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Monodromy Groups of Lagrangian Tori in $\mathbb{R}^4$

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1. Introduction

In this paper we work in the standard symplectic 4-space $\left(\mathbb{R}^4,\ \omega = \sum_{j=1}^{2} dx_j \wedge dy_j\right)$ unless otherwise mentioned. Let $L \overset{\iota}{\hookrightarrow} (\mathbb{R}^4, \omega)$ be an embedded Lagrangian torus with respect to the standard symplectic 2-form ω. The Lagrangian condition means that the pull-back 2-form $\iota^*\omega = 0 \in \Omega^2(L)$ vanishes on L. Gromov [7] proved that L is not *exact*—that is, the pull-back 1-form $\iota^*\lambda$ of a primitive λ of $\omega = d\lambda$ represents a nontrivial class in the cohomology group $H^1(L, \mathbb{R})$.

Let $\mathrm{Diff}_0^c(\mathbb{R}^4)$ denote the group of orientation-preserving diffeomorphisms with compact support on $\mathbb{R}^4$ that are isotopic to the identity map. We are interested in studying various types of self-isotopies of L. It is well known that to a smooth isotopy L_s, $s \in [0,1]$, between two embedded tori L_0, L_1 we may associate a family of maps $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ with $\phi_0 = \mathrm{id}$ such that $\phi_s(L) = L_s$. We will make no distinction between L_s and the associated maps ϕ_s from now on.

A path $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ with $0 \le s \le 1$ and $\phi_0 = \mathrm{id}$ associates to a fixed torus L a family of tori $L_s : \phi_s(L)$ in $\mathbb{R}^4$. The family of maps $\phi_t \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ is called a *smooth self-isotopy* of L if $\phi_1(L) = L$. Moreover, if all L_s are Lagrangian with respect to ω (ω-Lagrangian) then ϕ_s is called a *Lagrangian self-isotopy* of L. This is equivalent to saying that L is $\phi_s^*\omega$-Lagrangian. Suppose in addition that the cohomology class of $\iota^*\phi_s^*\lambda$ is independent of s; then ϕ_s is called a *Hamiltonian self-isotopy* of L. Equivalently, ϕ_s is Hamiltonian if it is generated by a Hamiltonian vector field. Each self-isotopy ϕ_s of L associates to an isomorphism

$$(\phi_1)_*\colon H_1(L, \mathbb{Z}) \to H_1(L, \mathbb{Z}),$$

which is called a *smooth* (resp., *Lagrangian, Hamiltonian*) monodromy of L if ϕ_t is smooth (resp., Lagrangian, Hamiltonian). The group of all smooth monodromies of L is called the *smooth monodromy group* (SMG) of L and is denoted by $\mathcal{S}(L)$. Likewise, $\mathcal{L}(L)$ and $\mathcal{H}(L)$ denote, respectively, the *Lagrangian monodromy group* (LMG) and the *Hamiltonian monodromy group* (HMG) of L. It is easy to see that $\mathcal{H}(L) \subset \mathcal{L}(L) \subset \mathcal{S}(L)$. Although here we focus only on Lagrangian 2-tori, the groups $\mathcal{H}(L)$, $\mathcal{L}(L)$, and $\mathcal{S}(L)$ are defined for any embedded Lagrangian submanifold L of any dimension.

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The interest in such monodromy groups is to study the Lagrangian knot problem [6] from a different perspective. If L and L' are smoothly isotopic, then clearly their smooth monodromy groups are isomorphic. Similar conclusions hold for the Lagrangian and the Hamiltonian cases as well. In [17] we studied $\mathcal{H}(L)$ for L either a monotone Clifford torus or a Chekanov torus. The latter was constructed (and called a *special torus*) by Chekanov in [3]. We proved that these two tori are distinguished by their spectrums associated to their Hamiltonian monodromy groups [17]. Another result concerning $\mathcal{H}(L)$ was obtained by Hu, Lalonde, and Leclercq in their preprint [8], where it was proved that the Hamiltonian monodromy group $\mathcal{H}(L)$ is trivial for any weakly exact Lagrangian submanifold L of a symplectic manifold. In this paper we focus instead on $\mathcal{L}(L)$ and $\mathcal{S}(L)$.

Recall from [13] that the Maslov class $\mu = \mu_L \in H^1(L,\mathbb{Z})$ of a Lagrangian torus $L \subset \mathbb{R}^4$ is nonzero with divisibility 2. Clearly, an element $h \in \mathcal{L}(L)$ must satisfy $\mu \circ h = \mu$. Note that, in general symplectic manifolds, $h \in \mathcal{L}(L)$ must also preserve the *linking class* $\ell_L \in H^1(L,\mathbb{Z})$ (see [5] and Section 2) whenever defined. However, since $\ell_L = 0$ for any embedded $L \subset \mathbb{R}^4$ [5], this requirement imposes no further restriction on $\mathcal{L}(L)$. Let G_μ denote the formal subgroup of all group isomorphisms $g\colon H_1(L,\mathbb{Z}) \to H_1(L,\mathbb{Z})$ such that $\mu \circ g = \mu$. Clearly $\mathcal{L}(L)$ is a subgroup of G_μ. Our first result is the following theorem.

THEOREM 1.1. *Assume that T is a Clifford torus. Then $\mathcal{L}(T) = G_\mu$.*

The group G_μ is freely generated by two generalized reflections f_0, f_1 (see (2)–(4) in Section 4) with $f_i(\gamma_0) = -\gamma_0$, where $\gamma_0 \in H_1(T,\mathbb{Z})$ is a primitive class with $\mu_T(\gamma_0) = 0$. Therefore, G_μ is isomorphic to the infinite dihedral group D_∞ [9].

For the smooth counterpart, our next theorem is due to the vanishing of ℓ_L.

THEOREM 1.2. *Let $L_s = \phi_s(L_0)$ for $0 \le s \le 1$ and $\phi_0 = \mathrm{id}$ be a smooth isotopy between two Lagrangian tori $L_0, L_1 \subset \mathbb{R}^4$. Then, for any $\gamma \in H_1(L_0,\mathbb{Z})$,*

$$\mu(\phi_{1*}(\gamma)) - \mu(\gamma) \in 4\mathbb{Z};$$

in other words,

$$\phi_1^*\mu - \mu \in H^1(L_0,\mathbb{Z}) \text{ has divisibility 4.}$$

Thus $\mathcal{S}(L)$ is a subgroup of

$$\mathcal{X} = \mathcal{X}_L := \{g \in \mathrm{Isom}(H_1(L,\mathbb{Z})) \mid \mu_L \circ g - \mu_L \in 4 \cdot H^1(L,\mathbb{Z})\}.$$

We determine $\mathcal{S}(L)$ for the case of a Clifford torus as follows.

THEOREM 1.3. *If T is a Clifford torus, then $\mathcal{S}(T) = \mathcal{X}_T$. In particular, $\mathcal{S}(T)$ is generated by $\mathcal{L}(T)$ and a reflection along a class $\gamma \in H_1(T,\mathbb{Z})$ with $\mu_T(\gamma) = 2$.*

It turns out that any smooth isotopy between a Lagrangian torus and a Clifford torus can be modified at either end by a self-isotopy to match the Maslov classes at both ends. We have the following result.

PROPOSITION 1.4. *Let $L \subset \mathbb{R}^4$ be an embedded Lagrangian torus smoothly isotopic to a Clifford torus T. Then there exists a smooth isotopy $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$,*

$s \in [0,1]$, *with* $\phi_0 = \mathrm{id}$ *and* $\phi_1(T) = L$, *where* ϕ_1 *preserves the corresponding Maslov classes*; *that is,*

$$\phi_1^* \mu_L = \mu_T.$$

Moreover, one can modify ϕ_s *so that* $\phi_s(T \setminus D)$ *is Lagrangian for* $s \in [0,1]$, *where* $D \subset T$ *is an embedded disc.*

However, at the present stage we do not know how to improve $\phi_s(T)$ to a genuine Lagrangian isotopy between T and L. To achieve that goal, it seems necessary (and perhaps enough) to have a better understanding of the isotopy of Lagrangian discs with prescribed boundary conditions.

We remark that Mohnke [12] showed that all embedded Lagrangian tori in $\mathbb{R}^4$ are smoothly isotopic to a Clifford torus. Also, Ivrii [10] showed that any embedded Lagrangian torus in $\mathbb{R}^4$ is Lagrangian isotopic to a Clifford torus. Both authors used pseudoholomorphic curve techniques [7] and methods of symplectic field theory [1; 4].

The rest of the paper is organized as follows. In Section 2 we review necessary background on the Maslov class and the linking class. In Section 3 we discuss framings of the symplectic normal bundle of a loop in $\mathbb{R}^4$ and also the change of framings under diffeomorphisms. Theorem 1.1 is proved in Section 4. Theorem 1.2 is proved in the beginning of Section 5; this is followed by the proof of Theorem 1.3, which consists of Propositions 5.3–5.4. Proposition 1.4 is proved in Section 6. We will use the convention $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ throughout the paper.

2. Maslov Class and Linking Class

Because we are concerned with monodromies of self-isotopies of a Lagrangian torus, we should first discuss two relevant classes in $H^1(L,\mathbb{Z})$: the *Maslov class* $\mu = \mu_L$ (see [11] for more details) and the *linking class* $\ell = \ell_L$. The latter is defined (and denoted by σ) in [5].

Maslov Class

The Maslov class μ is defined as follows. Given $\gamma \in H_1(L,\mathbb{Z})$, let $C \subset L$ be an immersed curve representing γ. Then the tangent bundle $T_C L$ over C is a closed path of Lagrangian planes and hence a cycle in the Grassmannian of Lagrangian planes in the symplectic vector space $\mathbb{R}^4$. In that case, $\mu(\gamma)$ is defined to be the *Maslov index* of the cycle $T_C L$.

Theorem 2.1 [13]. *The Maslov class* μ *of a Lagrangian torus* $L \subset \mathbb{R}^4$ *is nontrivial and is of divisibility 2.*

Example 2.2. Consider a Clifford torus

$$T = T_{a,b} := \{(ae^{it_1}, be^{it_2}) \in \mathbb{C}^2 \mid t_1, t_2 \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}\}.$$

Let $\gamma_1 \in H_1(T,\mathbb{Z})$ be the class represented by the curve $\{(ae^{it_1}, b) \in \mathbb{C}^2 \mid t_1 \in \mathbb{R}/2\pi\mathbb{Z}\}$ and let $\gamma_2 \in H_1(T,\mathbb{Z})$ be the class represented by $\{(a, be^{it_2}) \in \mathbb{C}^2 \mid t_2 \in \mathbb{R}/2\pi\mathbb{Z}\}$. Then $\mu_T(\gamma_1) = 2 = \mu_T(\gamma_2)$.

The inequality $\mu_L \neq 0$ implies that the Lagrangian monodromy group $\mathcal{L}(L)$ can be only a proper subgroup of $\mathrm{Isom}(H_1(L,\mathbb{Z})) \cong \mathrm{GL}(2,\mathbb{Z})$.

Linking Class

The linking class $\ell = \ell_L \in H^1(L,\mathbb{Z})$ is defined as follows. Take v to be any nonvanishing vector field on L that is *homotopically trivial*; in other words, v is homotopic to some v' in the space of nonvanishing vector fields on L such that v' generates the kernel of a nonvanishing closed 1-form on L. Let J be an ω-compatible almost complex structure on $\mathbb{R}^4$. Then $\ell(\gamma) := lk(C + \varepsilon Jv, L)$ is defined to be the linking number with L of the push-off of C in the direction of Jv, where $C \subset L$ is an immersed curve representing the class γ.

The class ℓ is independent of the choices involved. That $\ell(\gamma)$ is independent of J can be seen as follows. First of all, the space of ω-compatible almost complex structures is contractible and, since L is Lagrangian, Jv is transversal to L for any ω-compatible J. So in particular we can take J to be J_0, the standard complex structure on $\mathbb{R}^4$. Second, the independence of v follows from the observation that vector fields generating the kernels of nonvanishing closed 1-forms on L are homotopic as nowhere vanishing vector fields. Finally, if C and C' are two representatives of γ then, since $H_1(L)$ is abelian, C and C' are free homotopic. Hence $\ell(\gamma)$ is independent of the choices of v, J, and C with the prescribed conditions.

EXAMPLE 2.3. Let $C \subset L$ be an embedded closed curve representing a nontrivial class $\gamma \in H_1(L,\mathbb{Z})$. Parameterize C by $t \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ so that its tangent vector field $\dot{C}(t)$ is nonvanishing. Then $\dot{C}(t)$ extends to a homotopically trivial vector field v on L. For example, we can view L as an S^1-bundle over S^1 with fibers representing the class $[C] \in H_1(L,\mathbb{Z})$, and C is one of the fibers. Then take v to be a nonvanishing vector field tangent to the fibers.

THEOREM 2.4 [5]. *The linking class $\ell_L = 0$ for any embedded Lagrangian torus $L \subset \mathbb{R}^4$.*

REMARK 2.5. Given an embedded torus $L \subset \mathbb{R}^4$, we consider the set $\mathcal{J}^+(L)$ of almost complex structures J defined on $T_L\mathbb{R}^4$ such that $J(TL) \pitchfork TL$ and J is compatible with the orientation of $\mathbb{R}^4$. The homotopic class of such a J is isomorphic to $H^1(L,\mathbb{Z}) \cong \mathbb{Z}^2$. Similarly to ℓ_L for L being Lagrangian, each J associates to a linking class $\ell_L(J) \in H^1(L,\mathbb{Z})$ defined by linking numbers $\ell_L(J)(\gamma) := lk(C + \varepsilon Jv, L)$, where C and v are as defined previously.

Then $\ell_L(J_0) = \ell_L = 0$ if L is Lagrangian and J_0 is the standard complex structure (or any ω-compatible one). It will be shown later that the vanishing of ℓ implies, for ϕ_s as in Theorem 1.2, that $((\phi_1)_* J_0)|_{L_1}$ and $J_0|_{L_1}$ are homotopic in $\mathcal{J}^+(L_1)$. Hence for any embedded oriented closed curve $C \subset L_0$ we have $(\phi_1)_* N^\omega_C = N^\omega_{\phi_1(C)}$ up to a smooth isotopy rel L_1, which leads to Theorem 1.2. Here N^ω_C is the symplectic normal bundle as defined in the beginning of the next section.

3. Loops in $\mathbb{R}^4$ and Their Framings

Before moving on to Lagrangian tori in $\mathbb{R}^4$, it helps to have a closer look at loops in $\mathbb{R}^4$.

A loop in $\mathbb{R}^4$ is an embedded 1-dimensional submanifold diffeomorphic to S^1. The pull-back of ω on a loop vanishes, so a loop is an isotropic submanifold. Take a loop $C \subset \mathbb{R}^4$. We fix an orientation of C, fix a trivialization of $C \cong S^1 = \mathbb{R}/2\pi\mathbb{Z}$, and write $\dot{C}(t)$ for the tangent vector of C at $C(t)$.

Symplectic Normal Bundle

Let us recall some basic properties of the normal bundle N of C. The bundle N splits as

$$N = (T^*C) \oplus N^\omega,$$

where N^ω, called the *symplectic normal* bundle of C, is the trivial $\mathbb{R}^2$-bundle over C defined by

$$N^\omega := \{(C(t), v) \mid t \in S^1,\ v \in N|_{C(t)},\ \omega(\dot{C}(t), v) = 0\}.$$

By Weinstein's isotropic neighborhood theorem (see [11; 15; 16]), there exists a tubular neighborhood $U \subset \mathbb{R}^4$ of C, a tubular neighborhood $V \subset N$ of the zero section of the normal bundle $C \subset \mathbb{R}^4$, and a symplectomorphism with $C \subset U$ identified with the zero section of N:

$$(U \subset \mathbb{R}^4, \omega) \to (V \subset N = T^*C \times \mathbb{R}^2,\ \omega_C \times \omega_{\text{can}}).$$

Here $\omega_{\text{can}} = dx \wedge dy$ is the standard symplectic 2-form on $\mathbb{R}^2$, $\omega_C = dt \wedge dt^*$ is the canonical symplectic 2-form on T^*C, and t^* is the fiber coordinate of T^*C dual to t. The symplectic normal bundle N^ω is identified with $\{(t, 0, x, y) \in S^1 \times \mathbb{R} \times \mathbb{R}^2\}$.

Next we explore some properties of N^ω that will be applied in later sections.

Lagrangian Tori Associated to a Loop. Let

$$D^\omega \subset N^\omega$$

denote the associated symplectic normal disc bundle with fiber an open disc $\{(x, y) \in \mathbb{R}^2, x^2 + y^2 < \varepsilon\}$ with some positive radius ε. With the symplectomorphism near C described as before, the boundary $L = L_C := \partial D^\omega$ is an embedded Lagrangian torus in $\mathbb{R}^4$ provided that $\varepsilon > 0$ is small enough. Note that, for each sufficiently small ε, L_C with $\varepsilon > 0$ fixed is unique up to a Hamiltonian isotopy.

It is well known that any two loops in $\mathbb{R}^4$ are smoothly isotopic. The following proposition can be easily verified.

Proposition 3.1. *Let C_s, $0 \le s \le 1$, be a smooth isotopy of loops. Let D_s^ω denote the symplectic normal disc bundle of C_s with fiber radius $\varepsilon_s > 0$, and let $L_s := \partial D_s^\omega$. Then there exists an $\varepsilon > 0$ such that L_s is a Lagrangian isotopy of* *embedded Lagrangian tori provided that* $0 < \varepsilon_s < \varepsilon$. *In particular, if* $C_0 = C_1$ *as a set and* $\varepsilon_0 = \varepsilon_1$, *then* $D_0^\omega = D_1^\omega$ *and we get a Lagrangian self-isotopy of* $L_0 = \partial D_0^\omega$.

In Section 4 we will use this observation to construct Lagrangian self-isotopies of a Clifford torus.

Framings of N^ω

DEFINITION 3.2. To a nonvanishing section (i.e., a framing) σ of N^ω one can associate an S^1-family of Lagrangian planes

$$\dot{C}(t) \wedge \sigma(t), \quad t \in S^1;$$

we denote the corresponding Maslov index by

$$\mu_C(\sigma) := \mu(\dot{C}(t) \wedge \sigma(t)) \in 2\mathbb{Z}.$$

Note that $\mu_C(\sigma)$ depends only on the orientation of C and the homotopy class of σ among framings of N^ω.

If we fix a trivialization $\Phi\colon N^\omega \to C \times \mathbb{R}^2 = C \times \mathbb{C}^1$, then the homotopy classes of framings of N^ω can be identified with $[S^1, \mathbb{R}^2 \setminus \{0\}] = [S^1, S^1] = \mathbb{Z}$. Hence, for a map $\theta\colon S^1 \to S^1$ of degree m, the Maslov index associated to the section $\sigma'(t) := e^{i\theta(t)}\sigma(t)$ is $\mu_C(\sigma') = \mu_C(\sigma) + 2m$. In particular, there is a framing σ^0 of N^ω such that $\mu_C(\sigma^0) = 0$. We call σ^0 a *0-framing* of C, and it is unique up to homotopy. Likewise, for each $m \in \mathbb{Z}$ there is a framing σ^m of N^ω, with σ^m unique up to homotopy, such that $\mu_C(\sigma^m) = 2m$.

DEFINITION 3.3. We call σ^m an *m-framing* of N^ω or an *m-framing* of C.

The homotopy classes of framings of N^ω are classified by the framing number $\mu_C(\sigma)/2$.

EXAMPLE 3.4. Let $C \subset L$ be a simple closed curve representing the class $\gamma \in H_1(L, \mathbb{Z})$ of a Lagrangian torus. Let v be a nonvanishing section of $N_C^\omega \cap T_C L$. Then v is a $(\mu(\gamma)/2)$-framing of N_C^ω.

PROPOSITION 3.5. *Let* C_s, $s \in [0,1]$, *be a smooth isotopy between loops* C_0 *and* C_1. *Write* $C_s = \phi_s(C_0)$, *where* $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ *with* $\phi_0 = \mathrm{id}$. *Let* N_s^ω *and* σ_s^m *denote, respectively, the symplectic normal bundle and the m-framing of* C_s.

(i) *If* $(\phi_1)_* N_0^\omega = N_1^\omega$, *then*

$$\mu_{C_1}((\phi_1)_*\sigma_0^m) - \mu_{C_1}(\sigma_1^m) = \mu_{C_1}((\phi_1)_*\sigma_0^0) - \mu_{C_1}(\sigma_1^0) \in 4\mathbb{Z}.$$

(ii) *If* $\mu_{C_1}((\phi_1)_*\sigma_0^m) = \mu_{C_1}(\sigma_1^m) = 2m$ *then, up to a perturbation of* ϕ_s, *we may assume that* $(\phi_s)_* N_0^\omega = N_s^\omega$ *and* $(\phi_s)_*\sigma_0^m = \sigma_s^m$.

Proof. (i) First consider the case $m = 0$. Fix a trivialization $S^1 \cong \mathbb{R}/2\pi\mathbb{Z} \to C_0$ for C_0. This trivialization, when composed with ϕ_s, becomes a trivialization of C_s. By applying Weinstein's isotropic neighborhood theorem, we may symplectically identify a neighborhood of $C_s \in \mathbb{R}^4$ with a neighborhood of the zero section of the normal bundle N_s of C_s. We can trivialize $N_s = S^1 \times \mathbb{R} \times \mathbb{R}^2$ with coordinates (t, t^*, x, y) so that

- $C_s = S^1 \times \{0\} \times \{0\}$,
- $N_s^\omega = S^1 \times \{0\} \times \mathbb{R}^2$, and
- $\sigma_s^0(t) = (t, 0, \varepsilon, 0)$ for some $\varepsilon > 0$.

Then, for each s, the differential $(\phi_s)_*(t)$ at $C_0(t)$ with $t \in S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ can be viewed as a smooth loop in $\mathrm{GL}^+(4, \mathbb{R})$:

$$(\phi_s)_*(t) \in \begin{pmatrix} 1 & * \\ 0 & \mathrm{GL}^+(3,\mathbb{R}) \end{pmatrix}, \quad (\phi_0)_*(t) = \mathrm{id}, \quad (\phi_1)_*(t) \in \begin{pmatrix} 1 & * & 0 \\ 0 & c(t) & 0 \\ 0 & * & \mathrm{GL}(2,\mathbb{R}) \end{pmatrix}.$$

Note that, since $(\phi_1)_*(t)$ is an isomorphism, it follows that $c(t) \neq 0$ for $t \in S^1$. We view $(\phi_0)_*(t) = \mathrm{id}$ as a constant loop in $\mathrm{GL}^+(4, \mathbb{R})$ parameterized by t. Then $(\phi_s)_*(t), 0 \leq s \leq 1$, when viewed as a family of parameterized loops in $\mathrm{GL}^+(4, \mathbb{R})$, is a free homotopy between $(\phi_0)_*(t)$ and $(\phi_1)_*(t)$. This implies that $(\phi_1)_*(t)$ is free homotopic to the trivial class of

$$\pi_1(\mathrm{GL}^+(4, \mathbb{R})) \cong \pi_1(\mathrm{GL}^+(3, \mathbb{R})) = \mathbb{Z}_2.$$

The lower 3×3 block of the matrix form of $(\phi_s)_*(t)$ is invertible. We can therefore perturb ϕ_s by composing it with some suitable family of maps in $\mathrm{Diff}_0^c(\mathbb{R}^4)$, each of them fixing C_s pointwise and with the condition $(\phi_1)_* N_0^\omega = N_1^\omega$ preserved under the perturbation, so that the perturbed ϕ_s satisfy

$$(\phi_s)_*(t) \in \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{GL}^+(3,\mathbb{R}) \end{pmatrix} \quad \text{with} \quad (\phi_0)_*(t) = \mathrm{Id}$$

and either $(\phi_1)_*(t) = A(t)$ or $(\phi_1)_*(t) = A'(t)$, where

$$A(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos kt & -\sin kt \\ 0 & 0 & \sin kt & \cos kt \end{pmatrix}, \quad A'(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \cos kt & \sin kt \\ 0 & 0 & \sin kt & -\cos kt \end{pmatrix} \tag{1}$$

for some $k \in \mathbb{Z}$. Note that $A'(t)$ is free homotopic to $A(t)$ by a 180° rotation along the subspace spanned by its second and third column vectors. We can interchange the two cases $(\phi_1)_*(t) = A(t)$ and $(\phi_1)_*(t) = A'(t)$ by composing with ϕ_1 such a rotation along C_1.

Now the equality $[(\phi_1)_*(t)] = 0$ in $\pi_1(\mathrm{GL}^+(4, \mathbb{R}))$ implies that $k \in 2\mathbb{Z}$. Hence $\mu_{C_1}((\phi_1)_*\sigma_0^0) = 2k + \mu(\sigma_1^0) = 2k \in 4\mathbb{Z}$.

The equality $\mu_{C_1}((\phi_1)_*\sigma_0^m) - \mu_{C_1}(\sigma_1^m) = \mu_{C_1}((\phi_1)_*\sigma_0^0) - \mu_{C_1}(\sigma_1^0)$ follows from the property that $\sigma_s^m(t) = e^{imt}\sigma_s^0(t)$ up to homotopy.

(ii) The proof follows from the perturbation of ϕ_s constructed in (i). □

4. Lagrangian Monodromy Group of a Clifford Torus

In general, the LMG $\mathcal{L}(L)$ must preserve both the Maslov class μ_L and the linking class ℓ_L whenever defined. However, for $L \subset \mathbb{R}^4$ the class $\ell_L = 0$ is automatically preserved. In this section we determine the LMG of a Clifford torus in $\mathbb{R}^4$.

Identify $\mathbb{R}^4 \cong \mathbb{C}^2$. For $a, b > 0$, the *Clifford torus* $T_{a,b}$ is defined to be

$$T = T_{a,b} := \{(z_1, z_2) \mid |z_1| = a,\ |z_2| = b\}.$$

We fix a basis $\{\gamma_1, \gamma_2\}$ of $H_1(T, \mathbb{Z})$ such that

- γ_1 is represented by the cycle $\{(ae^{it}, b) \mid t \in \mathbb{R}/2\pi\mathbb{Z}\}$ and
- γ_2 is represented by the cycle $\{(a, be^{it}) \mid t \in \mathbb{R}/2\pi\mathbb{Z}\}$.

Then $\gamma_1 = \binom{1}{0}$ and $\gamma_2 = \binom{0}{1}$ when expressed as column vectors. We also denote $\gamma_0 := -\gamma_1 + \gamma_2$. Then $\mu(\gamma_0) = 0$ and $\gamma_0 = \binom{-1}{1}$ as a column vector. Likewise, the Maslov class $\mu \in H^1(T, \mathbb{Z})$ is expressed as a row vector $\mu = (2\ \ 2)$.

The mapping class group of T is then isomorphic to $\mathrm{GL}(2, \mathbb{Z})$, the group of 2×2 matrices with integral coefficients and with determinant ± 1. Let

$$G_\mu := \{g \in \mathrm{GL}(2, \mathbb{Z}) \mid \mu \circ g = \mu\}.$$

A direct computation shows that $G_\mu = G_\mu^+ \sqcup G_\mu^-$, where

$$G_\mu^+ = \left\{g_n := \begin{pmatrix} 1-n & -n \\ n & 1+n \end{pmatrix} \,\middle|\, n \in \mathbb{Z}\right\}, \tag{2}$$

$$G_\mu^- = \left\{f_n := \begin{pmatrix} 1-n & 2-n \\ n & -1+n \end{pmatrix} \,\middle|\, n \in \mathbb{Z}\right\}. \tag{3}$$

Elements of G_μ^+ are of determinant 1, and elements of G_μ^- are of determinant -1. Also, $g_n = (g_1)^n$ for g_1 a generator of $G_\mu^+ \cong \mathbb{Z}$. On the other hand, G_μ^- comprises elements of order 2 in G_μ. Geometrically, $g_n = (g_1)^n$ is the $(-n)$-Dehn twist along γ_0 and each f_n is a generalized reflection with $f_n(\gamma_0) = -\gamma_0$. Note that

$$f_0^2 = e = f_1^2, \quad (f_1 f_0)^n = g_n, \quad (f_0 f_1)^n = g_{-n} = (g_n)^{-1}, \quad g_n f_m = f_{n+m}$$

(here e denotes the identity element of G_μ). Therefore,

$$G_\mu = \langle f_0, f_1 \mid f_0^2 = e = f_1^2 \rangle \cong D_\infty \tag{4}$$

is freely generated by the two elements f_0, f_1 of order 2 and is isomorphic to the infinite dihedral group D_∞ [9].

Note that if $L_s = \phi_s(T)$, $s \in [0, 1]$, is a Lagrangian self-isotopy of T such that $L_0 = L_1 = T$ and $\phi_0 = \mathrm{id}$, then the induced isomorphism $(\phi_1)_* : H_1(T, \mathbb{Z}) \to H_1(T, \mathbb{Z})$ is an element of G_μ. That is, the LMG $\mathcal{L}(T)$ is a subgroup of G_μ.

PROPOSITION 4.1. *The LMGs of $T_{a,b}$ and $T_{a',b'}$ are isomorphic.*

Proof. Identify the ordered pairs (a, b) and (a', b') with the coordinates of two points in the first quadrant of the $\mathbb{R}^2$-plane. Take a smooth path $c(s) = (c_1(s), c_2(s))$, $s \in [0, 1]$, in the first quadrant so that $c(0) = (a, b)$ and $c(1) = (a', b')$. Then $T_{c(s)}$ is a Lagrangian isotopy of Clifford tori between $T_{a,b}$ and $T_{a',b'}$. $\square$

THEOREM 4.2. *The LMG of a Clifford torus T is $\mathcal{L}(T) = G_\mu$.*

Proof. We will explicitly construct Lagrangian self-isotopies of T with monodromies f_0 and f_1, respectively. Then $\mathcal{L}(T) = G_\mu$ by equation (4).

Case 1: The monodromy $f_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Recall that in [17] a Lagrangian self-isotopy for $T_{b,b}$ was constructed with monodromy f_1 (denoted by $\tilde{f}_1$ in [17]). For completeness we repeat the construction here. First let us consider the path in the unitary group $U(2)$ defined by

$$A_s := \begin{pmatrix} \cos\frac{\pi s}{2} & -\sin\frac{\pi s}{2} \\ \sin\frac{\pi s}{2} & \cos\frac{\pi s}{2} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}), \quad 0 \le s \le 1.$$

Here A_s acts on $\mathbb{C}^2$ and is the time-s map of the Hamiltonian vector field $X = \frac{\pi}{2}(x_1\partial_{x_2} - x_2\partial_{x_1} + y_1\partial_{y_2} - y_2\partial_{y_1})$ with $\omega(X,\cdot) = -dH$ for $H = \frac{\pi}{2}(x_2y_1 - x_1y_2)$. Observe that $A_1(T_{a,b}) = T_{b,a}$ and $(A_1)_* = f_1$ on $H_1(T_{b,b},\mathbb{Z})$. Fix $b > 0$ and modify H to get a C^∞-function $\tilde{H}$ with compact support such that $\tilde{H} = H$ on $\{|z_1| \le 2b, |z_2| \le 2b\}$. Let ϕ_s be the time-s map of the flow of the Hamiltonian vector field associated to $\tilde{H}$. Then $\phi_1(T_{b,b}) = (T_{b,b})$ and $(\phi_1)_* = (A_1)_* = f_1$ on $H_1(T_{b,b},\mathbb{Z})$. Now extend this self-isotopy of $T_{b,b}$ by conjugating it smoothly via a Lagrangian isotopy between $T_{a,b}$ and $T_{b,b}$ as described in Proposition 4.1. We may assume that the basis $\{\gamma_1,\gamma_2\}$ of $T_{b,b}$ is transported to the basis $\{\gamma_1,\gamma_2\}$ of $T_{a,b}$ along the latter isotopy. Readers can check now that the extended isotopy induces a Lagrangian self-isotopy of $T_{a,b}$ with monodromy f_1.

Case 2: The monodromy $f_0 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$. For $s \in [0,1]$ consider the family of diffeomorphisms $\Psi_s : \mathbb{R}^4 \to \mathbb{R}^4$,

$$\Psi_s(x_1,y_1,x_2,y_2) := (x_1\cos\pi s - y_2\sin\pi s, y_1, x_2, y_2\cos\pi s + x_1\sin\pi s).$$

Note that $\Psi_s \in \mathrm{SO}(4,\mathbb{R})$ are rotations on the (x_1y_2)-plane with the (y_1x_2)-plane fixed. Consider the simple closed curve C_0 defined by

$$\{(x_1 = 0,\ y_1 = 0,\ x_2 = b\cos t,\ y_2 = b\sin t) \in \mathbb{R}^4 \mid t \in [0,2\pi]\}.$$

Define $C_s(t) := \Psi_s(C_0)(t)$ for C_s, $s \in [0,1]$, a smooth family of curves. Note that C_1 equals C_0 but with the reversed orientation. Recall from Proposition 3.1 that for $\varepsilon > 0$ small enough, the Lagrangian torus boundary L_s of the symplectic, radius-ε normal disc bundle D_s^ω of C_s is embedded in $\mathbb{R}^4$ with core curve C_s. Note that $L_0 = T_{\varepsilon,b} = L_1$ as sets, so we obtain a Lagrangian self-isotopy of $T_{\varepsilon,b}$ for $\varepsilon > 0$ small enough. This self-isotopy of $T_{\varepsilon,b}$ reverses the orientation of $T_{\varepsilon,b}$, so the corresponding monodromy f is an element of G_μ^- with determinant -1 when expressed as a matrix. Note that Ψ_1 reverses the orientation of the core curve C_0 of D_0^ω. Since $\gamma_2 \subset \partial D_0^\omega = T_{\varepsilon,b}$ is longitudinal, this reversal implies that f sends γ_2 to $-\gamma_2 + m\gamma_1$ for some $m \in \mathbb{Z}$. Then a comparison with the formula for f_n in (3) yields $f = f_0 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & -1 \end{smallmatrix}\right)$ and $m = 2$.

Now, similarly to what was done in Case 1, extend the Lagrangian self-isotopy of $T_{\varepsilon,b}$ into an Lagrangian self-isotopy of $T_{a,b}$ through Clifford tori. The corresponding monodromy is f_0. This completes the proof. □

REMARK 4.3. If we take C_0 to be the curve

$$\{(x_1 = a\cos t,\ y_1 = a\sin t,\ x_2 = 0,\ y_2 = 0) \in \mathbb{R}^4 \mid t \in [0, 2\pi]\},$$

then Ψ_s will induce a Lagrangian self-isotopy of $T_{a,\varepsilon}$ with monodromy $f_2 = \left(\begin{smallmatrix} -1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$. The reader can check that $G_\mu = \langle f_1, f_2 \mid f_1^2 = e = f_2^2 \rangle$. Hence $\mathcal{L}(T) = G_\mu$ again.

5. Smooth Monodromy Group of a Clifford Torus

We start by proving Theorem 1.2.

Proof of Theorem 1.2. By the linearity of $(\phi_1)_*$ and μ, it is enough to prove the theorem for the case when $\gamma \in H_1(L_0, \mathbb{Z})$ is primitive.

Fix a positive basis $\{\gamma_1, \gamma_2\}$ for $H_1(L_1, \mathbb{Z})$ with $\mu(\gamma_1) = 2 = \mu(\gamma_2)$. Given a primitive class $\gamma \in H_1(L_0, \mathbb{Z})$, we have $(\phi_1)_*(\gamma) = n_1\gamma_1 + n_2\gamma_2$ for some $n_1, n_2 \in \mathbb{Z}$. Let $C_0 \subset L_0$ be an embedded curve representing the class γ, and let $C_s := \phi_s(C_0)$. We denote by N_s and N_s^ω (respectively) the normal bundle and the symplectic normal bundle of C_s. By assumption, C_1 represents the class $n_1\gamma_1 + n_2\gamma_2$.

Let σ_0 denote a nonvanishing section of the $\mathbb{R}^1$-bundle $(T_{C_0}L_0) \cap N_0^\omega$ over C_0. Then σ_0 is a $(\mu(\gamma)/2)$-framing of N_0^ω. Extend σ_0 to a smooth family σ_s with $0 \le s \le 1$, so that σ_s is a $(\mu(\gamma)/2)$-framing of N_s^ω. Let $m := \mu(\gamma)/2$.

Recall that J_0 is the standard complex structure over $\mathbb{R}^4 \cong \mathbb{C}^2$. Fix a trivialization for $N_s \cong S^1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ by taking $\{J_0\dot{C}_s(t), \sigma_s(t), J_0\sigma_s(t)\}$ as the basis of the fiber of N_s at $C_s(t)$, so that the coordinate (t, t^*, x, y) represents the fiber $t^*J_0\dot{C}_s(t) + x\sigma_s(t) + yJ_0\sigma_s(t)$.

Now let $\eta_s := \phi_s(\sigma_0)$. Observe that η_1 is a nonvanishing section of $N_1^\omega \cap T_{C_1}L_1$ and an $(n_1 + n_2)$-framing of N_1^ω. Let $k := n_1 + n_2$.

Recall that σ_1 is an m-framing of N_1^ω. Up to a homotopy of σ_s if necessary, we may assume the following:

- for each s, $\eta_s = \sigma_s$ at $t = 0$;
- for $t \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$, $\eta_1(t) = \sigma_1(t)\cos(k - m)t + J_0\sigma_1(t)\sin(k - m)t$.

Then, for each s, ϕ_s associates to a smooth map $\Phi_s \colon S^1 \to \mathrm{GL}^+(4, \mathbb{R})$, where

$$\Phi_s(t) := (\phi_s)_*(t) \in \begin{pmatrix} 1 & * \\ 0 & \mathrm{GL}^+(3, \mathbb{R}) \end{pmatrix},$$

$$\Phi_0(t) = \mathrm{id}, \qquad \Phi_1(t) = \begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 0 & * \\ 0 & * & \cos(k - m)t & * \\ 0 & * & \sin(k - m)t & * \end{pmatrix}.$$

The second and fourth columns of Φ_1 represent $(\phi_1)_*(J_0\dot{C}_0)$ and $(\phi_1)_*(J_0\sigma_0)$, respectively.

Extend $\dot{C}_0$ to a homotopically trivial nonvanishing vector field u_0 on L_0, and let $u_s := (\phi_s)_*u_0$. Then $u_1|_{C_1} = \dot{C}_1$. By continuity and $\ell_{L_0} = 0$ we have

$$lk(C_1 + \varepsilon \cdot (\phi_1)_*J_0u_0, L_1) = lk(C_0 + \varepsilon J_0u_0, L_0) = 0. \tag{5}$$

Similarly, since $\ell_{L_1} = 0$, it follows that

$$lk(C_1 + \varepsilon J_0(\phi_1)_* u_0, L_1) = lk(C_1 + \varepsilon J_0 u_1, L_1) = 0. \tag{6}$$

Note that (5) and (6) hold for any class $[C_0]$ and thus $[C_1] = (\phi_1)_*[C_0]$, which shows that $(\phi_1)_* J_0|_{L_1}$ is homotopic to $J_0|_{L_1}$ in $\mathcal{J}^+(L_1)$ as defined in Remark 2.5. In particular, $(\phi_1)_* J_0 u_0$ is homotopic to $J_0 u_1$ as nonvanishing sections of the normal bundle N_{L_1} of $L_1 \subset \mathbb{R}^4$. So up to an L_1-fixing isotopy we may assume that, along C_1, $(\phi_1)_* J_0 \dot{C}_0 = J_0 \dot{C}_1$ and $(\phi_1)_* N^{\omega}_{C_0} = N^{\omega}_{C_1}$. That is, $\Phi_1 = (\phi_1)_*$ satisfies

$$\Phi_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(k-m)t & * \\ 0 & 0 & \sin(k-m)t & * \end{pmatrix} \in \mathrm{GL}^+(4, \mathbb{R}). \tag{7}$$

Now Φ_1 satisfies the hypothesis of Proposition 3.5(i) and so, by a similar argument as employed there, up to an L_1-fixing isotopy we have

$$\Phi_1(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(k-m)t & -\sin(k-m)t \\ 0 & 0 & \sin(k-m)t & \cos(k-m)t \end{pmatrix} \in \mathrm{GL}^+(4, \mathbb{R})$$

with

$$k - m \in 2\mathbb{Z}, \tag{8}$$

since the lower 3×3 block of Φ_1 is free homotopic to $\mathrm{id} \in \mathrm{GL}^+(3, \mathbb{R})$ with respect to the basis $\{J_0 \dot{C}_1, \sigma_1, J_0 \sigma_1\}$. This completes the proof. □

Corollary 5.1. *The SMG $\mathcal{S}(L)$ of an embedded Lagrangian torus $L \subset \mathbb{R}^4$ is contained in the subgroup $\mathcal{X} \subset \mathrm{Isom}(H^1(L, \mathbb{Z}))$ defined by*

$$\mathcal{X} := \{ g \in \mathrm{Isom}(H_1(L, \mathbb{Z})) \mid \mu_L \circ g - \mu_L \in 4 \cdot H^1(L, \mathbb{Z}) \}.$$

Corollary 5.2. *Let $L \subset \mathbb{R}^4$ be an embedded Lagrangian torus. Fix a positive basis $\{\gamma_1, \gamma_2\}$ for $H_1(L, \mathbb{Z})$ with $\mu(\gamma_1) = 2 = \mu(\gamma_2)$. Then, with respect to $\{\gamma_1, \gamma_2\}$, $\mathcal{X}$ is represented as*

$$\mathcal{X} = \mathcal{X}^o \sqcup \mathcal{X}^e \subset \mathrm{GL}(2, \mathbb{Z}),$$

where

$$\mathcal{X}^o := \left\{ \begin{pmatrix} 1+2p & 2s \\ 2r & 1+2q \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z} \right\}, \tag{9}$$

$$\mathcal{X}^e := \left\{ \begin{pmatrix} 2r & 1+2q \\ 1+2p & 2s \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}) \mid p, q, r, s \in \mathbb{Z} \right\}. \tag{10}$$

Proof. Recall that $\mu = \mu_L$ has divisibility 2. Express γ_1 and γ_2 as column vectors $\binom{1}{0}$ and $\binom{0}{1}$, respectively. For $g = (g_{ij}) \in \mathcal{X}$, that $\mu(g(\gamma_j)) - \mu(\gamma_j) \in 4\mathbb{Z}$ implies that both $2(g_{11} + g_{21}) - 2$ and $2(g_{12} + g_{22}) - 2$ are divisible by 4. Hence (i) g_{11} and g_{21} have different parity and (ii) g_{12} and g_{22} have different parity. Since $\det g = \pm 1$, the two even-valued entries of g can lie in neither the same column nor the same row of g; hence either $g \in \mathcal{X}^o$ or $g \in \mathcal{X}^e$. □

We now determine the group $\mathcal{S}(T)$ of a Clifford torus T. The proof is divided into three separate propositions.

PROPOSITION 5.3. *Recall the basis $\{\gamma_1, \gamma_2\}$ for $H_1(T_{a,b}, \mathbb{Z})$. Each of the following four types of elements of* $\mathrm{GL}(2, \mathbb{Z}) \cong \mathrm{Isom}(H_1(T_{a,b}, \mathbb{Z}))$ *can be realized as the monodromy of some smooth self-isotopy of $T_{a,b}$:*

(i) *a k-Dehn twist $\tau_1^k := \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ along γ_1 with $k \in 2\mathbb{Z} \setminus \{0\}$;*

(ii) *a k-Dehn twist $\tau_2^k := \left(\begin{smallmatrix} 1 & 0 \\ -k & 1 \end{smallmatrix}\right)$ along γ_1 with $k \in 2\mathbb{Z} \setminus \{0\}$;*

(iii) *the γ_1-reflection $\bar{r}_1 := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$;*

(iv) *the γ_2-reflection $\bar{r}_2 := \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$.*

Proof. Because the specific values of $a, b > 0$ are immaterial, we may take values of a, b that are convenient for the construction of a smooth self-isotopy. In the following we will denote a Clifford torus as T. Also, since the Lagrangian monodromy $f_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ swaps elements in (i) and (iii) with elements in (ii) and (iv), we need only prove the two cases (i) and (iii).

Let $C := \{(0, be^{it}) \mid t \in [0, 2\pi]\} \subset \mathbb{R}^4$.

Case (i): τ_1^k $(k \neq 0)$ is even. Let U be a tubular neighborhood of C, $U \cong B^3 \times S^1$. Parameterize U by $(\rho, \varphi, \theta, t)$ for $(\rho, \varphi, \theta) \in [0, \rho_0] \times S^2$ the spherical coordinates of the 3-ball B^3, where ρ is the radial coordinate, (φ, θ) denotes the spherical coordinates on S^2, and $(\rho_0, \pi/2, \theta, t)$ parameterizes the equator of the S^2-fiber over t. We also assume that $(\rho_0, \pi/2, \theta, t) \in S^1 \times S^1$ parameterizes T so that τ_1^k is represented by the map $\phi(\theta, t) = (\theta + kt, t)$. Extend ϕ over U to obtain

$$\tilde{\phi} \colon U \to U, \quad \tilde{\phi}(\rho, \varphi, \theta, t) = (\rho, \psi_t(\varphi, \theta), t) := (\rho, (\varphi, \theta + kt), t).$$

As a loop in SO(3) parameterized by t, the maps ψ_t represent the trivial class of $\pi_1(\mathrm{SO}(3))$ since we assume that k is even. Then there exists between ψ_t and the constant loop id a smooth homotopy $\psi_{s,t} \in \mathrm{SO}(3)$ with $s, t \in [0, 1] \times S^1$ such that $\psi_{0,t} = \mathrm{Id} = \psi_{s,0}$ and $\psi_{1,t} = \psi_t$. This induces a smooth homotopy $\tilde{\phi}_s$, $s \in [0, 1]$, between $\tilde{\phi}_1 = \tilde{\phi}$ and $\tilde{\phi}_0 = \mathrm{id}_U$ with

$$\tilde{\phi}_s(\rho, (\varphi, \theta), t) := (\rho, \psi_{s,t}(\varphi, \theta), t).$$

Let X_s be the time-dependent vector field on U that generates the isotopy $\tilde{\phi}_s$; that is, $\frac{d\tilde{\phi}_s}{ds} = X_s \circ \tilde{\phi}_s$ and $\tilde{\phi}_0 = \mathrm{id}$. Note that X_s is tangent to ∂U. Extend X_s over $\mathbb{R}^4$ smoothly with compact support. Denote the time-1 map of the extended X_s as ϕ'. Then $\phi' \in \mathrm{Diff}_0^c(\mathbb{R}^4)$ is isotopic to the identity map and $\phi'|_L = \phi$.

Case (iii): $\bar{r}_1$. Parameterize B^3 by Cartesian coordinates (x_1, y_1, x_2) with $x_1^2 + y_1^2 + x_2^2 \leq 1$ so that $T \subset U = B^3 \times S^1$ is parameterized by $\{(x_1, y_1, 0, t) \mid x_1^2 + y_1^2 = 1\}$. Without loss of generality, we may assume that $\bar{r}_1$ is represented by the map $\phi(x_1, y_1, 0, t) = (-x_1, y_1, 0, t)$ for $(x_1, y_1, 0, t) \in T$. Extend ϕ over U to get

$$\tilde{\phi} \colon U \to U, \quad \tilde{\phi}(x_1, y_1, x_2, t) = (\psi(x_1, y_1, x_2), t) := ((-x_1, y_1, -x_2), t).$$

The map $\psi = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \mathrm{SO}(3)$ is isotopic to the identity map. Let ψ_s be a smooth path in SO(3) with $s \in [0,1]$, $\psi_0 = \mathrm{Id}$, and $\psi_1 = \psi$. This path induces an isotopy $\tilde{\phi}_s : U \to U$, $s \in [0,1]$:

$$\tilde{\phi}_s((x_1, y_1, x_2), t) = (\psi_s(x_1, y_1, x_2), t).$$

Now, just as in Case (i), we extend $\tilde{\phi}_s$ over $\mathbb{R}^4$ with compact support to obtain $\phi' \in \mathrm{Diff}_0^c(\mathbb{R}^4)$, which is isotopic to the identity map, and $\phi'|_L = \phi$. This completes the proof. □

Let

$$\mathcal{R} \subset \mathrm{GL}(2,\mathbb{Z})$$

be the subgroup generated by elements of $\mathcal{L}(T) = G_\mu$ and by τ_j^2 and $\bar{r}_j$ for $j = 1, 2$. Clearly we have the following inclusions as subgroups:

$$\mathcal{R} \subset \mathcal{S}(T) \subset \mathcal{X}.$$

We will show that $\mathcal{X} \subset \mathcal{R}$ and hence that $\mathcal{R} = \mathcal{S}(T) = \mathcal{X}$. To begin with, consider the subgroup $\mathcal{E} \subset \mathrm{GL}(2,\mathbb{Z})$ generated by τ_1^2 and τ_2^2. It is shown by Sanov [14] that $\mathcal{E}$ is free (see also [2]) and that

$$\mathcal{E} = \left\{\begin{pmatrix} 1+4p & 2s \\ 2r & 1+4q \end{pmatrix} \in \mathrm{GL}(2,\mathbb{Z}) \mid p, q, r, s \in \mathbb{Z}\right\}.$$

Proposition 5.4. *The group $\mathcal{X}$ is contained in $\mathcal{R}$, so $\mathcal{R} = \mathcal{S}(T) = \mathcal{X}$.*

Proof. Since $\mathcal{X}^e = f_1 \mathcal{X}^o$ and $f_1 \in \mathcal{R}$, it suffices to prove that if $h \in \mathcal{X}^o$ then $h \in \mathcal{R}$. Our strategy here is to show that for $h \in \mathcal{X}^o$ there exists a suitable element $g \in \mathcal{R}$ such that $gh \in \mathcal{E}$. Then $h = g^{-1}(gh) \in \mathcal{R}$.

Write $h = \begin{pmatrix} 1+2p & 2s \\ 2r & 1+2q \end{pmatrix}$. We divide the proof into four cases according to the parity of p and q.

(i) If both p and q are even, then we already have $h \in \mathcal{E} \subset \mathcal{R}$.

(ii) If both p and q are odd, then

$$\begin{aligned}(\bar{r}_1\bar{r}_2)h &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1+2p & 2s \\ 2r & 1+2q \end{pmatrix} \\ &= \begin{pmatrix} 1-2(1+p) & -2s \\ -2r & 1-2(1+q) \end{pmatrix} \in \mathcal{E}.\end{aligned}$$

Hence $h \in \mathcal{R}$ because $\bar{r}_1, \bar{r}_2 \in \mathcal{R}$.

(iii) If p is odd and q is even, then

$$\bar{r}_1 h = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1+2p & 2s \\ 2r & 1+2q \end{pmatrix} = \begin{pmatrix} 1-2(1+p) & 2s \\ -2r & 1+2q \end{pmatrix} \in \mathcal{E}$$

and again we have $h \in \mathcal{R}$.

(iv) The case of p even and q odd is similar; simply observe that $\bar{r}_2 h \in \mathcal{E}$.

Thus we have proved that $\mathcal{X} \subset \mathcal{R}$ and hence $\mathcal{S}(T) = \mathcal{X} = \mathcal{R}$. □

PROPOSITION 5.5. *The group* $\mathcal{S}(T) \subset \mathrm{GL}(2,\mathbb{Z})$ *is generated by* f_1, f_2, *and* $\bar{r}_1$.

Proof. Recall that $\mathcal{S}(T) = \mathcal{R}$ is generated by $\bar{r}_j$ and τ_j^2 with $j = 1,2$ and by elements of G_μ. The group G_μ is generated by f_1 and f_0. Observe that

$$\tau_1^2 = \bar{r}_2 f_0, \quad \tau_2^2 = f_2\bar{r}_1 = f_1 f_0 f_1 \bar{r}_1, \quad \bar{r}_2 = f_1\bar{r}_1 f_1.$$

So indeed $\mathcal{S}(T)$ is generated by the three elements $f_0, f_1, \bar{r}_1$ of order 2. Note that $(\bar{r}_1 f_1)^{-1} = f_1\bar{r}_1 = -\bar{r}_1 f_1$ and $(\bar{r}_1 f_1)^2 = (f_1\bar{r}_1)^2 = -e$. The element $-e$ commutes with every element of $\mathcal{S}(T)$. □

This concludes the proof of Theorem 1.3.

6. Proof of Proposition 1.4

We divide the proof into two steps. In Step 1 we show that there exists a smooth isotopy ϕ_s with $\phi_1(T) = L$ such that $\phi_1^*\mu_L = \mu_T$. In Step 2 we modify ϕ_s so that $\phi_s(T \setminus D)$ is Lagrangian for all t.

Step 1. Let $\psi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$, $s \in [0,1]$, be a smooth isotopy with $\psi_0 = \mathrm{id}$ and $\psi_1(L) = T$. Then $\psi_1^*\mu_L - \mu_T \in 4 \cdot H^1(T,\mathbb{Z})$ by Theorem 1.2, from which it follows that $\psi_1^*\mu_L = \mu_T \circ g$ for some $g \in \mathcal{X}_T$. Since $\mathcal{X}_T = \mathcal{S}(T)$ by Proposition 5.4, there exists a smooth self-isotopy ψ_s' of T with $(\psi_1')_* = g^{-1}$ and hence $(\psi_1')^*(\psi_1^*\mu_L) = (\psi_1')^*(\mu_T \circ g) = \mu_T$.

Now define

$$\phi_s = \begin{cases} \psi_{2s}' & \text{for } 0 \le s \le 1/2, \\ \psi_{2s-1} \circ \psi_1' & \text{for } 1/2 \le s \le 1. \end{cases}$$

Then we have $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$, $\phi_0 = \mathrm{id}$, $\phi_1(T) = L$, and $\phi_1^*\mu_L = (\psi_1 \circ \psi_1')^*\mu_L = (\psi_1')^*\psi_1^*\mu_L = \mu_T$.

Let $L_s := \phi_s(T)$ for $s \in [0,1]$. Then $L_0 = T$ and $L_1 = L$.

Step 2. We can improve the smooth isotopy L_s so that it is indeed a Lagrangian isotopy outside a disc.

LEMMA 6.1. *Let* $L_s = \phi_s(L_0)$, $s \in [0,1]$, *be a smooth isotopy between a Clifford torus* $T = L_0$ *and a Lagrangian torus* $L = L_1$ *with* $\phi_s \in \mathrm{Diff}_0^c(\mathbb{R}^4)$, $\phi_0 = \mathrm{id}$, *and* $\phi_1^*\mu_L = \mu_T$. *Then there exist a smooth isotopy* $L_s' = \phi_s'(L_0')$ *between* $T = L_0'$ *and* $L = L_1'$ *and a disc* $D \subset T$ *such that* $L_s' \setminus \phi_s'(D)$ *is Lagrangian for all* $s \in [0,1]$.

Proof. Take two simple curves $\gamma, \gamma' \subset T$ that generate $H_1(T,\mathbb{Z})$, and suppose that γ intersects with γ' at exactly one point $p \in T$. Fix an orientation of T. We orient γ and γ' so that the homological intersection $\gamma \cdot \gamma'$ is 1. Denote $\gamma_s := \phi_s(\gamma)$ and $\gamma_s' := \phi_s(\gamma')$ with induced orientations. Also let $p_s := \phi_s(p)$.

We start with γ_s. Let $2m = \mu_T(\gamma_0) = \mu_L(\gamma_1)$. Let $\sigma_s^m \subset N_s^\omega$ denote the m-framing of the symplectic normal bundle N_s^ω of γ_s, so $\mu_{\gamma_s}(\sigma_s^m) = 2m$. Clearly we may take σ_0^m to be a nonvanishing section of the normal bundle $N_{\gamma/T}$ of $\gamma = \gamma_0 \subset T$. Likewise we may take $\sigma_1^m = (\phi_1)_*(\sigma_0^m)$ because $\phi_1^*\mu_L = \mu_T$.

Now trivialize the normal bundle N_s of γ_s as $N_s = S^1 \times \mathbb{R} \times \mathbb{R}^2$ with coordinates (t, t^*, x, y) so that (i) $\gamma_s = S^1 \times \{0\} \times \{0\}$, (ii) $N_s^\omega = S^1 \times \{0\} \times \mathbb{R}^2$, and (iii) $\sigma_s^m(t) = (t, 0, \varepsilon, 0)$ for some $\varepsilon > 0$. This is exactly the same setup used in the proof of Proposition 3.5(i) except that σ_s^0 is replaced by σ_s^m here. With respect to the trivialization of N_s the differential of ϕ_s along γ_s defines a loop with base point Id in the subgroup $A \subset \mathrm{GL}^+(4, \mathbb{R})$ comprising matrices of the form $\left(\begin{smallmatrix} 1 & * \\ 0 & \mathrm{GL}^+(3,\mathbb{R}) \end{smallmatrix}\right)$. Note that ϕ_0 and ϕ_1 correspond to the constant loop. Thus the total of the family ϕ_s corresponds to a smooth map $\Phi\colon I^2/\partial I \cong S^2 \to A$ with $I^2 = [0,1]_s \times [0, 2\pi]_t$ and $\Phi(s,t) := (\phi_s)_*(t)$. Since $\pi_2(A, \mathrm{Id}) \cong \pi_2(\mathrm{SO}(3, \mathbb{R}), \mathrm{Id}) = 0$, there exists a smooth homotopy $\Xi\colon (I^2/\partial I^2) \times [0,1] \to A$ such that $\Xi(\cdot, 0) = \Phi$, $\Xi(\cdot, 1) = \mathrm{Id}$, and $\Xi(p, u) = \mathrm{Id}$ for $p \in \partial I^2$ and for all $u \in [0,1]$.

This implies that, for each s, there is: a tubular neighborhood $U_s \subset \mathbb{R}^4$ of γ_s; a smooth family of maps $\phi_{s,u} \in \mathrm{Diff}_0^+(\mathbb{R}^4)$ with $\phi_{s,0} = \phi_s$, $\phi_{s,u} = \phi_s$ on γ_s, and $\mathbb{R}^4 \setminus U_s$; and $\phi_{i,u} = \phi_i$ for $i = 0, 1$ such that $\phi_{s,1}(T)$ is Lagrangian along γ_s—in other words, $T_{\gamma_s}\phi_{s,1}(T)$ is Lagrangian. By a further perturbation if necessary, we may assume that there exists a tubular neighborhood $V \subset T$ of γ_0 such that $\phi_{s,1}(V)$ is Lagrangian.

Now apply the same argument to γ_s' and $\phi_{s,1}$ as we did to γ_s and ϕ_s. The result is (i) an open neighborhood $Q \subset T$ of $\gamma \cup \gamma'$ with $D := T \setminus Q$ diffeomorphic to a 2-disc and (ii) a new isotopy $L_s' = \phi_s'(T)$ of $T = L_0$ and $L = L_1$ with $\phi_s' \in \mathrm{Diff}_0^c(\mathbb{R}^4)$, $\phi_0' = \mathrm{id}$, such that $Q_s := \phi_s'(Q) \subset L_s'$ is Lagrangian for $s \in [0,1]$. We may assume that the $C_s := \partial Q_s$ are smooth for all s. Finally, take $D = T \setminus Q$. $\square$

This completes the proof of Proposition 1.4.

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