and Statements of Main Results

Recall that a *divisorial discrete valuation* (Definition 2.2) ν on R is *centered at infinity* iff $\nu(f) < 0$ for some $f \in R$, or equivalently iff there is an *algebraic completion* $\bar{X}$ of $X := \operatorname{Spec} R$ (i.e. $\bar{X}$ is a complete algebraic variety containing X as a dense open subset) and an irreducible component C of $\bar{X} \setminus X$ such that ν is the order of vanishing along C. On the other hand, one way to *construct* algebraic completions of the affine variety X is to start with a *degree-like function* on R (the terminology is from [Mon10] and [Mon14]), that is, a function $\delta : R \to \mathbb{Z} \cup \{-\infty\}$ that satisfies the following "degree-like" properties:

P1. $\delta(f + g) \le \max\{\delta(f), \delta(g)\}$, and
P2. $\delta(fg) \le \delta(f) + \delta(g)$,

and construct the graded ring

$$R^\delta := \bigoplus_{d \ge 0} \{f \in R : \delta(f) \le d\} \cong \sum_{d \ge 0} \{f \in R : \delta(f) \le d\} t^d, \tag{2}$$

where t is an indeterminate. It is straightforward to see that $\bar{X}^\delta := \operatorname{Proj} R^\delta$ is a *projective completion* of X, provided that the following conditions are satisfied:

(Proj-1). R^δ is finitely generated as a k-algebra, and
(Proj-2). $\delta(f) > 0$ for all $f \in R \setminus k$.

A fundamental class of degree-like functions are *divisorial semidegrees*—these are precisely the *negatives* of divisorial discrete valuations centered at infinity, and they serve as "building blocks" of an important class of degree-like functions (see [Mon10; Mon14]). Therefore a natural question in this context is:

QUESTION 1.3.[1] Given a divisorial semidegree δ on R, how to determine if $\delta(f) > 0$ for all $f \in R \setminus k$? Or equivalently, given a divisorial discrete valuation ν on R centered at infinity, how to determine if $\nu(f) < 0$ for all $f \in R \setminus k$?

In this paper, we give a complete answer to Question 1.3 for the case $k = \mathbb{C}$ and $R = \mathbb{C}[x, y]$ (note that the answer for the case $R = \mathbb{C}[x]$ is obvious since the only discrete valuations centered at infinity on $\mathbb{C}[x]$ are those that map $x - \alpha \mapsto -1$ for some $\alpha \in \mathbb{C}$). More precisely, we consider the sequence of *key forms* (Definition 2.4) corresponding to semidegrees and show the following:

THEOREM 1.4. *Let δ be a divisorial semidegree on $\mathbb{C}[x, y]$, and let $g_0, \dots, g_{n+1}$ be the key forms of δ in (x, y)-coordinates. Then*

1. *δ is nonnegative on $\mathbb{C}[x, y] \setminus \mathbb{C}$ iff $\delta(g_{n+1})$ is nonnegative.*
2. *δ is positive on $\mathbb{C}[x, y] \setminus \mathbb{C}$ iff one of the following holds*:
   a) *$\delta(g_{n+1})$ is positive*, *or*
   b) *$\delta(g_{n+1}) = 0$ and $g_k \notin \mathbb{C}[x, y]$ for some k, $0 \le k \le n + 1$.*

   Moreover, condition 2b is equivalent to the following condition:

   b′) *$\delta(g_{n+1}) = 0$ and $g_{n+1} \notin \mathbb{C}[x, y]$.*

REMARK 1.5. The key forms of a semidegree δ on $\mathbb{C}[x, y]$ are counterparts in (x, y)-coordinates of the *key polynomials* of $\nu := -\delta$ introduced in [Mac36] (and computed in local coordinates near the *center* of ν). The basic ingredient of the proof of Theorem 1.4 is the algebraic contractibility criterion of [Mon16a, Thm. 4.1], which uses key forms. We note that key forms were already used in [FJ07][2] (without calling them by any special name).

REMARK 1.6. The requirement in Theorem 1.4 that δ be *divisorial* (i.e. $-\delta$ be a *divisorial* valuation) is unnecessary: the only technical issue stems from valuations with an infinite sequence of key polynomials, but one can determine the sign of such a valuation by applying Theorem 1.4 to a divisorial valuation that "approximates" it sufficiently closely.

REMARK 1.7. The key forms of a semidegree can be computed explicitly from any of the alternative presentations of the semidegree (see e.g. [Mon16a, Algorithm 5.1] for an algorithm to compute key forms from the *generic Puiseux series* (Definition 2.13) associated to the semidegree). Therefore Theorem 1.4 gives an *effective* way to determine if a given semidegree is positive or nonnegative on $\mathbb{C}[x, y]$.

[1] The analogous question regarding Property (Proj-1) for $R = \mathbb{C}[x, y]$ and δ a divisorial semidegree is completely settled in [MN14] where the results of this paper are also applied to the *moment problem* of planar semialgebraic sets.

[2] Under the assumptions of Lemma A.12 of [FJ07], the polynomials U_j constructed in Section A.5.3 of [FJ07] are precisely the key forms of $-\nu$.

Trees of valuations centered at infinity on $\mathbb{C}[x, y]$ were considered in [FJ07] along with a parameterization of the tree called *skewness* α. The notion of skewness has an "obvious" extension[3] to the case of semidegrees, and using this definition, one of the assertions of [FJ07, Thm. A.7] can be reformulated as the statement that the following identity holds for a certain *subtree* of semidegrees δ on $\mathbb{C}[x, y]$:

$$\alpha(\delta) = \inf\left\{ \frac{\delta(f)}{d_\delta \deg(f)} : f \text{ is a nonconstant polynomial in } \mathbb{C}[x, y] \right\}, \tag{3}$$

where

$$d_\delta := \max\{\delta(x), \delta(y)\}. \tag{4}$$

It is observed in [Jon12, p. 121] that in general relation in (3) is satisfied with $\leq$, and "it is doubtful that equality holds in general". Example 3.1 shows that the equality indeed does not hold in general. It is not hard to see that $\alpha(\delta)$ can be expressed in terms of $\delta(g_{n+1})$ (see (17)), and using that expression, we give a characterization of the semidegrees for which (3) holds:

THEOREM 1.8. *Let δ be a semidegree on $\mathbb{C}[x, y]$, and $g_0, \dots, g_{n+1}$ be the corresponding key forms.*

1. (3) *holds iff one of the following assertions is true*:
   a) $\delta(g_{n+1}) \geq 0$, *or*
   b) $\delta(g_{n+1}) < 0$ *and* $g_k \in \mathbb{C}[x, y]$ *for all* k, $0 \leq k \leq n+1$, *or*
   Moreover, condition 1b is equivalent to the following condition:
   b′) $\delta(g_{n+1}) < 0$ *and* $g_{n+1} \in \mathbb{C}[x, y]$.
2. *The* inf *in right-hand side of* (3) *can be replaced by* min *iff* $g_{n+1} \in \mathbb{C}[x, y]$ *iff* $g_k \in \mathbb{C}[x, y]$ *for all* k, $0 \leq k \leq n+1$; *in this case the minimum is achieved with* $f = g_{n+1}$.

REMARK 1.9. It is possible to give a *geometric* characterization of the semidegrees δ for which (3) holds. Indeed, [CPR05] introduced the notion of compactifications of $\mathbb{C}^2$ that *admit systems of numerical curvettes*. In Section 1.2 and Remark 1.14, we construct two compactifications $\bar{X}$ and $\tilde{X}$ of $\mathbb{C}^2$ associated to δ. [Mon13, Thm. 3.2] (which uses the results of this article) shows that (3) holds iff $\bar{X}$ (or equivalently, $\tilde{X}$) admits a system of numerical curvettes.

Our final result is the following corollary of the arguments in the proof of Theorem 1.4, which answers a question of Professor Peter Russell.[4]

[3] In [FJ07] the skewness α was defined only for valuations ν centered at infinity that satisfied $\min\{\nu(x), \nu(y)\} = -1$. Here for a semidegree δ, we define $\alpha(\delta)$ to be the skewness of $-\delta/d_\delta$ (where d_δ is as in (4)) in the sense of [FJ07].

[4] Prof. Russell's question was motivated by the correspondence established in [Mon16a, Thm. 4.3] between normal algebraic compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity and algebraic curves in $\mathbb{C}^2$ with one place at infinity. Since the *semigroups of poles* of planar curves with one place at infinity are very *special* (see e.g. [Abh78; SS94]), he asked if similarly the semigroups of values of semidegrees that determine normal algebraic compactifications of $\mathbb{C}^2$ can be similarly distinguished from the semigroup of values of general semidegrees. Whereas

COROLLARY 1.10. *Let* δ *be a semidegree on* $\mathbb{C}[x, y]$. *Define*

$$S_\delta := \{(\deg(f), \delta(f)) : f \in \mathbb{C}[x, y] \setminus \{0\}\} \subseteq \mathbb{Z}^2, \tag{5}$$

and let $\mathcal{C}_\delta$ *be the cone over* S_δ *in* $\mathbb{R}^2$. *Then*

1. *The following are equivalent*:
   a) $\mathcal{C}_\delta$ *is a closed subset of* $\mathbb{R}^2$.
   b) g_k *is a polynomial for all* k, $0 \le k \le n+1$.
   c) g_{n+1} *is a polynomial.*
2. *The following are equivalent*:
   a) δ *determines an analytic compactification of* $\mathbb{C}^2$.
   b) *The positive* x*-axis is not contained in the closure* $\bar{\mathcal{C}}_\delta$ *of* $\mathcal{C}_\delta$ *in* $\mathbb{R}^2$.
3. *The following are equivalent*:
   a) δ *determines an algebraic compactification of* $\mathbb{C}^2$.
   b) $\mathcal{C}_\delta$ *is closed in* $\mathbb{R}^2$, *and the positive* x*-axis is not contained in* $\mathcal{C}_\delta$.
   c) S_δ *is a finitely generated semigroup, and* $(k, 0) \notin S_\delta$ *for all positive integer* k.

REMARK 1.11. The phrase "δ determines an algebraic (resp. analytic) compactification of $\mathbb{C}^2$" means "there exists a (necessarily unique) normal algebraic (resp. analytic) compactification $\bar{X}$ of $X := \mathbb{C}^2$ such that $C_\infty := \bar{X} \setminus X$ is an irreducible curve and δ is proportional to the order of pole along C_∞". In particular, δ determines an algebraic compactification of $\mathbb{C}^2$ if and only if δ satisfies conditions (Proj-1) and (Proj-2) for $k = \mathbb{C}$ and $R = \mathbb{C}[x, y]$.

REMARK 1.12. S_δ is isomorphic to the *global Enriques semigroup* (in the terminology of [CPRL02]) of the compactification of $\mathbb{C}^2$ from Proposition 2.10. Also, the assertions of Corollary 1.10 remain true if in (5) deg is replaced by any other semidegree that determines an algebraic compactification of $\mathbb{C}^2$ (e.g. a weighted degree with positive weights).

1.2. Cones of Curves on Compactifications of $\mathbb{C}^2$

In this section, we give geometric interpretations of Theorems 1.4 and 1.8 in terms of the *cone of curves* and the *nef cone* on a compactification of $\mathbb{C}^2$ related to a given semidegree.

DEFINITION 1.13. Let Y be a normal compact algebraic surface. A *(real) one cycle* on Y is a formal linear combination of irreducible curves on Y with coefficients in $\mathbb{R}$. It is possible (originally done by Mumford [Mum61]) to define an intersection product on pairs of cycles on Y by passing to the desingularization of Y. Two cycles C and C' are *numerically equivalent* if $(C, D) = (C', D)$ for each cycle D on Y. The (real) vector space of cycles modulo numerical equivalence is denoted $N_1(Y)$. The *cone* $\mathrm{NE}(Y)$ *of curves* on Y is the subset of $N_1(Y)$

Example 3.2 shows that they cannot be distinguished only by the values of the semidegree itself, Corollary 1.10 shows that this can be done if paired with the degree of polynomials.

consisting of (equivalence classes of) cycles with nonnegative coefficients. The *nef cone* $\mathrm{Nef}(Y)$ of Y is the set of (equivalence classes of) all cycles C on Y such that $(C, C') \geq 0$ for all irreducible curves C' on Y; in other words, $\mathrm{Nef}(Y)$ is the cone that is *dual* to $\mathrm{NE}(Y)$ via the intersection product.

Let $\mathbb{C}^2 \subseteq \mathbb{P}^2 = \mathbb{C}^2 \cup L$ be the usual compactification of $\mathbb{C}^2$ (where L is the "line at infinity"), and δ be a divisorial semidegree on $\mathbb{C}[x, y]$ centered at infinity. Assume that $\delta \neq \deg$. By local theory, there exists a *minimal* normal analytic compactification $\bar{X}$ of $\mathbb{C}^2$ with the following properties:

- the identification of $\mathbb{C}^2$ extends to a birational morphism $\pi : \bar{X} \to \mathbb{P}^2$,
- the center of δ on $\bar{X}$ is a curve at infinity.

It turns out (see Proposition 2.10) that the curve at infinity on $\bar{X}$ has precisely two irreducible components: one is the strict transform of L (call it C_1), and the other is the center of δ on $\bar{X}$ (call it C_2). If D is a curve on $\mathbb{C}^2$ defined by a polynomial $f \in \mathbb{C}[x, y]$, then its closure $\bar{D}$ in $\bar{X}$ is linearly equivalent to $\deg(f)C_1 + \delta(f)C_2$ as Weil divisors on $\bar{X}$. It follows that $N_1(\bar{X})$ is generated by the equivalence classes of C_1 and C_2. Theorem 1.4 is related to the question of whether (the equivalence classes of) C_1 and C_2 generate the cone $\mathrm{NE}(\bar{X})$ of curves on $\bar{X}$. More precisely, an ingredient of Theorem 1.4 is the following result.

THEOREM 1.4′. *Let $g_0, \dots, g_{n+1}$ be the key forms of δ in (x, y)-coordinates. Then*

1. *$\delta(g_{n+1})$ equals a negative rational number times the self intersection number of C_1.*
2. *$\delta(g_{n+1}) \geq 0$ iff C_1 and C_2 generate* $\mathrm{NE}(\bar{X})$. *Let l_i be the half-line of all nonnegative real multiples of C_i, $1 \leq i \leq 2$. Then*:
   a) *l_2 determines an edge of* $\mathrm{NE}(\bar{X})$.
   b) *l_1 determines an edge of* $\mathrm{NE}(\bar{X})$ *iff $\delta(g_{n+1}) \geq 0$.*
   c) *C_1 is in the interior of* $\mathrm{NE}(\bar{X})$ *iff $\delta(g_{n+1}) < 0$.*

Similarly, Theorem 1.8 is related to properties of the nef cone $\mathrm{Nef}(\bar{X})$ of $\bar{X}$. More precisely, let $g_0, \dots, g_{n+1}$ be the key forms of δ in (x, y)-coordinates. Define the $\mathbb{Q}$-Cartier divisors $C_1^* := C_1 + d_\delta C_2$ and $C_2^* := C_1 + \frac{m_\delta \delta(g_{n+1})}{d_\delta} C_2$ on $\bar{X}$, where d_δ is as in (4), and

$$m_\delta := \gcd(\delta(g_0), \dots, \delta(g_n)). \tag{6}$$

Let l_i^* be the half-line of all nonnegative real multiples of C_i^*, $1 \leq i \leq 2$.

THEOREM 1.8′. *$\delta(g_{n+1}) \geq 0$ iff C_1^* and C_2^* generate* $\mathrm{Nef}(\bar{X})$. *More precisely*:

1. *l_1^* determines an edge of* $\mathrm{Nef}(\bar{X})$.
2. *l_2^* determines an edge of* $\mathrm{Nef}(\bar{X})$ *iff $\delta(g_{n+1}) \geq 0$.*
3. *$C_2^* \notin \mathrm{Nef}(\bar{X})$ iff $\delta(g_{n+1}) < 0$.*

REMARK 1.14. Consider the minimal resolution of singularities $\tilde{\pi} : \tilde{X} \to \bar{X}$. Let $\tilde{E}$ be the union of the exceptional curves of $\tilde{\pi}$ with the strict transform of C_1.

Then assertion 1 of Theorem 1.4′ implies that

$$\delta(g_{n+1}) > 0 \quad \text{iff } (C_1, C_1) < 0$$
$$\text{iff the intersection matrix of } \tilde{E} \text{ is negative definite,} \tag{7}$$
$$\delta(g_{n+1}) \geq 0 \quad \text{iff } (C_1, C_1) \leq 0$$
$$\text{iff the intersection matrix of } \tilde{E} \text{ is nonpositive definite.} \tag{8}$$

In particular, the property that $\delta(g_{n+1}) > 0$ (resp. $\delta(g_{n+1}) \geq 0$) is equivalent to the purely numerical criterion (7) (resp. (8)), which is completely determined by the weighted configuration of projective lines on $\tilde{X} \setminus \mathbb{C}^2$.

REMARK 1.15. Let $g_0, \ldots, g_{n+1}$ be the key forms of δ in (x, y)-coordinates, and let $\tilde{X}$ be as in Remark 1.14. Let $\tilde{C}_1, \ldots, \tilde{C}_k$ be the irreducible components of $\tilde{X} \setminus \mathbb{C}^2$, and for each j, $0 \leq j \leq k$, let δ_j be the semidegree on $\mathbb{C}[x, y]$ associated to $\tilde{C}_j$. It is not hard to see that the last key form of δ_j is g_{i_j} for some i_j, $1 \leq i_j \leq n+1$. Moreover, in the case that $\delta(g_{n+1}) \geq 0$, it turns out that $\delta_j(g_{i_j}) \geq 0$ for each j, $1 \leq j \leq k$. Theorem 1.4 then implies that $\mathrm{NE}(\tilde{X})$ is (the simplicial cone) generated by $\tilde{C}_1, \ldots, \tilde{C}_k$. Combining this with Remark 1.14, it follows that if $\delta(g_{n+1}) \geq 0$, then the cones $\mathrm{NE}(\tilde{X})$ and $\mathrm{Nef}(\tilde{X})$ are (simplicial and) completely determined by the weighted configuration of projective lines on $\tilde{X} \setminus \mathbb{C}^2$. However, if $\delta(g_{n+1}) < 0$, then Example 3.3 shows that in general the weighted configuration of projective lines on $\tilde{X} \setminus \mathbb{C}^2$ does not determine $\mathrm{NE}(\tilde{X})$ or $\mathrm{Nef}(\tilde{X})$.

2. Preliminaries

NOTATION 2.1. Throughout the rest of the article, we write $X := \mathbb{C}^2$ with polynomial coordinates (x, y) and denote by deg the usual degree in (x, y)-coordinates. We also write $\bar{X}^{(0)}$ for the copy of $\mathbb{P}^2$ into which X is embedded via the map $(x, y) \mapsto [1 : x : y]$. Note that the semidegree on $\mathbb{C}[x, y]$ corresponding to the line at infinity on $\bar{X}^0$ is simply deg.

2.1. Divisorial Discrete Valuations, Semidegrees, Key Forms, and Associated Compactifications

DEFINITION 2.2 (Discrete valuations). A discrete valuation on $\mathbb{C}(x, y)$ is a map $\nu : \mathbb{C}(x, y) \setminus \{0\} \to \mathbb{Z}$ such that, for all $f, g \in \mathbb{C}(x, y) \setminus \{0\}$,

1. $\nu(f + g) \geq \min\{\nu(f), \nu(g)\}$, and
2. $\nu(fg) = \nu(f) + \nu(g)$.

A discrete valuation ν on $\mathbb{C}(x, y)$ is called *divisorial* iff there exist a normal algebraic surface Y_ν equipped with a birational map $\sigma : Y_\nu \to \bar{X}^0$ and a curve C_ν on Y_ν such that, for all nonzero $f \in \mathbb{C}[x, y]$, $\nu(f)$ is the order of vanishing of $\sigma^*(f)$ along C_ν. The *center* of ν on $\bar{X}^0$ is $\sigma(C_\nu)$; ν is said to be *centered at infinity* (with respect to (x, y)-coordinates) iff the center of ν on $\bar{X}^0$ is contained in $\bar{X}^0 \setminus X$; equivalently, ν is centered at infinity iff there is a nonzero polynomial $f \in \mathbb{C}[x, y]$ such that $\nu(f) < 0$.

DEFINITION 2.3 (Semidegrees). A *(divisorial) semidegree* on $\mathbb{C}(x, y)$ is a map $\delta : \mathbb{C}(x, y) \setminus \{0\} \to \mathbb{Z}$ such that $-\delta$ is a (divisorial) discrete valuation centered at infinity.

DEFINITION 2.4 (cf. definition of key polynomials in [FJ04, Def. 2.1], see also Remark 2.6). Let δ be a divisorial semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. A sequence of elements $g_0, g_1, \ldots, g_{n+1} \in \mathbb{C}[x, x^{-1}, y]$ is called the sequence of *key forms* for δ if the following properties are satisfied:

P0. $g_0 = x$, $g_1 = y$.

P1. Let $\omega_j := \delta(g_j)$, $0 \leq j \leq n+1$. Then

$$\omega_{j+1} < \alpha_j \omega_j = \sum_{i=0}^{j-1} \beta_{j,i} \omega_i \quad \text{for } 1 \leq j \leq n,$$

where

a) $\alpha_j = \min\{\alpha \in \mathbb{Z}_{>0} : \alpha\omega_j \in \mathbb{Z}\omega_0 + \cdots + \mathbb{Z}\omega_{j-1}\}$ for $1 \leq j \leq n$,

b) $\beta_{j,i}$ are integers such that $0 \leq \beta_{j,i} < \alpha_i$ for $1 \leq i < j \leq n$ (in particular, $\beta_{j,0}$ are allowed to be *negative*).

P2. For $1 \leq j \leq n$, there exists $\theta_j \in \mathbb{C}^*$ such that

$$g_{j+1} = g_j^{\alpha_j} - \theta_j g_0^{\beta_{j,0}} \cdots g_{j-1}^{\beta_{j,j-1}}.$$

P3. Let $y_1, \ldots, y_{n+1}$ be indeterminates, and ω be the *weighted degree* on $B := \mathbb{C}[x, x^{-1}, y_1, \ldots, y_{n+1}]$ corresponding to weights ω_0 for x and ω_j for y_j, $0 \leq j \leq n+1$ (i.e. the value of ω on a polynomial is the maximum "weight" of its monomials). Then, for every polynomial $g \in \mathbb{C}[x, x^{-1}, y]$,

$$\delta(g) = \min\{\omega(G) : G(x, y_1, \ldots, y_{n+1}) \in B, G(x, g_1, \ldots, g_{n+1}) = g\}. \tag{9}$$

THEOREM 2.5 ([Mon16a, Thm. 3.17], cf. [FJ04, Thm. 2.29]). *There is a unique and finite sequence of key forms for δ.*

REMARK 2.6. Let δ be as in Definition 2.4. Set $u := 1/x$ and $v := y/x^k$ for some k such that $\delta(y) < k\delta(x)$, and let $\tilde{g}_0 = u$, $\tilde{g}_1 = v$, $\tilde{g}_2, \ldots, \tilde{g}_{n+1} \in \mathbb{C}[u, v]$ be the *key polynomials* of $\nu := -\delta$ in (u, v)-coordinates. Then the key forms of δ can be computed from the $\tilde{g}_j$ as follows:

$$g_j(x, y) := \begin{cases} x & \text{for } j = 0, \\ x^{k \deg_v(\tilde{g}_j)} \tilde{g}_j(1/x, y/x^k) & \text{for } 1 \leq j \leq n+1. \end{cases} \tag{10}$$

Theorem 2.5 is an immediate consequence of the existence of key polynomials (see e.g. [FJ04, Thm. 2.29]).

EXAMPLE 2.7. Let p, q be integers such that $p > 0$, and δ be the weighted degree on $\mathbb{C}(x, y)$ corresponding to weights p for x and q for y. Then the key forms of δ are x, y.

EXAMPLE 2.8. Let $\varepsilon := q/2p$ for positive integers p, q such that $q < 2p$, and δ_ε be the semidegree on $\mathbb{C}(x, y)$ defined as follows:

$$\begin{aligned}\delta_\varepsilon(f(x, y)) &:= 2p \deg_x(f(x, y)|_{y=x^{5/2}+x^{-1}+\xi x^{-5/2-\varepsilon}}) \\ &\text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},\end{aligned} \tag{11}$$

where ξ is a new indeterminate, and $\deg_x$ is the degree in x. Note that $\delta_\varepsilon = -2p\nu_\varepsilon$, where ν_ε is from Example 1.2 (we multiplied by $2p$ to simply make the semidegree integer valued). Then the sequence of key forms of δ_ε is x, y, $y^2 - x^5$, $y^2 - x^5 - 2x^{-1}y$.

The following property of key forms can be proved in a straightforward way from their defining properties.

PROPOSITION 2.9. *Let δ and $g_0, \dots, g_{n+1}$ be as in Definition 2.4, and d_δ and m_δ be as in respectively* (4) *and* (6). *Then*

$$m_\delta \delta(g_{n+1}) \leq d_\delta^2. \tag{12}$$

Moreover, we have equality in (12) *iff* $\delta = \deg$.

Let $\bar{X}^0 := \mathbb{P}^2$ be the usual compactification of $\mathbb{C}^2$ given by $(x, y) \hookrightarrow [1 : x : y]$. If δ is a divisorial valuation on $\mathbb{C}[x, y]$ centered at infinity, then by definition there is a compactification $\bar{X}^1$ of $\mathbb{C}^2$ such that δ is the order of pole along an irreducible curve $C \subseteq \bar{X}^1 \setminus \mathbb{C}^2$. Without loss of generality, we may assume that $\bar{X}^1$ is nonsingular and there is a morphism $\pi : \bar{X}^1 \to \bar{X}^0$ that is identity on $\mathbb{C}^2$. Assume that δ is not the degree in (x, y)-coordinates. Then C is an *exceptional curve* of π (i.e. $\pi(C)$ is a point). Let $\bar{X}$ be the surface obtained from $\bar{X}^1$ by contracting all exceptional curves of π other than C (which is possible due to a criterion of Grauert [Băd01, Thm. 14.20]). Then $\bar{X} \setminus \mathbb{C}^2$ is the union of two irreducible curves, and the following result, which follows from results of [Mon16b], describes the matrix of intersection numbers of these curves in terms of the key forms of δ.

PROPOSITION 2.10 ([Mon16b, Props. 4.2 and 4.7]). *Given a divisorial semidegree δ on $\mathbb{C}[x, y]$ such that $\delta \neq$* deg *and $\delta(x) > 0$, there exists a unique compactification $\bar{X}$ of $\mathbb{C}^2$ such that*

1. *$\bar{X}$ is projective and normal.*
2. *$\bar{X}_\infty := \bar{X} \setminus X$ has two irreducible components C_1, C_2.*
3. *The semidegrees on $\mathbb{C}[x, y]$ corresponding to C_1 and C_2 are respectively* deg *and δ.*

Moreover, all singularities of $\bar{X}$ are rational (*which implies in particular that all Weil divisors are $\mathbb{Q}$-Cartier*). *Let $g_0, \dots, g_{n+1}$ be the key forms of δ. Then the* inverse *of the matrix of intersection numbers (C_i, C_j) of C_i and C_j, $1 \leq i, j \leq 2$, is*

$$\mathcal{M} = \begin{pmatrix} 1 & d_\delta \\ d_\delta & m_\delta \delta(g_{n+1}) \end{pmatrix}, \tag{13}$$

where d_δ and m_δ are as in respectively (4) *and* (6).

We will use the following result, which is an immediate corollary of [Mon16a, Thm. 4.3].

PROPOSITION 2.11. *Let δ, $\bar{X}$ and C_1, C_2 be as in Proposition 2.10. Let $g_0, \dots, g_{n+1}$ be the key forms of δ. Then the following are equivalent:*

1. *There is a (compact algebraic) curve C on $\bar{X}$ such that $C \cap C_1 = \emptyset$.*
2. *g_k is a polynomial for all k, $0 \le k \le n+1$.*
3. *g_{n+1} is a polynomial.*

The following follows from combining Theorems 4.1 and 4.3 of [Mon16a]:

THEOREM 2.12. *Let δ be a divisorial semidegree on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$, and $g_0, \dots, g_{n+1}$ be the key forms of δ. Then δ determines a normal algebraic compactification of $\mathbb{C}^2$ (in the sense of Remark 1.11) iff $\delta(g_{n+1}) > 0$ and g_{n+1} is a polynomial.*

2.2. *Descending Puiseux Series*

NOTE. The proofs of Theorems 1.4, 1.4′, and 1.8′ do *not* use the material of this subsection. Proposition 2.20 and Corollary 2.22 are used in the proof of $\delta(g_{n+1}) < 0$ case of Theorem 1.8.

DEFINITION 2.13 (Descending Puiseux series). The field of *descending Puiseux series* in x is

$$\mathbb{C}\langle\langle x \rangle\rangle := \bigcup_{p=1}^{\infty} \mathbb{C}((x^{-1/p})) = \left\{ \sum_{j \le k} a_j x^{j/p} : k, p \in \mathbb{Z},\, p \ge 1 \right\},$$

where for each integer $p \ge 1$, $\mathbb{C}((x^{-1/p}))$ denotes the field of Laurent series in $x^{-1/p}$. Let $\phi = \sum_{q \le q_0} a_q x^{q/p}$ be a descending Puiseux series where p is the *polydromy order* of ϕ, that is, p is the smallest positive integer such that $\phi \in \mathbb{C}((x^{-1/p}))$. Then the *conjugates* of ϕ are $\phi_j := \sum_{q \le q_0} a_q \zeta^q x^{q/p}$, $1 \le j \le p$, where ζ is a primitive pth root of unity. The usual factorization of polynomials in terms of Puiseux series implies the following:

THEOREM 2.14. *Let $f \in \mathbb{C}[x, y]$. Then there are unique (up to conjugacy) descending Puiseux series $\phi_1, \dots, \phi_k$, a unique nonnegative integer m, and $c \in \mathbb{C}^*$ such that*

$$f = cx^m \prod_{i=1}^{k} \prod_{\phi_{ij} \text{ is a conjugate of } \phi_i} (y - \phi_{ij}(x)).$$

The relation between descending Puiseux series and semidegrees is given by the following proposition, which is a reformulation of the corresponding result for Puiseux series and valuations [FJ04, Prop. 4.1].

PROPOSITION 2.15 ([Mon16b, Thm. 1.2]). *Let δ be a divisorial semidegree on $\mathbb{C}(x, y)$ such that $\delta(x) > 0$. Then there exist a* descending Puiseux polynomial (*i.e. a descending Puiseux series with finitely many terms*) $\phi_\delta \in \mathbb{C}\langle\langle x\rangle\rangle$ *and a rational number* $r_\delta < \mathrm{ord}_x(\phi_\delta)$ *such that, for every polynomial* $f \in \mathbb{C}[x, y]$,

$$\delta(f) = \delta(x)\deg_x(f(x, y)|_{y=\phi_\delta(x)+\xi x^{r_\delta}}), \tag{14}$$

where ξ *is an indeterminate.*

Definition 2.16. If ϕ_δ and r_δ are as in Proposition 2.15, then we say that $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ is the *generic descending Puiseux series* associated with δ.

Example 2.17. Let p, q be integers such that $p > 0$, and δ be the weighted degree on $\mathbb{C}(x, y)$ corresponding to weights p for x and q for y. Then $\tilde{\phi}_\delta = \xi x^{q/p}$ (i.e. $\phi_\delta = 0$).

Example 2.18. Let δ_ε be the semidegree from Example 2.8. Then $\tilde{\phi}_\delta = x^{5/2} + x^{-1} + \xi x^{-5/2}$.

The following result, which is an immediate consequence of [Mon16b, Prop. 4.2, Assertion 2], connects descending Puiseux series of a semidegree with the geometry of associated compactifications.

Proposition 2.19. *Let* $\delta, \bar{X}, C_1, C_2$ *be as in Proposition 2.10, and let* $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ *be the generic descending Puiseux series associated with* δ. *Assume in addition that* δ *is* not *a weighted degree, that is,* $\phi_\delta(x) \neq 0$. *Pick* $f \in \mathbb{C}[x, y] \setminus \{0\}$ *and let* C_f *be the curve on* $\bar{X}$ *that is the closure of the curve defined by* f *on* $\mathbb{C}^2$. *Then* $C_f \cap C_1 = \emptyset$ *iff the descending Puiseux factorization of* f *is of the form*

$$f = \prod_{i=1}^{k} \prod_{\phi_{ij} \text{ is a conjugate of } \phi_i} (y - \phi_{ij}(x)),$$
$$\text{where each } \phi_i \text{ satisfies } \phi_i(x) - \phi_\delta(x) = c_i x^{r_\delta} + \text{l.o.t.} \tag{15}$$

for some $c_i \in \mathbb{C}$ (*where l.o.t. denotes lower-order terms in* x).

The following result gives some relations between descending Puiseux series and key forms of semidegrees and follows from standard properties of key polynomials (in particular, the first three assertions follow from [Mon16a, Props. 3.17, 3.21, and 5.3], and the last assertion follows from the first one; a particular case of the last assertion (namely, the case $\delta(y) \le \delta(x)$) was proved in [Mon16b, Identity (4.6)]).

Proposition 2.20. *Let* δ *be a divisorial semidegree on* $\mathbb{C}(x, y)$ *such that* $\delta(x) > 0$. *Let* $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ *be the generic descending Puiseux series associated to* δ, *and* $g_0, \ldots, g_{n+1}$ *be the key forms of* δ. *Then*:

1. *There is a descending Puiseux series* ϕ *with*

$$\phi(x) - \phi_\delta(x) = cx^{r_\delta} + \text{l.o.t.}$$

for some $c \in \mathbb{C}$ (where l.o.t. denotes lower order terms in x) such that the descending Puiseux factorization of g_{n+1} is of the form

$$g_{n+1} = \prod_{\phi^* \text{ is a conjugate of } \phi} (y - \phi^*(x)). \tag{16}$$

2. *Let the* Puiseux pairs [Mon16a, *Definition 3.2*] *of ϕ_δ be $(q_1, p_1), \dots, (q_l, p_l)$ (if $\phi_\delta \in \mathbb{C}((1/x))$, then simply set $l = 0$). Set $p_0 := 1$. Then*

$$\deg(g_{n+1}) = \begin{cases} 1 & \text{if } \phi_\delta = 0, \\ \max\{1, \deg_x(\phi_\delta)\} p_0 p_1 \cdots p_l & \text{otherwise.} \end{cases}$$

3. *Write r_δ as $r_\delta = q_{l+1}/(p_0 \cdots p_l p_{l+1})$, where p_{l+1} is the smallest integer ≥ 1 such that $p_0 \cdots p_l p_{l+1} r_\delta$ is an integer. Let d_δ and m_δ be as in respectively* (4) *and* (6). *Then*

$$m_\delta = p_{l+1},$$
$$d_\delta = \begin{cases} \max\{p_1, q_1\} & \text{if } \phi_\delta = 0, \\ \max\{1, \deg_x(\phi_\delta)\} p_0 p_1 \cdots p_{l+1} & \text{otherwise.} \end{cases}$$

4. *Let the skewness $\alpha(\delta)$ of δ be defined as in footnote 3. Then*

$$\alpha(\delta) = m_\delta \delta(g_{n+1})/d_\delta^2 = \begin{cases} \frac{\min\{p_1, q_1\}}{\max\{p_1, q_1\}} = \min\{\delta(x), \delta(y)\}/d_\delta & \text{if } \phi_\delta = 0, \\ \frac{\delta(g_{n+1})}{d_\delta \deg(g_{n+1})} & \text{otherwise.} \end{cases} \tag{17}$$

The following lemma is a consequence of assertion 1 of Proposition 2.20 and the definition of generic descending Puiseux series of a semidegree. It follows via a straightforward but cumbersome induction on the number of *Puiseux pairs* of the *descending Puiseux roots* of f, and we omit the proof.

LEMMA 2.21. *Let δ be a divisorial semidegree on $\mathbb{C}(x, y)$ such that $\delta(x) > 0$. Let $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ be the generic descending Puiseux series associated to δ, and $g_0, \dots, g_{n+1}$ be the key forms of δ. Then, for all $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$,*

$$\frac{\delta(f)}{\deg(f)} \geq \frac{\delta(g_{n+1})}{\deg(g_{n+1})}. \tag{18}$$

Now assume in addition that δ is not *a weighted degree, that is, $\phi_\delta(x) \neq 0$. Then equality holds in* (18) *iff f has a descending Puiseux factorization as in* (15).

Combining Propositions 2.11 and 2.19 and Lemma 2.21 yields the following:

COROLLARY 2.22. *Consider the setup of Proposition 2.11. Assume in addition that δ is* not *a weighted degree. Then assertions 1–3 of Proposition 2.11 are equivalent to the following statement:*

4. *There exists $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$ for which equality holds in* (18).

3. Proofs

Proof of Theorem 1.4′. Let $\bar{X}$ be the projective compactification of $X := \mathbb{C}^2$ from Section 1.2. With the notation in Proposition 2.10, the matrix of intersection numbers (C_i, C_j) of C_i and C_j, $1 \le i, j \le 2$, is

$$\mathcal{I} = \frac{1}{d_\delta^2 - m_\delta \delta(g_{n+1})} \begin{pmatrix} -m_\delta \delta(g_{n+1}) & d_\delta \\ d_\delta & -1 \end{pmatrix}. \tag{19}$$

In particular, $(C_1, C_1) = -\frac{m_\delta}{d_\delta^2 - m_\delta \delta(g_{n+1})} \delta(g_{n+1})$. Since $\delta \neq \deg$ (by the assumptions of Theorem 1.4′), assertion 1 of Theorem 1.4′ follows from Proposition 2.9. It follows similarly that $(C_2, C_2) < 0$, so that [Kol96, Lemma II.4.12][5] implies that l_2 determines an edge of $\mathrm{NE}(\bar{X})$, which implies assertion 2a. Now observe that

$$\begin{aligned} \delta(g_{n+1}) \ge 0 \quad &\Rightarrow \quad (C_1, C_1) \le 0 \quad \text{(assertion 1)} \\ &\Rightarrow \quad l_1 \text{ determines an edge of } \mathrm{NE}(\bar{X}) \\ &\qquad\ \text{[Kol96, Lemma II.4.12]}. \end{aligned} \tag{20}$$

On the other hand, $\delta(g_{n+1}) < 0 \Rightarrow (C_1, C_1) > 0$ (assertion 1), which implies that there exists $\beta \in \mathbb{Q}$ such that with respect to the basis $(C_1, C_2 + \beta C_1)$ of $N_1(\bar{X})$, the intersection form is of the form $x_1^2 - x_2^2$. [Kol96, Lemma II.4.12] then implies that C_1 is in the interior of $\mathrm{NE}(\bar{X})$. The preceding sentence, together with (20), implies assertions 2b and 2c. The first statement of assertion 2 follows from assertions 2a, 2b, and 2c. □

Proof of Theorem 1.4. Without loss of generality, we may (and will) assume that $\delta \neq \deg$ and use the notation in Theorem 1.4′. Pick $f \in \mathbb{C}[x, y] \setminus \{0\}$ and let $\bar{D}_f$ be the closure in $\bar{X}$ of the curve D_f defined by f in $\mathbb{C}^2$, so that $\bar{D}_f \sim \deg(f) C_1 + \delta(f) C_2$. Consequently, $\delta(f) \ge 0$ for all $f \in \mathbb{C}[x, y] \setminus \{0\}$ iff $\mathrm{NE}(\bar{X})$ is generated by C_1 and C_2. Assertion 1 then follows from assertion 2 of Theorem 1.4′.

We now prove assertion 2. Proposition 2.11 implies that assertions 2b and 2b′ are equivalent. Therefore by assertion 1 it suffices to show that either 2a or 2b′ implies that δ is positive on $\mathbb{C}[x, y] \setminus \mathbb{C}$. Now if 2a holds, then $(C_1, C_1) < 0$ (Theorem 1.4′). A criterion of Grauert (adapted to the case of normal surfaces in [Sak84, Thm. 1.2]) then implies that C_1 is *contractible,* that is, there is a map $\pi : \bar{X} \to \bar{X}'$ of normal analytic surfaces such that $\pi(C_1)$ is a point and $\pi|_{\bar{X} \setminus C_1}$ is an isomorphism. In particular, for each $f \in \mathbb{C}[x, y]$, $\delta(f)$ is the order of pole of f along the irreducible curve at infinity on the compactification $\bar{X}'$ of $X := \mathbb{C}^2$, and consequently δ is positive on $\mathbb{C}[x, y] \setminus \mathbb{C}$, as required. Now assume that 2b′ holds. Then Theorem 1.4′ implies that $(C_1, C_1) = 0$. Assume (to the contrary of our goal) that there exists $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$ such that $\delta(f) = 0$. Then we have

[5] Even though [Kol96, Lemma II.4.12] is proved for only *nonsingular* surfaces, its proof goes through for arbitrary normal surfaces using the intersection theory due to [Mum61].

$(\bar{D}_f, C_1) = (\deg(f)C_1, C_1) = 0$, so that $\bar{D}_f \cap C_1 = \emptyset$. Proposition 2.11 then implies that g_{n+1} is a polynomial, which contradicts 2b′. It follows that δ is positive on $\mathbb{C}[x, y] \setminus \mathbb{C}$, which completes the proof of assertion 2. □

Proof of Theorem 1.8′. A straightforward computation using the entries of the intersection matrix $\mathcal{I}$ from (19) shows that

$$(C_i^*, C_j) = \delta_{ij}, \tag{21}$$

where δ_{ij} is the Kronecker delta. Since $(C_1^* + \varepsilon C_2, C_2) < 0$ for all $\varepsilon > 0$ and since l_2 is an edge of $\mathrm{NE}(\bar{X})$, identity (21) immediately implies assertion 1. To complete the proof of Theorem 1.8′, it suffices to prove the ($\Leftarrow$) direction of assertions 2 and 3. Now if $\delta(g_{n+1}) \geq 0$, then $\mathrm{NE}(\bar{X})$ is generated by C_1 and C_2 (assertion 2 of Theorem 1.4′), so that (21) implies that l_2^* is also an edge of $\mathrm{Nef}(\bar{X})$. This implies the ($\Leftarrow$) direction of assertion 2. On the other hand, if $\delta(g_{n+1}) < 0$, then C_1 is in the interior of $\mathrm{NE}(\bar{X})$ (assertion 2c of Theorem 1.4′). Since $(C_2^*, C_1) = 0$, it follows that $C_2^* \notin \mathrm{Nef}(\bar{X})$, which implies the ($\Leftarrow$) direction of assertion 3, as required to complete the proof of Theorem 1.8′. □

Proof of Theorem 1.8. Without loss of generality, we may (and will) assume that $\delta \neq \deg$ and use the notation in Theorems 1.2 and 1.8′. Let ϕ_δ be as in Proposition 2.20. Consider first the case that $\phi_\delta = 0$. Then $n = 0$, and the key forms of δ are $g_0 = x$ and $g_1 = y$ (Example 2.7). On the other hand, (17) implies that (3) holds, so that Theorem 1.8 is true in this case. Therefore we may (and will) assume that $\phi_\delta \neq 0$ and divide the proof into separate cases depending on $\delta(g_{n+1})$.

Case 1: $\delta(g_{n+1}) \geq 0$. In this case, C_2^* is on an edge of $\mathrm{Nef}(\bar{X})$ (assertion 2 of Theorem 1.8′). Since any nef divisor is a limit of ample divisors and large multiples of ample divisors have global sections, it follows that there exist $f_1, f_2, \ldots \in \mathbb{C}[x, y]$ such that $\bar{D}_{f_k} \sim r_k(C_1 + s_k C_2)$ for some $r_k, s_k \in \mathbb{Q}_{>0}$ such that $\lim_{k\to\infty} s_k = \frac{m_\delta \delta(g_{n+1})}{d_\delta}$ (where the $\bar{D}_{f_k}$ are defined as in the proof of Theorem 1.4). Identity (17) then implies that we have equality in (3) in this case.

Case 2: $\delta(g_{n+1}) < 0$. In this case, C_1 is in the interior of $\mathrm{NE}(\bar{X})$ (Theorem 1.4′, assertion 2c). [Kol96, Lemma II.4.12] (adapted to the case of normal surfaces as in footnote 5) implies that $\mathrm{NE}(\bar{X})$ has an edge of the form $\{r(C_1 - aC_2) : r \geq 0\}$ for some $a \in \mathbb{Q}_{>0}$ and, moreover, that there exists $r > 0$ such that $rC_1 - arC_2 \sim \bar{D}_g$ for some $g \in \mathbb{C}[x, y]$. Then $\deg(g) = r$ and $\delta(g) = -ar$. Pick $f \in \mathbb{C}[x, y] \setminus \mathbb{C}$. Since the "other edge" of $\mathrm{NE}(\bar{X})$ is spanned by C_2 (Theorem 1.4′, assertion 2a), it follows that $\bar{D}_f \sim sC_2 + t(C_1 - aC_2)$ for some $s \in \mathbb{Q}_{\geq 0}$ and $t \in \mathbb{Q}_{>0}$, and therefore

$$\frac{\delta(f)}{\deg(f)} = \frac{s - ta}{t} \geq -a = \frac{\delta(g)}{\deg(g)}.$$

It follows that

$$\inf\left\{\frac{\delta(f)}{d_\delta \deg(f)} : f \in \mathbb{C}[x, y] \setminus \mathbb{C}\right\} = \frac{\delta(g)}{d_\delta \deg(g)}. \tag{22}$$

On the other hand, (17) implies that

$$\alpha(\delta) = \frac{\delta(g_{n+1})}{d_\delta \deg(g_{n+1})}.$$

Lemma 2.21 and Corollary 2.22 then imply that we have an equality in (3) iff g_{n+1} is a polynomial.

The assertions in Theorem 1.8 now follow from the conclusions of the two cases. □

Proof of Corollary 1.10. We continue to assume that $\delta \neq \deg$ and use the notation in the proof of Theorem 1.8. Identify $\mathrm{Nef}(\bar{X})$ with its image in $\mathbb{R}^2$ via the map $a_1 C_1 + a_2 C_2 \mapsto (a_1, a_2)$. Note that:

(A) The "upper edge" of $\mathrm{Nef}(\bar{X})$ is $l_1^* = \{r(1, d_\delta) : r \in \mathbb{R}_{\geq 0}\}$ (Theorem 1.8′), and $l_1^* \subseteq C_\delta$ (since $(1, d_\delta) = (\deg(f), \delta(f))$, where f is a general linear polynomial in (x, y)).

(B) C_δ contains the "lower edge" of $\mathrm{Nef}(\bar{X})$ iff g_{n+1} is a polynomial iff g_k is a polynomial for all k, $0 \leq k \leq n+1$ (follows by combining Theorem 1.8, Lemma 2.21, and Corollary 2.22).

Since $\mathrm{Nef}(\bar{X})$ is a closed cone and since C_δ contains the *ample cone* of $\bar{X}$, these observations imply assertion 1. For assertion 2, note that δ determines an analytic compactification of $\mathbb{C}^2$

iff C_1 is *contractible*

iff $(C_1, C_1) < 0$ (by Grauert's criterion [Sak84, Thm. 1.2])

iff $\delta(g_{n+1}) > 0$ (Theorem 1.4′, assertion 1).

Since the arguments in the proof of Theorem 1.8 show that $\delta(g_{n+1}) \leq 0$ iff the closure of C_δ contains the positive x-axis, this completes the proof of assertion 2. The equivalence of assertions 3a and 3b follows from assertion 1 and Theorem 2.12. Since 3c clearly implies 3b, it remains to show that 3b $\Rightarrow$ S_δ is finitely generated. Since C_δ is a rational cone, 3b implies that $\bar{S}_\delta := C_\delta \cap \mathbb{Z}^2$ is finitely generated. Since $\bar{S}_\delta$ is *integral* over S_δ (i.e. for every $s \in \bar{S}_\delta$, there is a positive integer m such that $ms \in S_\delta$), it follows that S_δ is also finitely generated, as required to complete the proof of the corollary. □

Example 3.1 (An example where (3) does not hold). Let δ be the semidegree on $\mathbb{C}(x, y)$ defined as follows:

$$\delta(f(x, y)) := \deg_x(f(x, y)|_{y = x^{-1} + \xi x^{-2}}) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},$$

where ξ is an indeterminate. Then the key forms of δ are $x, y, y - x^{-1}$. Moreover,

$$d_\delta = \max\{\delta(x), \delta(y)\} = \max\{1, -1\} = 1,$$

$$m_\delta = \gcd(\delta(x), \delta(y), \delta(y - x^{-1})) = \gcd(1, -1, -2) = 1,$$

and therefore (17) implies that

$$\alpha(\delta) = \delta(y - x^{-1}) / \deg(y - x^{-1}) = -2. \tag{23}$$

Now consider the surface $\bar{X}$ from Proposition 2.10. Then the matrix $\mathcal{M}$ (from Proposition 2.10) and the intersection matrix $\mathcal{I}$ of C_1 and C_2 are:

$$\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \qquad \mathcal{I} = \mathcal{M}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}. \tag{24}$$

In the notation in the proof of Theorem 1.4, we have $\bar{D}_y \sim \deg(y)C_1 + \delta(y)C_2 = C_1 - C_2$. It follows from (24) that $(C, C) = -1/3 < 0$, so that [Kol96, Lemma II.4.12] implies that C spans an edge of the cone of curves on $\bar{X}$, that is, the polynomial g from Case 2 in the proof of Theorem 1.8 is y. It then follows from identities (22) and (23) that

$$\inf\left\{ \frac{\delta(f)}{d_\delta \deg(f)} : f \in \mathbb{C}[x, y] \setminus \mathbb{C} \right\} = \frac{\delta(y)}{d_\delta \deg(y)} = -1 > \alpha(\delta).$$

EXAMPLE 3.2 (The *semigroup of values* does not distinguish semidegrees that determine algebraic compactifications of $\mathbb{C}^2$). Let δ be the semidegree on $\mathbb{C}(x, y)$ defined as follows:

$$\delta(f(x, y)) := 2\deg_x(f(x, y)|_{y=x^{5/2}+x^{-1}+\xi x^{-3/2}}) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},$$

where ξ is an indeterminate. Then the key forms of δ are x, y, $y^2 - x^5$, $y^2 - x^5 - 2x^{-1}y$ with corresponding δ-values 2, 5, 3, 1. Since the δ-value of the last key polynomial is *positive,* it follows from the arguments in the proof of Corollary 1.10 that δ determines an analytic compactification of $\mathbb{C}^2$. But the last key form of δ is *not* a polynomial, so that the compactification determined by δ is *nonalgebraic* (Theorem 2.12). On the other hand, it follows from our computation of the values of δ and Corollary 2.22 that the semigroup of values of δ on polynomials is

$$N_\delta := \{\delta(f) : f \in \mathbb{C}[x, y]\} = \{2, 3, 4, \dots\}.$$

Now let δ' be the weighted degree on (x, y)-coordinates corresponding to weights 2 for x and 3 for y. Then δ' determines an *algebraic* compactification of $\mathbb{C}^2$, namely, the weighted projective surface $\mathbb{P}^2(1, 2, 3)$. But $N_\delta = N_{\delta'}$.

EXAMPLE 3.3 ($\mathrm{NE}(\tilde{X})$ or $\mathrm{Nef}(\tilde{X})$ is not determined by purely numerical conditions if $\delta(g_{n+1}) < 0$). Let δ' be the semidegree on $\mathbb{C}(x, y)$ defined as follows:

$$\delta'(f(x, y)) := \deg_x(f(x, y)|_{y=\xi x^{-2}}) \quad \text{for all } f \in \mathbb{C}(x, y) \setminus \{0\},$$

where ξ is an indeterminate; in other words, δ' is the weighted degree on $\mathbb{C}[x, y]$ corresponding to weights 1 for x and -2 for y. Then the key forms of δ' are x, y. Moreover,

$$d_{\delta'} = \max\{\delta'(x), \delta'(y)\} = \max\{1, -1\} = 1,$$
$$m_{\delta'} = \gcd(\delta'(x), \delta'(y)) = \gcd(1, -2) = 1.$$

Let $\bar{X}'$ be the surface associated to δ' via the construction in Proposition 2.10. Then the matrix $\mathcal{I}'$ of curves C_1' and C_2' at infinity on $\bar{X}'$ is identical to $\mathcal{I}$ from (24), and it is straightforward to see that the weighted dual graphs of the curves

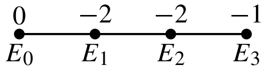

Figure 1 Dual graph of curves at infinity on $\tilde{X}$ and $\tilde{X}'$

at infinity (with respect to $\mathbb{C}^2 = \operatorname{Spec}(\mathbb{C}[x, y])$) on the minimal resolutions $\tilde{X}$ and $\tilde{X}'$ of respectively $\bar{X}$ and $\bar{X}'$ are also identical; see Figure 1 (here E_0 (resp. E_3) corresponds to the strict transform of C_1 (resp. C_2) in the case of $\tilde{X}$ and to the strict transform of C'_1 (resp. C'_2) in the case of $\tilde{X}'$).

On the other hand, if $\bar{D}'_y$ is the closure of the x-axis in $\bar{X}'$, then $\bar{D}'_y \sim \deg(y)C'_1 + \delta'(y)C'_2 = C'_1 - 2C'_2$. Since $\bar{D}_y = C_1 - C_2$ determines an edge of $\mathrm{NE}(\bar{X})$, it follows that $\mathrm{NE}(\bar{X}) \not\cong \mathrm{NE}(\bar{X}')$ (via the natural isomorphism $N_1(\bar{X}) \cong N_1(\bar{X}')$ given by the mapping $C_1 \mapsto C'_1$, $C_2 \mapsto C'_2$). Consequently, it follows that the cones of curves and nef cones of $\tilde{X}$ and $\tilde{X}'$ are also not isomorphic.

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