Michigan Math. J. 64 (2015), 475–485

The Complex Structure of the Teichmüller Space

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Abstract. The Teichmüller space of a topological surface X is a space that parameterizes complex structures on X up to the action of homeomorphisms that are isotopic to the identity. This space itself has a complex structure defined in terms of Beltrami differentials and quasi-conformal mappings. For X a surface of genus g and m punctures, n geodesics $A_1, \dots, A_n$ ($n = 6g - 6 + 2m$) can be chosen so that their hyperbolic translation lengths $(L(A_1), \dots, L(A_n))$ give a local parameterization of the Teichmüller space.

In this paper we describe the almost complex structure as a real matrix acting on the tangent space with basis $(\partial/\partial L(A_1), \dots, \partial/\partial L(A_n))$. In the cotangent space the natural Hermitian scalar product of the associated quadratic differentials $(\Theta_{A_1}, \dots, \Theta_{A_n})$ determines a skew-symmetric real matrix C and a symmetric matrix S. We prove that the matrix of the complex structure is SC^{-1}.

Introduction

The Teichmüller space of a topological surface X is a space that parameterizes complex structures on X up to the action of homeomorphisms that are isotopic to the identity. This space itself has a complex structure, which is defined in terms of Beltrami differentials and quasi-conformal mappings. We describe the relationship of this complex structure in terms of the variation of the lengths of geodesics on a variable surface X_τ. We will view such surfaces as a quotient space of the upper half-plane factored by variable fixed-point-free Fuchsian groups Γ_τ. The upper half-plane has a hyperbolic metric whose corresponding geodesics are semicircles and half-lines orthogonal to $\mathbb{R}$.

A hyperbolic element A in Γ has a unique geodesic axis and a well-defined translation length $L(A)$ where

$$\cos h\left(\frac{L(A)}{2}\right) = \frac{1}{2}|\operatorname{trace}(A)|.$$

For X a surface of genus g and m punctures, the group elements of Γ are expressed as real analytic functions of finitely many group elements $A_1, \dots, A_n$ ($n = 6g - 6 + 2m$); and the n-tuple $(L(A_1), \dots, L(A_n))$ gives a local coordinate chart of the Teichmüller space. (For these classical facts, see, e.g., Gardiner [2, pp. 153–157], Ahlfors [1].)

Received January 16, 2014. Revision received April 24, 2015.
Research partially supported by Fondecyt 1130445.

The important problem of how to describe the complex structure as a real matrix acting on the tangent space with basis $(\partial/\partial L(A_1), \dots, \partial/\partial L(A_n))$ was stated by Wolpert [6; 8]. The purpose of this paper is to give just such a description, the explicit nature of which, moreover, might well lead to applications of Teichmüller theory to dynamical systems and three-dimensional topology (see, e.g., [4]); we show here an insight into Abelian varieties.

1. The Tangent Space to the Teichmüller Space

Let X be a compact Riemann surface whose universal cover is the upper-half-plane $\mathbb{H}$ with covering group a fixed-point-free Fuchsian group Γ.

The tangent space at X of the Teichmüller space is identified with equivalence classes of Γ invariant Beltrami differentials as follows. Denote by $M(\Gamma)$ the space of complex valued, measurable, essentially bounded functions $\mu(z)$ on $\mathbb{H}$ satisfying the invariance property

$$\mu(A(z))\overline{A'(z)}/A'(z) = \mu(z)$$

for all A in Γ. Let $Q(\Gamma)$ be the space of holomorphic functions $h(z)$ on $\mathbb{H}$ satisfying the transformation law

$$h(A(z))A'(z)^2 = h(z).$$

If Δ is a fundamental domain, then the pairing

$$\int_\Delta \mu h$$

is well defined, and for the null space $N(\Gamma)$ of Beltrami differentials orthogonal to all $Q(\Gamma)$, the finite-dimensional complex vector spaces $M(\Gamma)/N(\Gamma)$ and $Q(\Gamma)$ are the tangent and cotangent spaces to the Teichmüller space.

We consider now the fundamental construction for each μ in $M_1(\Gamma)$, the open unit ball in $M(\Gamma)$. The Beltrami equation

$$\begin{cases} \omega_{\bar{z}} = \mu(z)\omega_z, & z \text{ in } \mathbb{H}, \\ \omega_{\bar{z}} = \overline{\mu(\bar{z})}\omega_z, & z \text{ in } \mathbb{L}, \end{cases}$$

has a unique solution ω_μ fixing 0, 1, ∞. In this case, $\Gamma_\mu = \omega_\mu(\Gamma)\omega_\mu^{-1}$ is again a Fuchsian group, and $\mathbb{H}/\Gamma_\mu$ is the deformed surface.

Moreover, for real ε,

$$\omega_{\varepsilon_\mu}(z) = z + \varepsilon G(z) + o(\varepsilon^2),$$

where $G(z) = F(z) + \overline{F(\bar{z})}$ and

$$F(z) = -\frac{z(z-1)}{\pi} \iint_{\mathbb{H}} \mu(\zeta) \frac{1}{\zeta(\zeta-1)(\zeta-z)} \left(\frac{d\zeta\, \overline{d\zeta}}{-2i} \right). \tag{1.1}$$

These formulas are the basis of the infinitesimal approach to Teichmüller theory.

Denote by Π_2 the space of complex polynomials of degree at most two. The group Γ acts on the right on Π_2 via

$$(pA)(z) = p(A(z))/A'(z).$$

A cocycle $\chi : \Gamma \to \Pi_2$ is a function such that

$$\chi(A_1 \cdot A_2) = \chi_{A_1} \cdot A_2 + \chi_{A_2},$$

and a coboundary is a function given by

$$p \cdot A - p.$$

The vector space of cocyles modulo coboundaries is the space $H^1(\Gamma, \Pi_2)$.

With each Beltrami differential μ in $M(\Gamma)$, we associate a tangent vector $t(\mu)$ (or $\partial/\partial t(\mu)$) in $H^1(\Gamma, \Pi_2)$ by the following procedure.

The function $F(z)$ defined in (1.1) is the unique solution of

$$\frac{\partial F}{\partial \bar{z}} = \begin{cases} \mu & \text{on } \mathbb{H}, \\ 0 & \text{on } \mathbb{L}, \end{cases}$$

vanishing at 0, 1 and $0(|z|^2)$ at ∞.

For A in Γ, $F(A(z))/A'(z)$ satisfies the same equation, and it follows that

$$F(z) - F(A(z))/A'(z) = p_A(z) \tag{1.2}$$

is a polynomial of degree at most two.

The tangent vector in $H^1(\Gamma, \Pi_2)$ is

$$t(\mu)(A) = p_A(z) + \overline{p_A(\bar{z})}. \tag{1.3}$$

The complex structure in terms of Beltrami differentials is simple: $\mu \to i\mu$. Then

$$t(i\mu)(A) = i(p_A(z) - \overline{p_A(\bar{z})}) \tag{1.4}$$

as seen in (1.1) if we replace μ by $i\mu$.

These are quadratic polynomials with real coefficients.

The tangent space to the Teichmüller space is thus identified with a subspace V in $H^1(\Gamma, \Pi_2)$; we will find V explicitly and describe the complex structure in terms of these cocyles. For formulas (1.1) and (1.3), see, for example, Gardiner [2]. For $H^1(\Gamma, \Pi_2)$, see Kra [3].

2. The Fenchel–Nielsen Deformation

Let A in Γ be the transformation $A(z) = \lambda z$, $\lambda > 0$. Choose $\varphi(\theta)$, $\theta = \arg z$, a continuous function with compact support in $(0, \pi)$ such that

$$\int_0^{\pi} \varphi(\theta)\, d\theta = \frac{1}{2}.$$

The formula

$$\omega = z \exp\left(2\varepsilon \int_0^{\theta} \varphi\right), \quad \varepsilon \text{ real},$$

defines a quasi-conformal automorphism of $\mathbb{H}$ with Beltrami differential

$$\mu_\varepsilon(z) = \frac{i\varepsilon\varphi(\theta)e^{2i\theta}}{1 - i\varepsilon\varphi(\theta)} = \varepsilon\mu_0(z) + o(\varepsilon), \tag{2.1}$$

where $\mu_0(z) = i\varphi(\theta)e^{2i\theta}$. To obtain an element in $M(\Gamma)$, we average over the group via

$$\mu_A(z) = \sum_{\langle A\rangle\backslash\Gamma} \mu_\varepsilon(B(z))\overline{B'(z)}/B'(z). \tag{2.2}$$

We extend these constructions to a general element A in Γ with fixed points $p < q$ via

$$h(z) = \frac{z-p}{-z+q}, \qquad \hat{\mu}_\varepsilon = (\mu_\varepsilon \circ h)\bar{h}'/h' \tag{2.3}$$

and

$$\mu_A = \sum_{\langle A\rangle\backslash\Gamma} (\hat{\mu}_\varepsilon \circ B)\bar{B}'/B'.$$

On the other hand, in the cotangent space, we define dual concepts.

For $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, define $\omega_A = (tr^2 A - 4)(cz^2 + (d-a)z - b)^{-2}$, a quadratic differential for the cyclic group $\langle A\rangle$.

The Petersson series is defined by

$$\theta_A = \sum_{\langle A\rangle\backslash\Gamma} (\omega_A \circ B)B'^2.$$

We now have the following fact:

$$t(\mu_A) \text{ is equivalent modulo } N(\Gamma) \text{ to } \frac{i}{\pi}(\operatorname{Im} z)^2\bar{\theta}_A. \tag{2.4}$$

From this it is apparent that $t(\mu_A)$ is independent of the particular function $\varphi(\theta)$ chosen, and, in fact, in the foregoing integral formulas, we will assume that $\varphi = (1/2)\delta(\pi/2)$ (the Dirac delta distribution at $\pi/2$).

If L_B is the translation length of B, then

$$t(\mu_A)(L_B) = \sum_{p\in\alpha\cdot\beta} \cos\theta_p. \tag{2.5}$$

For simple closed curves α and β, this formula represents the infinitesimal change of the length L_B given an unit infinitesimal twist along the curve α.

Here α and β are the projections of the axis of A and B on the surface M, $\alpha\cdot\beta$ is the geometric intersection of α and β, and θ_p is the angle at each intersection point, measured from α to β as the x-axis crosses the y-axis.

The skew-symmetric form

$$\left(\frac{2}{\pi^2}\right)\operatorname{Im}\int_\Delta \theta_A\bar{\theta}_B(\operatorname{Im} z)^2$$

is equal to

$$c(A, B) = \sum_{p\in\alpha\cdot\beta} \cos\theta_p. \tag{2.6}$$

For all details, see Wolpert [6; 7].

The symmetric form

$$\operatorname{Re}\int_\Delta \theta_A \bar{\theta}_B (\operatorname{Im} z)^2$$

equals

$$s(A, B) = \frac{2}{\pi}\left(\delta_{A,B} L_A + \sum_{\langle A\rangle\backslash\Gamma/\langle B\rangle} c\left(\log\left|\frac{c+1}{c-1}\right| - 2\right)\right). \tag{2.7}$$

Here $c = \cos\theta_p$ at each intersection point of $\alpha \cdot \beta$ or $c = \cos h\delta$, where δ is the hyperbolic distance from the axis of A to each disjoint axis congruent to the axis of B. ($\delta_{A,B}$ is the Kronecker symbol equal to 1 if $A = B$ and 0 if $A \neq B$.)

We shall recover formulas (2.6) and (2.7) with calculations involving only Beltrami differentials, without any reference to quadratic differentials; this is necessary since the complex structure ($\mu \to i\mu$) is defined in the tangent, rather than in the cotangent space. In these computations we shall assume α and β to be simple closed curves.

For all preliminaries and formulas in this section, see Wolpert [6; 7; 8; 9; 10] and Riera [5].

3. The Complex Structure in the Tangent Space

In this section we let $(A_1, \dots, A_n)$ $(n = 6g - 6 + 2m)$ be hyperbolic elements in Γ such that

(i) $A_1(z) = \lambda_1 z$, $\lambda_1 > 1$.
(ii) A_j fixes p_j, q_j, $p_j < q_j$, p_j repelling, q_j attracting, and $L_j = \log\lambda_j$, $\lambda_j > 1$, the translation length $(1 < j \leq n)$.

Even though the results are simpler to state if we assume that no A_j fixes ∞, condition (i) is more natural.

The next two lemmas, in which the key fact in the theory is that 0, 1, ∞ are distinguished points, are the only technical results that will be needed in what follows.

Lemma 3.1. *Let A be a hyperbolic transformation with fixed points p, q $(p < q)$, both different from* 0, ∞. *Set*

$$\mu(z) := \sum_{n\in\mathbb{Z}} \hat{\mu}_0(\lambda^n z) \quad (\lambda > 1),$$

where $\hat{\mu}_0$ is the first-order term of $\hat{\mu}_\varepsilon$, the A-invariant Fenchel–Nielsen differential (2.3), *and let*

$$F(z) = -\frac{z(z-1)}{\pi}\iint_{\mathbb{H}} \mu(\zeta)\frac{1}{\zeta(\zeta-1)(\zeta-z)}\left(\frac{d\zeta\, d\bar{\zeta}}{-2i}\right).$$

Then

(i) *If the axis of A intersects the imaginary axis at an angle θ, then*

$$F(z) - \frac{F(\lambda z)}{\lambda} = +\frac{z}{2}\left(\cos\theta + \frac{i}{\pi}\left(\cos\theta\log\frac{1+\cos\theta}{1-\cos\theta} - 2\right)\right);$$

(ii) *If the axis of A is a non-Euclidean distance δ from the imaginary axis*

$$F(z)-\frac{F(\lambda z)}{\lambda}=-\frac{z}{2\pi i}\left(\cos h\delta\log\left(\frac{\cos h\delta+1}{\cos h\delta-1}\right)-2\right).$$

(*If* $\lambda<1$, *then the signs in the right-hand sides are to be reversed.*)

Proof. With a natural change of variables, we may write $F(z)$ as

$$-\frac{1}{\pi}\iint_{\mathbb{H}}\hat{\mu}_0(\eta)\sum_{\eta}\frac{\lambda^n z(z-1)}{\eta(\lambda^n\eta-1)(\lambda^n\eta-z)}\left(\frac{d\eta\, d\bar{\eta}}{-2i}\right),$$

and, similarly, $F(\lambda z)/\lambda$ equals

$$-\frac{1}{\pi}\iint_{\mathbb{H}}\hat{\mu}_0(\eta)\sum_{n}\frac{\lambda^n z(\lambda z-1)}{\eta(\lambda^{n+1}\eta-1)(\lambda^n\eta-z)}\left(\frac{d\eta\, d\bar{\eta}}{-2i}\right).$$

The difference of the sums involved is the telescopic series

$$\frac{z}{\eta}\sum_{n}\frac{\lambda^n(1-\lambda)}{(\lambda^n\eta-1)(\lambda^{n+1}\eta-1)}=\begin{cases} z/\eta^2, & \lambda>1,\\ -z/\eta^2, & 0<\lambda<1.\end{cases}$$

To evaluate the integral

$$-\frac{z}{\pi}\iint_{\mathbb{H}}\frac{\hat{\mu}_0(\eta)}{\eta^2}\left(\frac{d\eta\, d\bar{\eta}}{-2i}\right),$$

we make the change of variables $\eta=(p\zeta-q)(\zeta-1)$, $\zeta=\rho e^{i\theta}$ and assume that $\varphi(\theta)=(1/2)\delta(\pi/2)$ to obtain

$$-\frac{iz}{2\pi}(p-q)^2\int_0^{+\infty}\frac{\rho\, d\rho}{(p\rho i-q)^2(\rho i-1)^2}.$$

A straightforward residue calculation then shows that this last integral has the value

$$\begin{aligned}&\frac{iz}{2\pi}\frac{(p-q)^2}{p^2}\sum\operatorname{Res}\left(\frac{w\log w}{(w+iq/p)^2(w+i)^2}\right)\\ &\quad=\frac{iz}{2\pi}\left(\frac{q+p}{q-p}\left(\log\left(-i\frac{q}{p}\right)-\frac{3\pi}{2}i\right)-2\right).\end{aligned}$$

(i) If $p<0<q$, then $(q+p)/(q-p)=\cos\theta$ and $\log(-iq/p)=\log|q/p|+i\pi/2$, thus proving formula (i).

(ii) If p,q have the same sign, then $(q+p)/(q-p)=\cos h\delta$ and $\log(-iq/p)=\log|q/p|+i3\pi/2$, proving formula (ii). □

LEMMA 3.2. *Let A have fixed points* $0,\infty$ *and multiplier* $\lambda>1$, *and let* $\mu_0(\zeta)$ *be as in* (2.1). *Set*

$$F(z)=\frac{-z(z-1)}{\pi}\iint_{\mathbb{H}}\mu_0(\zeta)\frac{1}{\zeta(\zeta-1)(\zeta-z)}\left(\frac{d\zeta\, d\bar{\zeta}}{-2i}\right).$$

Then

$$F(z)-F(\lambda z)/\lambda=-z\log\frac{\lambda}{2\pi i}.$$

Proof. The computation is similar to that used in the proof of Lemma 3.1, except that there are no sums.

The integral obtained is

$$\frac{z(\lambda-1)}{\pi}\iint_{\mathbb{H}}\mu_0(\zeta)\frac{1}{\zeta(\zeta-1)(\lambda\zeta-1)}\left(\frac{d\zeta d\bar{\zeta}}{-2i}\right)$$
$$=\frac{z}{2\pi}\frac{(\lambda-1)}{\lambda}\sum\mathrm{Res}\left(\frac{\log w}{(w+i)(w+i/\lambda)}\right),$$

and the result follows. □

Proposition 3.1. *Let $p(A_j)$ be the cocycle in $H^1(\Gamma,\Pi_2)$ defined in (1.2). Then*

(i) *for $j>1$,*

$$p(A_j)(A_1)=\frac{z}{2}(c(A_j,A_1)+is(A_j,A_1));$$

(ii) *for $k\neq 1, j$,*

$$p(A_j)(A_k)=\frac{1}{2}(z-p_k)(z-q_k)(c(A_j,A_k)+is(A_j,A_k)).$$

Proof.

(i) The cocycle $p(A_j)$ is defined via integration by the Beltrami differential

$$\mu_{A_j}(z)=\sum_{\langle A_j\rangle\backslash\Gamma}\hat{\mu}_{g^{-1}(A_j)}(z)$$
$$=\sum_{\langle A_j\rangle\backslash\Gamma/\langle A_1\rangle}\sum_n\hat{\mu}_{g^{-1}(A_j)}(\lambda_1^n z),$$

and the result follows from Lemma 3.1 and the definition of $c(A_j,A_1)$ and $s(A_j,A_1)$ in (2.6) and (2.7). The second sum is over all g in the double cosets modulo $\langle A_j\rangle$ on the left and $\langle A_1\rangle$ on the right.

(ii) Let h be a Moebius transformation that takes the imaginary axis to the axis of A_k, namely

$$h(w)=\frac{-p_k w+q_k}{-w+1}=z.$$

Also if $\hat{\mu}_{\varepsilon,A_j}$ is the A_j invariant differential as in (2.3), then denote by $F(z)$ the solution of

$$\frac{\partial F}{\partial\bar{z}}=\sum_n\hat{\mu}_{\varepsilon,A_j}(A_k^n)\bar{A}_k^{n'}/A_k^{n'}$$

vanishing at 0, 1 and $o(|z|^2)$ at ∞. Then

$$\frac{\partial(F\circ h/h')}{\partial\bar{w}}=\sum_n\hat{\mu}_{\varepsilon,h^{-1}A_j}(\lambda_k^n w).$$

To obtain the solution of this last equation vanishing at 0, 1 and $o(|w|^2)$ at ∞, we consider

$$\hat{F}(w)=F(h(w))/h'(w)-(a+bw+cw^2)$$

with

$$a = \frac{F(h(0))}{h'(0)} = \frac{F(q_k)}{q_k - p_k},$$
$$c = \frac{F(h(\infty))}{h'(\infty)} = \frac{F(p_k)}{q_k - p_k},$$

and b an appropriate constant.

From Lemma 3.1 we have

$$\hat{F}(w) - \frac{\hat{F}(\lambda_k w)}{\lambda_k} = w\Omega$$

with Ω the geometric constant in terms of the imaginary axis and $h^{-1}A_j$; this equals the constant in terms of axis of A_k, A_j.

Hence,

$$\begin{aligned} w\Omega = {} & F(z)(h^{-1})'(z) - F(A_k(z))/A_k'(z)(h^{-1})'(z) \\ & - (a + bw + cw^2) + \frac{a + b\lambda_k w + cw_k^2\lambda_k^2}{\lambda_k}, \end{aligned}$$

and therefore

$$\begin{aligned} (z - p_k)(z - q_k)\Omega = {} & F(z) - F(A_k(z))/A_k'(z) \\ & - \frac{\lambda_k - 1}{(q_k - p_k)^2}\left(\frac{1}{\lambda_k}F(q_k)(z - p_k)^2 - F(p_k)(z - q_k)^2\right). \end{aligned}$$

We may replace z by p_k in this last identity to obtain

$$F(p_k)\left(1 - \frac{1}{A_k'(p_n)}\right) = 0.$$

But since A_k is hyperbolic, $A_k'(p_k) \neq 1$, and it follows that $F(p_k) = 0$. Likewise, $F(q_k) = 0$, and the formula is proven. □

PROPOSITION 3.2. *Let $p(A_j)$ be the cocycle in $H^1(\Gamma, \Pi_2)$ as defined before. Then*

(iii) $p(A_1)(A_1) = i(z/2)s(A_1, A_1)$;

(iv) *For* $j > 1$, $p(A_j)(A_j) = (i/2)(z - p_j)(z - q_j)s(A_j, A_j)$.

Proof. Similar to that of Proposition 3.1. □

COROLLARY 3.3. *Set* $c_{j_k} = c(A_j, A_k)$, $s_{j_k} = s(A_j, A_k)$, *and* $C = (c_{j_k})$, $S = (s_{j_k})$. *Then*

(i) $t(A_j)(A_1) = c_{j_1}z$, $t(iA_j)(A_1) = -s_{j_1}z$;

(ii) *For* $p > 1$, $t(A_j)(A_k) = c_{j_k}(z - p_k)(z - q_k)$, $t(iA_j)(A_k) = -s_{j_k}(z - p_k) \times (z - q_k)$.

Proof.

(i) Follows from $t(A_j)(A_1) = p(A_j)(A_1) + \overline{p(A_j)(A_1)}$ and similarly for the other identities.

□

PROPOSITION 3.4. *Let* (A_j), $1 \leq j \leq 6g-6+2m$, *be hyperbolic elements in* Γ *such that the skew-symmetric matrix* C *is invertible.*

Then $(\partial/\partial L_j)$ *is a basis of the tangent space of* $T(\Gamma)$, *and there exists a neighborhood where* (L_j) *is a coordinate chart.*

Proof. All indices run from 1 to $6g-6+2m$.

We first prove that $(\partial/\partial t_j)$ is a basis. Indeed, if

$$\sum a_j\, \partial/\partial t_j = 0,$$

then we may apply this to L_k so that (see (2.5))

$$\sum a_j c_{j_k} = 0,$$

and therefore $a_j = 0$ for all j.

Since C is the matrix that changes $(\partial/\partial t_j)$ to $(\partial/\partial L_j)$, we obtain the first conclusion. We now refer to constructions of a coordinate chart following Gardiner [2, Chapter 8.3], or Wolpert [7, Theorem 3.4]. The lengths L_j^* of A_j^* are given in such a way that it is clear that every element of Γ is expressible in terms of them, so that (L_j^*) is indeed a (local) coordinate chart. In particular, the mapping $\varphi : (L_j^*) \to (L_j)$ is C^∞ (even real analytic). Since in the tangent space to the Teichmüller space bases correspond to bases under $d\varphi$, it follows that φ is a diffeomorphism in a whole neighborhood. □

THEOREM 3.5. *Let* (A_j), $1 \leq j \leq 6g-6+2m$, *be hyperbolic elements in* Γ *such that the matrix* C *is invertible. Then the complex structure in the tangent space is given in terms of the basis* $(\partial/\partial t_j)$ *by the matrix*

$$R = C^{-1}S$$

and in terms of the basis $(\partial/\partial L_j)$ *by*

$$\hat{R} = SC^{-1}.$$

Proof. As before, let $t(A_j) = t(\mu(A_j))$ be the Fenchel–Nielsen tangent deformation in $H^1(\Gamma, \Pi_2)$. The complex structure is defined by multiplication of the Beltrami differentials by i, so that

$$t(i\mu(A_j)) = r(t(A_j)) = \sum_l r_{lj} t(A_l).$$

Then, for $k > 1$,

$$\begin{aligned} t(i\mu(A_j))(A_k) &= -s_{j_k}(z-p_k)(z-q_k) \\ &= \sum_l r_{lj} t(A_l)(A_k) \\ &= \sum_l r_{lj} c_{lk}(z-p_k)(z-q_k) \end{aligned}$$

$$= -\left(\sum_l c_{kl} r_{lj}\right)(z - p_k)(z - q_k).$$

For $k = 1$, the computation is similar. Thus, $S = CR$ or $R = C^{-1}S$.

Since the change for basis from $(\partial/\partial t_j)$ to $(\partial/\partial l_j)$ is given by C, we have

$$\hat{R} = CRC^{-1}, \qquad \hat{R} = SC^{-1}.$$

Finally, in order to see this result in different perspective, we recall Hermann Weyl's interpretation of the Riemann matrix of a compact Riemann surface. Let $(\alpha_1, \ldots, \alpha_{2g})$ be a basis of $H_1(M, \mathbb{Z})$ with intersection product $c_{ij} = -\alpha_i \cdot \alpha_j$.

A basis over $\mathbb{R}$ of the space of analytic differentials $(dw_1, \ldots, dw_{2g})$ in $H^{1,0}(M, \mathbb{C})$ is said to be dual to the basis of curves if

$$\operatorname{Re} \int_{\alpha_j} dw_i = c_{ij}.$$

In this setting the Riemann relations imply that the matrix

$$\int_{\alpha_j} dw_i = s_{ij}$$

is positive definite and symmetric. Multiplication by i in the vector space of differentials is represented in terms of basis dw_i by a square matrix R with $R^2 = -Id$.

We then have the fundamental relation $R = C^{-1}S$ together with $C' = -C$ and $S' = S$. Knowledge of R is equivalent, under an appropriate linear change of coordinates, to the Riemann matrix of the surface. □

In view of Theorem 3.5, we might therefore wonder to what extent does the tangent space at a point in the Teichmüller space play a similar role as the Jacobi variety; does this matrix R determine the analytic type of the surface X?

NOTE. As suggested by the referee, the answer to this question is "yes". For the Teichmüller space, knowing the matrix R is equivalent to knowing the almost complex structure. And since by the Bers embedding the almost complex structure is integrable, the matrix R determines the complex analytic structure.

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