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A Multiplicative Formula for Structure Constants in the Cohomology of Flag Varieties

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1. Introduction

Let G be a connected, simply connected, semisimple complex algebraic group and let $P \subseteq Q$ be a pair of parabolic subgroups. Consider the induced sequence of flag varieties

$$Q/P \hookrightarrow G/P \twoheadrightarrow G/Q. \tag{1}$$

The goal of this paper is to give a simple multiplicative formula connecting the structure coefficients for the cohomology ring of the three flag varieties in (1) with respect to their Schubert bases. Let W be the Weyl group of G and let $W_P \subseteq W_Q \subseteq W$ denote the Weyl groups of P and Q, respectively. Let $W^P \subseteq W$ denote the set of minimal-length coset representatives in W/W_P. For any $w \in W^P$, let $\bar{X}_w \subseteq G/P$ denote the corresponding Schubert variety and let $[X_w] \in H^*(G/P) = H^*(G/P, \mathbb{Z})$ denote the Schubert class of $\bar{X}_w$. It is well known that the Schubert classes $\{[X_w]\}_{w \in W^P}$ form an additive basis for cohomology. Similarly, we have Schubert classes $[X_u] \in H^*(G/Q)$ for $u \in W^Q$ and $[X_v] \in H^*(Q/P)$ for $v \in W^P \cap W_Q$. The letters w, u, and v will be used to denote Schubert varieties in G/P, G/Q, and Q/P, respectively. In Lemma 2.1 we show that, for any $w \in W^P$, there is a unique decomposition $w = uv$ with $u \in W^Q$ and $v \in W^P \cap W_Q$. Fix $s \geq 2$ and, for any $w_1, \dots, w_s \in W^P$ such that $\sum_{k=1}^s \operatorname{codim} X_{w_k} = \dim G/P$, define the associated structure coefficient (or structure constant) to be the integer c_w for

$$[X_{w_1}] \cdots [X_{w_s}] = c_w[\mathrm{pt}] \in H^*(G/P).$$

The following theorem is the first result of this paper.

Theorem 1.1. *Let $w_1, \dots, w_s \in W^P$, and let $u_k \in W^Q$ and $v_k \in W^P \cap W_Q$ be defined by $w_k = u_k v_k$. Assume that*

$$\sum_{k=1}^{s} \operatorname{codim} X_{w_k} = \dim G/P \quad \textit{and} \quad \sum_{k=1}^{s} \operatorname{codim} X_{u_k} = \dim G/Q. \tag{2}$$

If $c_w, c_u, c_v \in \mathbb{Z}_{\geq 0}$ are defined by

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$$\prod_{k=1}^{s}[X_{w_k}] = c_w[\mathrm{pt}], \quad \prod_{k=1}^{s}[X_{u_k}] = c_u[\mathrm{pt}], \quad \prod_{k=1}^{s}[X_{v_k}] = c_v[\mathrm{pt}]$$

in $H^*(G/P), H^*(G/Q), H^*(Q/P)$, *respectively, then* $c_w = c_u \cdot c_v$.

Note that the dimensional conditions in (2) imply that $\sum_{k=1}^{s} \operatorname{codim} X_{v_k} = \dim Q/P$ and hence the associated structure constant c_v is well-defined.

To prove Theorem 1.1, we study the geometry of (1). Fix a maximal torus H and a Borel subgroup B such that $H \subseteq B \subseteq P$. It is well known that, if $\prod_{k=1}^{s}[X_{w_k}] = c_w[\mathrm{pt}]$, then the number of points in the intersection of generic translates

$$|g_1 X_{w_1} \cap \cdots \cap g_s X_{w_s}| = c_w. \tag{3}$$

We show that, for a generic choice of $(\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$, the intersection given in (3) projects onto the intersection $\bigcap_{k=1}^{s} g_k X_{u_k} \subseteq G/Q$ with each fiber of the projection containing exactly c_v points. The techniques used in the proof are inspired by Belkale's work in [1].

1.1. Levi-Movability

The main application of Theorem 1.1 is to show that the product formula applies to "Levi-movable" s-tuples $(w_1, \ldots, w_s) \in (W^P)^s$. Let L_P denote the Levi subgroup of P containing H. Belkale and Kumar give the following definition in [2].

DEFINITION 1.1. The s-tuple $(w_1, \ldots, w_s) \in (W^P)^s$ is Levi-movable, or L_P-movable, if

$$\sum_{k=1}^{s} \operatorname{codim} X_{w_k} = \dim G/P$$

and if, for generic $(l_1, \ldots, l_s) \in (L_P)^s$, the intersection

$$l_1 w_1^{-1} X_{w_1} \cap \cdots \cap l_s w_s^{-1} X_{w_s}$$

is transverse at $eP \in G/P$.

If $(w_1, \ldots, w_s)$ is Levi-movable, then the associated structure constant is not zero. These statements become equivalent if we also assume that $(w_1, \ldots, w_s)$ satisfies a system of linear equalities given in [2, Thm. 15(b)] (these equalities are also given in Proposition 4.1). The following theorem is the second result of this paper.

THEOREM 1.2. *Let* $(w_1, \ldots, w_s)$ *be* L_P*-movable, and let* $u_k \in W^Q$, $v_k \in W^P \cap W_Q$ *be defined by* $w_k = u_k v_k$. *Then the following statements hold*:

(i) $(u_1, \ldots, u_s)$ *is* L_Q*-movable*;
(ii) $(v_1, \ldots, v_s)$ *is* $L_{(L_Q \cap P)}$*-movable.*

A consequence of Theorem 1.2 is that, if $(w_1, \ldots, w_s)$ is L_P-movable, then the product formula in Theorem 1.1 can be applied to its associated structure constant because the conditions in (2) are satisfied. Moreover, since $(u_1, \ldots, u_s)$ and $(v_1, \ldots, v_s)$ are also Levi-movable, we can again apply the product formula to decompose their associated structure constants. This reduces the problem of computing structure constants associated to any Levi movable s-tuple to those coming from the cohomology of flag varieties G/P, where P is maximal parabolic subgroup of G.

The author has proved a special case of Theorems 1.1 and 1.2 for type-A flag varieties in [16, Thm. 3] and for type-C flag varieties in his thesis [15]. The techniques used to prove Theorem 1.1 are direct generalizations of those used in [15; 16]. However, the proof of Theorem 1.2 is different from the proof for the type-A and type-C cases in previous papers. The results of Theorems 1.1 and 1.2 were also obtained simultaneously by Ressayre in [13]. We remark that Ressayre's proof of these theorems differs from those presented in this paper.

Unfortunately, the converse of Theorem 1.2 is false. Counterexamples already exist for two-step flag varieties of type A. In the following corollary, we give a "numerical" converse that can be recovered if we assume $(w_1, \dots, w_s)$ satisfies the numerical conditions given in [2, Thm. 15(b)]. These conditions are also stated in Proposition 4.1.

Corollary 1.2. *Let $(w_1, \dots, w_s) \in (W^P)^s$, and let $u_k \in W^Q$ and $v_k \in W^P \cap W_Q$ be defined by $w_k = u_k v_k$. Assume that the following statements are true*:

(i) *$(u_1, \dots, u_s)$ is L_Q-movable*;
(ii) *$(v_1, \dots, v_s)$ is $L_{(L_Q \cap P)}$-movable*;
(iii) *$(w_1, \dots, w_s)$ satisfies the numerical conditions given in Proposition 4.1.*

Then $(w_1, \dots, w_s)$ is L_P-movable.

This corollary is a direct consequence of Theorem 1.1 and [2, Thm. 15]. We remark that Corollary 1.2 can also be established by work outside this paper. In particular, [2, Thm. 15] and [11, Prop. 11] would also imply Corollary 1.2.

1.2. Representation Theory and Tensor Product Invariants

In this section we state a corollary of Theorems 1.1 and 1.2 in regards to representation theory of the group G. Let $X(H)$ denote the character group of the maximal torus H, and let $X^+(H)$ denote the set of dominant characters with respect to the Borel subgroup B. For any dominant character $\lambda \in X^+(H)$ of G, let V_λ denote the corresponding irreducible finite-dimensional representation of G of highest weight λ. For any $s \geq 2$, define

$$\Gamma(s, G) := \{(\lambda_1, \dots, \lambda_s) \in X^+(H)^s \otimes_{\mathbb{Z}} \mathbb{Q} \mid \exists N > 1,\ (V_{N\lambda_1} \otimes \cdots \otimes V_{N\lambda_s})^G \neq 0\}.$$

The set $\Gamma(s, G)$ forms a convex cone in the vector space $X^+(H)^s \otimes_{\mathbb{Z}} \mathbb{Q}$ and has been studied in the context of Horn's problem on generalized eigenvalues [2; 5; 6]. The set $\Gamma(s, G)$ was initially characterized by Klyachko [8] in type A and later in all types by Berenstein and Sjamaar [3]. These characterizations consist of a list of inequalities parameterized by nonzero products of Schubert classes. Knutson, Tao, and Woodard [9] determined a minimal set of inequalities for type A. Belkale and Kumar [2] showed that, for all types, it is enough to consider the set of inequalities corresponding to Levi-movable s-tuples with associated structure coefficient equal to 1. More recently, Ressayre [14] showed that this set of inequalities is actually minimal. Let Δ denote the set of simple roots of G, and let $\Delta(P)$ denote the simple roots associated to the parabolic subgroup $P \subseteq G$. For any $\alpha \in \Delta$, let $\omega_{\alpha^\vee}$ denote the corresponding fundamental coweight.

THEOREM 1.3 [2; 14]. *If $(w_1, \dots, w_s) \in W^P$ is L_P-movable with associated structure constant $c_w = 1$, then the set of $(\lambda_1, \dots, \lambda_s) \in \Gamma(s, G)$ such that*

$$\sum_{k=1}^{s} \omega_{\alpha^\vee}(w_k^{-1}\lambda_k) = 0 \quad \forall \alpha \in \Delta \backslash \Delta(P)$$

is a face of $\Gamma(s, G)$ whose codimension is of cardinality $|\Delta \backslash \Delta(P)|$. Moreover, any face of $\Gamma(s, G)$ that intersects the interior of the dominant chamber $X^+(H)^s \otimes_{\mathbb{Z}} \mathbb{Q}$ can be described as above, and the list of faces of codimension 1 is irredundant.

Let $F(w_1, \dots, w_s) \subseteq \Gamma(s, G)$ be the face of $\Gamma(s, G)$ associated to the Levi movable s-tuple $(w_1, \dots, w_s) \in (W^P)^s$ with $c_w = 1$. Applying Theorems 1.1 and 1.2 yields the following corollary.

COROLLARY 1.3. *Let $(w_1, \dots, w_s) \in (W^P)^s$ be L_P-movable with associated structure constant $c_w = 1$, and let $w_k = u_k v_k$ for $u_k \in W^Q$ and $v_k \in W^P \cap W_Q$. Then $F(w_1, \dots, w_s)$ is a face of $F(u_1, \dots, u_s)$.*

Proof. From Theorems 1.1 and 1.2 it follows that $(u_1, \dots, u_s)$ is L_Q-movable and that $c_w = c_u \cdot c_v = 1$, where c_w, c_u, c_v are the structure constants associated to $(w_k)_{k=1}^s, (u_k)_{k=1}^s, (v_k)_{k=1}^s$, respectively. Hence $c_u = 1$ and, by Theorem 1.3, $F(u_1, \dots, u_s)$ is a face of $\Gamma(s, G)$ of codimension $|\Delta \backslash \Delta(Q)|$. It suffices to show that if $(\lambda_1, \dots, \lambda_s) \in F(w_1, \dots, w_s)$ then $(\lambda_1, \dots, \lambda_s) \in F(u_1, \dots, u_s)$. Let $\alpha \in \Delta \backslash \Delta(Q) \subseteq \Delta \backslash \Delta(P)$. Now, for any $w \in W^P$ and rational dominant weight λ, we have

$$\omega_{\alpha^\vee}(w^{-1}\lambda) = uv\omega_{\alpha^\vee}(\lambda) = u\omega_{\alpha^\vee}(\lambda) = \omega_{\alpha^\vee}(u^{-1}\lambda)$$

because $v \in W_Q$ acts trivially on any $\omega_{\alpha^\vee}$, where $\alpha \in \Delta \backslash \Delta(Q)$. This proves the corollary. □

1.3. Generalizations to Branching Schubert Calculus

In this section we give generalizations of Theorems 1.1 and 1.2. We remark that the generalization of Theorem 1.1 was also independently obtained by Ressayre in [13]. Let $\tilde{G}$ be any connected semisimple subgroup of G, and fix maximal tori and Borel subgroups $\tilde{H} \subseteq \tilde{B} \subseteq \tilde{G}$ and $H \subseteq B \subseteq G$ such that $\tilde{H} = H \cap \tilde{G}$ and $\tilde{B} = B \cap \tilde{G}$. For any parabolic subgroup $P \subseteq G$ containing B, we define the parabolic subgroup $\tilde{P} := P \cap \tilde{G}$ of $\tilde{G}$. Consider the $\tilde{G}$-equivariant embedding of flag varieties

$$\phi_z \colon \tilde{G}/\tilde{P} \hookrightarrow G/P$$

defined by $\phi(g\tilde{P}) := gP$. The problem concerning "branching Schubert calculus" is to compute the pullback

$$\phi^*([X_w]) = \sum_{\tilde{w}\in\tilde{W}^P} c_w^{\tilde{w}}[X_{\tilde{w}}]$$

in terms of the Schubert basis in $H^*(\tilde{G}/\tilde{P})$. If $\dim X_w = \dim G/P - \dim \tilde{G}/\tilde{P}$, then $\phi^*([X_w]) = c_w[\mathrm{pt}]$ for some $c_w \in \mathbb{Z}_{\geq 0}$. Consider the diagonal embedding $\phi\colon \tilde{G}/\tilde{P} \hookrightarrow (\tilde{G}/\tilde{P})^s$, and let $[X_{w_1}\times\cdots\times X_{w_s}]$ be a Schubert class in $H^*((\tilde{G}/\tilde{P})^s)$. Then

$$\phi^*([X_{w_1}\times\cdots\times X_{w_s}]) = \prod_{k=1}^{s}[X_{w_k}].$$

Hence the problem of branching Schubert calculus is a generalization of the usual Schubert calculus.

Let Q be a parabolic subgroup that contains P, and define $\tilde{Q} := Q\cap\tilde{G}$ to be the corresponding parabolic subgroup of $\tilde{G}$. The embedding ϕ induces the maps

$$\phi_1\colon \tilde{G}/\tilde{Q} \hookrightarrow G/Q \quad \text{and} \quad \phi_2\colon \tilde{Q}/\tilde{P} \hookrightarrow Q/P$$

given by $\phi_1(g\tilde{Q}) := gQ$ and $\phi_2 := \phi|_{\tilde{Q}/\tilde{P}}$. The following result is an analogue of Theorem 1.1.

THEOREM 1.4. *Let $w = uv \in W^P$, where $u\in W^Q$ and $v\in W^P\cap W_Q$. Assume that*

$$\dim X_w = \dim G/P - \dim\tilde{G}/\tilde{P} \quad \textit{and} \quad \dim X_u = \dim G/Q - \dim\tilde{G}/\tilde{Q}. \tag{4}$$

If $c_w, c_u, c_v \in \mathbb{Z}_{\geq 0}$ are defined by

$$\phi^*([X_w]) = c_w[\mathrm{pt}], \quad \phi_1^*([X_u]) = c_u[\mathrm{pt}], \quad \phi_2^*([X_v]) = c_v[\mathrm{pt}]$$

in $H^*(\tilde{G}/\tilde{P}), H^*(\tilde{G}/\tilde{Q}), H^*(\tilde{Q}/\tilde{P})$, respectively, then $c_w = c_u\cdot c_v$.

The techniques used to prove Theorem 1.4 are the same as those used to prove Theorem 1.1, so we provide only a brief overview in Section 5.

As in Section 1.1, we give a special set of the $w\in W^P$ that satisfy the assumptions in Theorem 1.4 by generalizing the notion of Levi-movability.

DEFINITION 1.4. We say $w\in W^P$ is (L_P,ϕ)-movable if, for generic $l\in L_P$, the following induced map on tangent spaces is an isomorphism:

$$\phi_*\colon T_{e\tilde{P}}(\tilde{G}/\tilde{P}) \to \frac{T_{eP}(G/P)}{T_{eP}(lw^{-1}X_w)}.$$

If ϕ is the diagonal embedding, then $w = (w_1,\ldots,w_s)$ is (L_P,ϕ)-movable if and only if w is L_P-movable. We now give an analogue of Theorem 1.2. Let $\tilde{\mathfrak{h}}$ denote the Lie algebra of $\tilde{H}$, and let $\Delta_{\tilde{G}}\subset\tilde{\mathfrak{h}}^*$ denote the simple roots of $\tilde{G}$. Let $\Delta_{\tilde{Q}}\subseteq\Delta_{\tilde{G}}$ denote the set of simple roots corresponding to the parabolic subgroup $\tilde{Q}\subseteq\tilde{G}$. Let $\mathfrak{z}$ denote the Lie algebra of the center of L_Q.

THEOREM 1.5. *Assume there exists a vector $\tau \in \tilde{\mathfrak{h}} \cap \mathfrak{z}$ such that $\alpha(\tau) \geq 0$ for any $\alpha \in \Delta_{\tilde{G}}$, with equality if and only if $\alpha \in \Delta_{\tilde{Q}}$. Let $w = uv \in W^P$, where $u \in W^Q$ and $v \in W^P \cap W_Q$. If w is (L_P, ϕ)-movable, then the following statements hold*:

(i) *u is (L_Q, ϕ_1)-movable*;

(ii) *v is $(L_{(L_Q \cap P)}, \phi_2)$-movable.*

The existence of $\tau \in \tilde{\mathfrak{h}} \cap \mathfrak{z}$ in Theorem 1.5 is a restriction on the choice of $Q \subseteq G$. In the case of the diagonal embedding, the vector τ exists if and only if the parabolic subgroup $Q \subseteq G = \tilde{G}^s$ is of the form $Q = \tilde{Q}^s$ for some parabolic subgroup $\tilde{Q} \subset \tilde{G}$.

Theorem 1.5 implies that if $w \in W^P$ is (L_P, ϕ)-movable, then w satisfies the conditions in (4) and hence we can decompose the associated structure constant c_w. As with Theorem 1.4, the proof of Theorem 1.5 follows the same outline as the proof in the diagonal embedding case.

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2. Preliminaries

Let G be a connected, simply connected, semisimple complex algebraic group. Fix a Borel subgroup B and a maximal torus $H \subseteq B$. Let $W := N_G(H)/H$ denote the Weyl group of G, where $N_G(H)$ is the normalizer of H in G. Let $P \subseteq G$ be a standard parabolic subgroup (P contains B), and let L_P denote the Levi subgroup of P containing H. Denote the Lie algebras of G, H, B, P, L_P by the respective fraktur letters $\mathfrak{g}, \mathfrak{h}, \mathfrak{b}, \mathfrak{p}, \mathfrak{l}_P$.

Let $R \subseteq \mathfrak{h}^*$ be the set of roots and let $R^{\pm} \subseteq R$ denote the set of positive roots (negative roots). Let R_P denote the set of roots corresponding to $\mathfrak{l}_P$, and let $R_P^{\pm}$ denote the set of positive roots (negative roots) with respect to the Borel subgroup $B_P := B \cap L_P$ of L_P.

Let W^P be the set of minimal-length representatives of the coset space W/W_P, where W_P is the Weyl group of P (or L_P). For any $w \in W^P$, define the Schubert cell

$$X_w := BwP/P \subseteq G/P.$$

We denote the cohomology class of the closure $\bar{X}_w$ by $[X_w] \in H^*(G/P)$. We begin with some basic lemmas on the Weyl group W.

LEMMA 2.1. *The map $\eta\colon W^Q \times (W^P \cap W_Q) \to W^P$ given by $(u, v) \mapsto uv$ is well-defined and a bijection.*

Proof. Since $W = \bigsqcup_{u \in W^Q} uW_Q$, we have that $W/W_P = \bigsqcup_{u \in W^Q} uW_Q/W_P$. It suffices to show that if $v \in W^P \cap W_Q$ then $uv \in W^P$. Let $\ell\colon W \to \mathbb{Z}_{\geq 0}$ denote the length function on W. For any $v' \in W_P$ we have that

$$\ell(uvv') = \ell(u) + \ell(vv') = \ell(u) + \ell(v) + \ell(v') = \ell(uv) + \ell(v'),$$

since $u \in W^Q$, $vv' \in W_Q$, $v \in W^P$, and $v' \in W_P$. Hence $uv \in W^P$. □

Lemma 2.1 shows that, for any $w \in W^P$, there is a unique $u \in W^Q$ and $v \in W^P \cap W_Q$ such that $w = uv$. We will assume this relationship between w, u, v given any $w \in W^P$. If these groups' elements are indexed by $w_k \in W^P$, then we write $w_k = u_k v_k$ accordingly.

Note that the flag variety $Q/P \simeq L_Q/(L_Q \cap P)$ for L_Q the Levi subgroup of Q. Under this identification, the Schubert cell $X_v \simeq B_Q v(L_Q \cap P)/(L_Q \cap P)$.

Lemma 2.2. *For any $w = uv \in W^P$, we have $u^{-1}X_w \cap Q/P = X_v$.*

Proof. Let X'_w denote the subset of $L_Q/(L_Q \cap P)$ identified with $u^{-1}X_w \cap Q/P$ under the isomorphism $Q/P \simeq L_Q/(L_Q \cap P)$. Since $v \in W_Q$, it follows that

$$u^{-1}X_w \cap Q/P = (u^{-1}BuvP \cap Q)P/P = (u^{-1}Bu \cap Q)vP/P.$$

By [10, Exer. 1.3.E], the group $B_Q \subseteq u^{-1}Bu \cap Q$ and hence $B_Q v(L_Q \cap P)/(L_Q \cap P) \subseteq X'_w$. Since the B_Q-orbits of $L_Q/(L_Q \cap P)$ are in bijection with $W^P \cap W_Q$, the set X'_w cannot contain more than a single B_Q-orbit. This proves the lemma. □

3. Structure Coefficients and Transversality

In this section we prove Theorem 1.1. Assume that $(w_1, \ldots, w_s) \in (W^P)^s$ satisfy the conditions (2), and let $w_k = u_k v_k$ with respect to Lemma 2.1. We begin by considering the following G-variety. Define

$$Y = Y(u_1, \ldots, u_s) := \left\{ (\bar{g}; \bar{g}_1, \ldots, \bar{g}_s) \in G/Q \times (G/B)^s \;\middle|\; \bar{g} \in \bigcap_{k=1}^{s} g_k X_{u_k} \right\},$$

where the action G on Y is the diagonal action. We now prove that Y is smooth and irreducible. Define

$$\tilde{Y} := G \times_Q (Qu_1^{-1}B/B \times \cdots \times Qu_s^{-1}B/B).$$

Note that if $\bar{g} \in g_k X_{u_k}$ then by [2, Lemma 1] we have $g^{-1}g_k = q_k u_k^{-1}$ for some $q_k \in Q$. Because translated Schubert varieties of the form $qu_k^{-1}X_{u_k}$ are precisely those that contain the identity, we can view $\tilde{Y}$ as the parameter set of all intersections $\bigcap_{k=1}^{s} g_k X_{u_k}$ that contain the identity up to translation paired with a point in G.

Lemma 3.1. *The G-equivariant map $\xi : \tilde{Y} \to Y$ given by*

$$\xi((g; \overline{q_1 u_1^{-1}}, \ldots, \overline{q_s u_s^{-1}})) = (\bar{g}; \overline{gq_1 u_1^{-1}}, \ldots, \overline{gq_s u_s^{-1}}) \tag{5}$$

is well-defined and an isomorphism. Moreover, Y is smooth and irreducible.

Proof. If ξ is an isomorphism, then the irreducibility and smoothness of Y follows from the irreducibility of smoothness of $\tilde{Y}$. The fact that ξ is an isomorphism is a consequence of [12, Lemma 6.1]. □

LEMMA 3.2. *For any $u \in W^Q$, the map $Qu^{-1}B/B \to Q/B$ given by $\overline{qu^{-1}} \mapsto \bar{q}$ is well-defined and Q-equivariant.*

Proof. Let $q_1, q_2 \in Q$ such that $q_1u^{-1}B = q_2u^{-1}B$. Then $uq_2^{-1}q_1u^{-1} \in B$. It suffices to show that $q_2^{-1}q_1 \in B$—in other words, that $Q \cap u^{-1}Bu \subseteq B$. By [4, Prop. 2.1], the intersection $Q \cap u^{-1}Bu$ is connected; hence it is enough to show that $\mathfrak{q} \cap u^{-1}\mathfrak{b} \subseteq \mathfrak{b}$. We look at the set of roots $R_Q \cap u^{-1}R^+$ corresponding to $\mathfrak{q} \cap u^{-1}\mathfrak{b}$. Since $u \in W^Q$, we have that $uR_Q^+ \subseteq R^+$ and $uR_Q^- \subseteq R^-$. Thus

$$R_Q \cap u^{-1}R^+ = u^{-1}(uR_Q \cap R^+) = u^{-1}(uR_Q^+) \subseteq R^+.$$

This proves the lemma. $\square$

Assume we have $(\bar{g}; \bar{g}_1, \ldots, \bar{g}_s) \in G/P \times (G/B)^s$ such that $\bar{g} \in \bigcap_{k=1}^s g_k X_{w_k}$. It is easy to see that $(\bar{g}Q; \bar{g}_1, \ldots, \bar{g}_s) \in Y$. Since $eP \in g^{-1}g_k X_{w_k}$, it follows from [2, Lemma 1] that $g^{-1}g_k X_{w_k} = p_k v_k^{-1} u_k^{-1} X_{w_k}$ for some $p_k \in P$. Set $q_k = p_k v_k^{-1} \in Q$. By Lemma 2.2,

$$g^{-1}g_k X_{w_k} \cap Q/P = q_k(u_k^{-1}X_{w_k} \cap Q/P) = q_k X_{v_k}.$$

We consider the points of Y that satisfy the following property.

DEFINITION 3.3. We say that $(\bar{g}; \bar{g}_1, \ldots, \bar{g}_s) \in Y$ has property P1 if:

(i) $\bigcap_{k=1}^s (g^{-1}g_k X_{w_k} \cap Q/P)$ is transverse at every point in the intersection in Q/P; and

(ii) for any $(q_1, \ldots, q_s) \in Q^s$ such that $g^{-1}g_k X_{u_k} = q_k u_k^{-1} X_{u_k} \subseteq G/Q$ for all k, the intersection

$$\bigcap_{k=1}^s q_k X_{v_k} = \bigcap_{k=1}^s q_k \bar{X}_{v_k} \subseteq Q/P.$$

PROPOSITION 3.4. *The set of points in Y with property P1 contains a nonempty G-stable open subset.*

Proof. By Kleiman's transversality [7] there exists a nonempty open set $O \subseteq (Q/B)^s$ such that, for any $(q_1, \ldots, q_s) \in O$, the following conditions are satisfied:

- $\bigcap_{k=1}^s q_k X_{v_k} \subseteq Q/P$ is transverse at every point in the intersection; and
- $\bigcap_{k=1}^s q_k X_{v_k} = \bigcap_{k=1}^s q_k \bar{X}_{v_k}$.

Moreover, we can choose O to be stable under the diagonal action of Q on $(Q/B)^s$ by replacing O with $\bigcup_{q \in Q} qO$. Consider the map

$$\tilde{\xi} : Y \to G \times_Q (Q/B)^s$$

defined by $\tilde{\xi} := \zeta \circ \xi^{-1}$, where

$$\zeta((g; \overline{q_1u_1^{-1}}, \ldots, \overline{q_su_s^{-1}})) := (g; \overline{q_1}, \ldots, \overline{q_s}).$$

By Lemma 3.2, the map $\tilde{\xi}$ is well-defined and G-equivariant. Clearly any

$$(g; g_1, \ldots, g_s) \in \tilde{\xi}^{-1}(G \times_Q O)$$

satisfies property P1. $\square$

Proof of Theorem 1.1. Assume that $c_u \neq 0$. We first show that there exists $(\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$ satisfying three conditions as follows.

(i) $\bigcap_{k=1}^s g_k X_{w_k}$ is transverse at every point of the intersection in G/P, and

$$\bigcap_{k=1}^{s} g_k X_{w_k} = \bigcap_{k=1}^{s} g_k \bar{X}_{w_k}.$$

(ii) $\bigcap_{k=1}^s g_k X_{u_k}$ is transverse at every point of the intersection in G/Q, and

$$\bigcap_{k=1}^{s} g_k X_{u_k} = \bigcap_{k=1}^{s} g_k \bar{X}_{u_k}.$$

(iii) For every $x \in \bigcap_{k=1}^s g_k X_{u_k}$, we have that $(x; \bar{g}_1, \ldots, \bar{g}_s) \in Y$ has property P1.

By Kleiman's tranversality [7], there exists an open subset $O_1 \subseteq (G/B)^s$ such that every point in O_1 satisfies conditions (i) and (ii). By Proposition 3.4, there exists a nonempty open subset $Y^\circ \subseteq Y$ such that every point in Y° has property P1. Consider the projection of Y onto its second factor,

$$\sigma : Y \to (G/B)^s.$$

Since $c_u \neq 0$, the morphism σ is dominant. Moreover, the fibers of σ are generically finite and hence $\dim Y = \dim(G/B)^s$. Since Y is irreducible, we have that

$$\dim \overline{\sigma(Y \backslash Y^\circ)} \leq \dim Y \backslash Y^\circ < \dim Y = \dim(G/B)^s.$$

Define the nonempty open set $O_2 := (G/B)^s \backslash (\overline{\sigma(Y \backslash Y^\circ)})$. Any $(\bar{g}_1, \ldots, \bar{g}_s) \in O_1 \cap O_2$ satisfies conditions (i)–(iii). Assume that $(\bar{g}_1, \ldots, \bar{g}_s) \in O_1 \cap O_2 \subseteq (G/B)^s$. Conditions (i) and (ii) imply that

$$\left| \bigcap_{k=1}^{s} g_k X_{w_k} \right| = c_w \quad \text{and} \quad \left| \bigcap_{k=1}^{s} g_k X_{u_k} \right| = c_u.$$

Consider the G-equivariant projection $\pi : G/P \twoheadrightarrow G/Q$. If $\bar{g} \in \bigcap_{k=1}^s g_k X_{u_k}$, then condition (iii) implies that $(\bar{g}; \bar{g}_1, \ldots, \bar{g}_s) \in Y$ has property P1. By Lemma 2.2, we have

$$\left| \bigcap_{k=1}^{s} g_k X_{w_k} \cap \pi^{-1}(\bar{g}) \right| = \left| \bigcap_{k=1}^{s} q_k u^{-1} X_{w_k} \cap Q/P \right| = \left| \bigcap_{k=1}^{s} q_k X_{v_k} \right| = c_v, \tag{6}$$

where we choose $q_k \in Q$ such that $g^{-1} g_k X_{w_k} = q_k u_k^{-1} X_{w_k}$.

If $c_w = 0$, then $\bigcap_{k=1}^s g_k X_{w_k} = \emptyset$. Equation (6) now implies that $c_v = 0$ and hence $c_w = c_u \cdot c_v$.

If $c_w \neq 0$ then we have the surjection

$$\pi\left(\bigcap_{k=1}^{s} g_k X_{w_k} \right) = \bigcap_{k=1}^{s} g_k X_{u_k},$$

and equation (6) again implies that $c_w = c_u \cdot c_v$.

Finally, if $c_u = 0$ then $c_w = 0$ since, for generic $(\bar{g}_1, \ldots, \bar{g}_s) \in (G/B)^s$, we have

$$\pi\left(\bigcap_{k=1}^{s} g_k X_{w_k}\right) \subseteq \bigcap_{k=1}^{s} g_k X_{u_k} = \emptyset.$$

Hence we still have $c_w = c_u \cdot c_v$. □

4. Applications to Levi-Movability

One application of Theorem 1.1 is computing structure coefficients that correspond to Levi-movable s-tuples in $(W^P)^s$. We begin with some preliminaries on Lie theory. Let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subset R^+$ be the set of simple roots of G, where n is the rank of G. Note that the set Δ forms a basis for $\mathfrak{h}^*$, and let $\{x_1, x_2, \ldots, x_n\} \subseteq \mathfrak{h}$ be the dual basis to Δ such that

$$\alpha_i(x_j) = \delta_{i,j}.$$

Let $\Delta(P) \subset \Delta$ denote the simple roots associated to P (i.e., the simple roots that generate R_P^+). For any parabolic subgroup P and $w \in W^P$, define the character

$$\chi_w^P := \rho - 2\rho^P + w^{-1}\rho;$$

here ρ is the half-sum of all the roots in R^+ and ρ^P is the half-sum of roots in R_P^+. The following proposition is proved in [2] using geometric invariant theory.

PROPOSITION 4.1 [2, Thm. 15]. *If $(w_1, \ldots, w_s)$ is L_P-movable, then for every $\alpha_i \in \Delta \backslash \Delta(P)$ we have*

$$\left(\left(\sum_{k=1}^{s} \chi_{w^k}^P\right) - \chi_1^P\right)(x_i) = 0.$$

Proof of Theorem 1.2. Recall that, by Lemma 2.1, for any $w \in W^P$ we have $w = uv$ such that $u \in W^Q$ and $v \in W^P \cap W_Q$. For any pair of parabolic subgroups $P \subseteq Q$, let $T^P := T_{eP}(G/P)$ and $T^{P,Q} := T_{eP}(Q/P)$. For any $w \in W^P$ and $p \in P$, we have the subspace $pT_w^P := T_{eP}(pw^{-1}X_w) \subseteq T^P$. The condition for Levi-movability is equivalent to the condition that the diagonal map

$$\phi \colon T^P \to \bigoplus_{k=1}^{s} T^P / l_k T_{w_k}^P$$

be an isomorphism for generic $(l_1, \ldots, l_s) \in (L_P)^s$. Consider the diagram

$$\begin{array}{ccccc} T^{P,Q} & \hookrightarrow & T^P & \twoheadrightarrow & T^Q \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi_1} \\ \displaystyle\bigoplus_{k=1}^{s} \frac{T^{P,Q}}{l_k T_{v_k}^{P,Q}} & \hookrightarrow & \displaystyle\bigoplus_{k=1}^{s} \frac{T^P}{l_k T_{w_k}^P} & \twoheadrightarrow & \displaystyle\bigoplus_{k=1}^{s} \frac{T^Q}{l_k v_k^{-1} T_{u_k}^Q} \end{array} \tag{7}$$

where ϕ_1 and ϕ_2 are the diagonal maps corresponding (respectively) to G/Q and Q/P. It suffices to show that if ϕ is an isomorphism then ϕ_1 and ϕ_2 are isomorphisms.

Fix $(l_1, \ldots, l_s) \in (L_P)^s$ so that ϕ is an isomorphism. By the commutativity of diagram (7), $\dim\operatorname{coker}\phi_1 = 0$ (since $\dim\operatorname{coker}\phi = 0$). If $\dim\ker\phi_1 = 0$ then ϕ_1 is an isomorphism, which proves part (i). Since ϕ is injective, ϕ_2 is also injective. By the snake lemma, we have that

$$\dim\ker\phi_1 = \dim\operatorname{coker}\phi_2 = 0.$$

Hence ϕ_2 is an isomorphism, which proves part (ii). We now prove that

$$\dim\ker\phi_1 = 0.$$

Since ϕ_1 is surjective, the map

$$\phi_1\colon T^Q/\ker\phi_1 \to \bigoplus_{k=1}^{s} \frac{T^Q}{l_i v_k^{-1} T^Q_{u_k}}$$

is an isomorphism. As a consequence, the induced map on top exterior powers,

$$\Phi_1\colon \det(T^Q/\ker\phi_1) \to \det\left(\bigoplus_{k=1}^{s} \frac{T^Q}{l_k v_k^{-1} T^Q_{u_k}}\right),$$

is nonzero. Identifying the character group $X(H)$ with the weight lattice in $\mathfrak{h}^*$ shows that $\mathfrak{h}$ acts on the complex line $\det(T^Q/\ker\phi_1)$ by the character $-\chi_1^Q - \beta$, where β is the sum of roots in $\ker\phi_1$. Similarly, $\mathfrak{h}$ acts diagonally on $\det\left(\bigoplus_{k=1}^{s} \frac{T^Q}{l_k v_k^{-1} T^Q_{u_k}}\right)$ by the character $-\sum_{i=1}^{s} \chi^Q_{u_i}$. It is easy to see that the map Φ_1 is equivariant with respect to the action of the center of L_Q. In particular, for any $\alpha_i \in \Delta\backslash\Delta(Q)$,

$$(\chi_1^Q + \beta)(x_i) = \sum_{k=1}^{s} \chi^Q_{u_k}(x_i).$$

For any $w = uv \in W^P$ and $\alpha_i \in \Delta\backslash\Delta(Q)$, we have

$$\begin{aligned}
\chi_w^P(x_i) &= (\rho - 2\rho^P)(x_i) + w^{-1}\rho(x_i)\\
&= \rho(x_i) - \rho(uvx_i)\\
&= (\rho - 2\rho^Q)(x_i) + u^{-1}\rho(x_i)\\
&= \chi_u^Q(x_i)
\end{aligned}$$

because the Weyl group W_Q acts trivially on x_i and $\rho^P(x_i) = \rho^Q(x_i) = 0$. Hence, by Proposition 4.1, we have

$$\beta(x_i) = \left(\left(\sum_{k=1}^{s} \chi^Q_{u_i}\right) - \chi_1^Q\right)(x_i) = \left(\left(\sum_{i=1}^{s} \chi^P_{w_i}\right) - \chi_1^P\right)(x_i) = 0$$

for all $\alpha_i \in \Delta\backslash\Delta(Q)$. But

$$\ker\phi_1 \subseteq T^Q = \bigoplus_{\alpha\in R^-\setminus R_Q^-} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ denotes the root space of $\mathfrak{g}$ corresponding to α. Hence $-\beta$ is a positive linear combination of positive simple roots in $\Delta\setminus\Delta(Q)$. This implies that $\dim\ker\phi_1 = 0$, which proves Theorem 1.2. $\square$

5. Branching Schubert Calculus

In this section we generalize Theorems 1.1 and 1.2 to the setting of branching Schubert calculus. These generalizations are stated in Theorems 1.4 and 1.5. Since the proofs are similar to those for Theorems 1.1 and 1.2, we leave several details to the reader. Let $\tilde{G}$ be any connected semisimple subgroup of G, and fix maximal tori and Borel subgroups $\tilde{H} \subseteq \tilde{B} \subseteq \tilde{G}$ and $H \subseteq B \subseteq G$ such that $\tilde{H} = H \cap \tilde{G}$ and $\tilde{B} = B \cap \tilde{G}$. As in Theorem 1.1, we consider a pair of parabolic subgroups $P \subseteq Q$ in G that contain B. Define the parabolic subgroups

$$\tilde{P} := P \cap \tilde{G} \quad \text{and} \quad \tilde{Q} := Q \cap \tilde{G},$$

and consider the maps

$$\begin{aligned} \phi &\colon \tilde{G}/\tilde{P} \hookrightarrow G/P,\\ \phi_1 &\colon \tilde{G}/\tilde{Q} \hookrightarrow G/Q,\\ \phi_2 &\colon \tilde{Q}/\tilde{P} \to Q/P \end{aligned}$$

defined by $\phi(g\tilde{P}) := gP$, $\phi_1(g\tilde{Q}) := gQ$, and $\phi_2 := \phi|_{\tilde{Q}/\tilde{P}}$. Consider the following commutative diagram:

$$\begin{tikzcd} \tilde{Q}/\tilde{P} \arrow[r, hook] \arrow[d, "\phi_2"] & \tilde{G}/\tilde{P} \arrow[r, two heads, "\pi"] \arrow[d, "\phi"] & \tilde{G}/\tilde{Q} \arrow[d, "\phi_1"] \\ Q/P \arrow[r, hook] & G/P \arrow[r, two heads] & G/Q \end{tikzcd} \tag{8}$$

For any $w \in W^P$ such that $\dim X_w = \dim G/P - \dim \tilde{G}/\tilde{P}$, we have the associated structure constant $c_w \in \mathbb{Z}_{\geq 0}$ defined by

$$\phi^*([X_w]) = c_w[\mathrm{pt}].$$

By Lemma 2.1, we can write $w = uv$ for $u \in W^Q$ and $v \in W^P \cap W_Q$. We show that if condition (4) is satisfied then $c_w = c_u \cdot c_v$, where

$$\phi_1^*([X_u]) = c_u[\mathrm{pt}] \quad \text{and} \quad \phi_2^*([X_v]) = c_v[\mathrm{pt}].$$

5.1. *Proof of Theorem 1.4*

If $w \in W^P$ satisfies condition (4), then there exists a nonempty open subset $O_1 \subseteq G/B$ such that, if $\bar{g} \in O_1$, then the cardinality of inverse images

$$|\phi^{-1}(gX_w)| = c_w \quad \text{and} \quad |\phi_1^{-1}(gX_u)| = c_u.$$

Consider the projection $\pi : \tilde{G}/\tilde{P} \to \tilde{G}/\tilde{Q}$. By the commutativity of diagram (8), we have that $\pi(\phi^{-1}(gX_w)) \subseteq \phi_1^{-1}(gX_u)$. Therefore, if $c_u = 0$ then $c_w = 0$.

Assume that $c_u \neq 0$. It suffices to show that (a) for generic $\bar{g} \in G/B$, the map π restricted to $\phi^{-1}(gX_w)$ is surjective when $c_w \neq 0$ and (b) for any $\bar{h} \in \phi_1^{-1}(gX_u)$ we have $|\pi^{-1}(\bar{h}) \cap \phi^{-1}(gX_w)| = c_v$. Following the proof of Theorem 1.1, we define the set

$$Y := \{(\bar{h}, \bar{g}) \in \tilde{G}/\tilde{Q} \times G/B \mid \phi(\bar{h}) \in gX_u\}.$$

By an analogue of Lemma 3.1, the set Y is a smooth irreducible $\tilde{G}$-variety. Similarly, by an analogue of Proposition 3.4, the set of points in Y with the following property P2 contains a nonempty open subset of Y.

DEFINITION 5.1. We say $(\bar{h}, \bar{g}) \in Y$ has property P2 if:

(i) the intersection $(h^{-1}gX_w \cap Q/P) \cap \phi_2(\tilde{Q}/\tilde{P})$ is transverse at every point in Q/P; and

(ii) for any $q \in Q$ such that $h^{-1}gX_u = qu^{-1}X_u \subseteq G/Q$, the intersection

$$qX_v \cap \phi_2(\tilde{Q}/\tilde{P}) = q\bar{X}_v \cap \phi_2(\tilde{Q}/\tilde{P}) \subseteq Q/P.$$

Let $Y^\circ \subseteq Y$ be a nonempty open set whose points have property P2, and let $\sigma : Y \to G/B$ denote the projection onto the second factor of Y. By the proof of Theorem 1.1, the set $O_2 := (G/B)\backslash\overline{\sigma(Y\backslash Y^\circ)}$ is an open subset of G/B. Moveover, if $g \in O_1 \cap O_2$ and $c_w \neq 0$, then $\pi(\phi^{-1}(gX_w)) = \phi_1^{-1}(gX_u)$. By [2, Lemma 1] we can choose $q \in Q$ such that $h^{-1}gX_w = qu^{-1}X_w$. By Lemma 2.2, for any $\bar{h} \in \phi_1^{-1}(gX_u)$ we have

$$\begin{aligned} |\pi^{-1}(\bar{h}) \cap \phi^{-1}(gX_w)| &= |qu^{-1}X_w \cap Q/P \cap \phi_2(\tilde{Q}/\tilde{P})| \\ &= |qX_v \cap \phi_2(\tilde{Q}/\tilde{P})| = c_v. \end{aligned} \tag{9}$$

If $c_w = 0$ then equation (9) implies that $c_v = 0$. In either case, $c_w = c_u \cdot c_v$. This proves Theorem 1.4.

5.2. *Proof of Theorem 1.5*

Let $\tilde{R}$ denote the set of roots of $\tilde{G}$ with respect to the torus $\tilde{H}$, and let $\tilde{R}^+$ denote the set of positive roots with respect to the Borel subgroup $\tilde{B}$. Let $\Delta_{\tilde{G}} := \{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m\} \subseteq \tilde{R}^+$ denote the simple roots of $\tilde{G}$, where m is the rank of $\tilde{G}$. Let $\{\tilde{x}_1, \ldots, \tilde{x}_m\} \subseteq \tilde{\mathfrak{h}}$ denote the dual basis to $\Delta_{\tilde{G}}$. For any parabolic subgroup $\tilde{Q} \subseteq \tilde{G}$ that contains $\tilde{B}$, let $\tilde{R}^+_{\tilde{Q}}$ denote the positive roots of $\tilde{Q}$ or $L_{\tilde{Q}}$ and let $\Delta_{\tilde{Q}} := \Delta_{\tilde{G}}(\tilde{Q}) \subseteq \Delta_{\tilde{G}}$ denote the corresponding simple roots. Consider the following diagram, which is analogous to (7). By an abuse of notation we will use ϕ, ϕ_1, ϕ_2 to denote the induced map on Lie algebras.

$$\begin{tikzcd} \tilde{T}^{P,Q} \arrow[r, hook] \arrow[d, "\phi_2"] & \tilde{T}^P \arrow[r, two heads] \arrow[d, "\phi"] & \tilde{T}^Q \arrow[d, "\phi_1"] \\ \dfrac{T^{P,Q}}{lT_v^{P,Q}} \arrow[r, hook] & \dfrac{T^P}{lT_w^P} \arrow[r, two heads] & \dfrac{T^Q}{lv^{-1}T_u^Q} \end{tikzcd} \tag{10}$$

Since $w \in W^P$ is (L_P, ϕ)-movable, the map ϕ is an isomorphism for general $l \in L_P$. By the snake lemma, it suffices to show that ϕ_1 is injective. Let $\beta \in \tilde{\mathfrak{h}}^*$ denote the sum of roots corresponding to $\ker\phi_1$. Following the proof of Theorem 1.2, it suffices to show that $\beta(\tilde{x}_i) = 0$ for all $\tilde{\alpha}_i \in \Delta_{\tilde{G}} \backslash \Delta_{\tilde{Q}}$, since $\ker\phi_1 \subseteq \tilde{T}^Q$. Consider the group

$$C := \tilde{H} \cap Z(L_Q),$$

where $Z(L_Q)$ denotes the center of L_Q. Observe that $C \subseteq Z(L_{\tilde{Q}})$ and that $\mathrm{Lie}(C) = \tilde{\mathfrak{h}} \cap \mathfrak{z}$, where $\mathfrak{z}$ denotes the Lie algebra of $Z(L_Q)$. Since $C \subseteq \tilde{H}$, we have induced C-module structures on $\tilde{T}^P$, $\tilde{T}^Q$, and $\tilde{T}^{P,Q}$. Similarly, since $C \subseteq Z(L_Q)$, we have induced C-module structures on T^P, T^Q, and $T^{P,Q}$. It is easy to see that the maps ϕ, ϕ_1, and ϕ_2 are C-equivariant. Since ϕ is an isomorphism and ϕ_1 is surjective, the induced C-equivariant maps

$$\Phi\colon \det(\tilde{T}^P) \to \det(T^P/lT^P_w)$$

and

$$\Phi_1\colon \det(\tilde{T}^Q/\ker\phi_1) \to \det(T^Q/lv^{-1}T^Q_u)$$

are nonzero. Let $i\colon \tilde{G} \hookrightarrow G$ denote the embedding of $\tilde{G}$ into G, and define the character

$$\tilde{\chi}^{\tilde{P}} := 2(\tilde{\rho} - \tilde{\rho}^{\tilde{P}}),$$

where $\tilde{\rho}$ is the half-sum of all roots in $\tilde{R}^+$ and $\tilde{\rho}^{\tilde{P}}$ is the half-sum of all roots in $\tilde{R}^+_{\tilde{P}}$. Then $\tilde{\mathfrak{h}}$ acts on $\det(\tilde{T}^P)$ by the character $-\tilde{\chi}^{\tilde{P}}$. For any $\tau \in \mathrm{Lie}(C)$ we have

$$\beta(\tau) = (i^*\chi^Q_u - \tilde{\chi}^{\tilde{Q}})(\tau) = (i^*\chi^P_w - \tilde{\chi}^{\tilde{P}})(\tau) = 0,$$

since the isomorphisms Φ and Φ_1 are C-equivariant. By the assumptions of Theorem 1.5 there exists a vector $\tau_0 \in \mathrm{Lie}(C)$ such that $\alpha(\tau_0) \geq 0$ for any $\alpha \in \Delta_{\tilde{G}}$, with equality if and only if $\alpha \in \Delta_{\tilde{Q}}$. This implies that $\beta(\tilde{x}_i) = 0$ for all $\tilde{\alpha}_i \in \Delta_{\tilde{G}} \backslash \Delta_{\tilde{Q}}$ and hence that $\dim\ker\phi_1 = 0$, proving Theorem 1.5.

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