Michigan Math. J. 51 (2003)

The Finer Geometry and Dynamics of the Hyperbolic Exponential Family

Mariusz Urbański & Anna Zdunik

1. Introduction

Given $\lambda \in \mathbb{C} \setminus \{0\}$, let the entire function $f_\lambda : \mathbb{C} \to \mathbb{C}$ be defined by the formula

$$f_\lambda(z) = \lambda e^z.$$

McMullen [Mc] proved that the Hausdorff dimension of the set of points escaping to infinity under forward iterates of f_λ is equal to 2. In this paper we thoroughly investigate the geometric (fractal) and dynamical structure of the complement (in the Julia set $J(f_\lambda)$) of this set, which will be denoted in the sequel by $J_r(f_\lambda)$. Although our results apply to all functions f_λ with attracting periodic cycles, we perform our analysis in great detail assuming that $\lambda \in (0, 1/e)$ and treat the general case briefly in Section 6. (In a forthcoming paper we treat in the same spirit a large class of nonhyperbolic functions f_λ, including the case when $\lambda \in [1/e, \infty)$.) Since f is periodic with period $2\pi i$, it is natural to identify points that differ by $2k\pi i$ and to consider (instead of f) the map F, our main technical device, defined on some strip P of height 2π. Armed with the map F and the concept of tightness, we prove the existence and uniqueness of a probability conformal measure m (with an exponent greater than 1) for F and a σ-finite conformal measure for f. This powerful tool enables us in turn to prove that h_λ, the Hausdorff dimension of the set $J_r(f_\lambda)$, is less than 2, that the h_λ-dimensional Hausdorff measure of $J_r(f_\lambda)$ is positive and finite on each horizontal strip, and that the h_λ-dimensional packing measure of $J_r(f_\lambda)$ is locally infinite at each point of $J_r(f_\lambda)$.

The fact that $h_\lambda < 2$ shows in particular that the equality of the hyperbolic dimension and the Hausdorff dimension, conjectured in the theory of iteration of rational functions, fails in the context of transcendental entire functions.

Turning toward dynamics, we prove the existence and uniqueness of a Borel probability F-invariant ergodic measure equivalent with the conformal measure m. We do this by applying first the method of M. Martens to show the existence of a σ-finite F-invariant conservative ergodic measure equivalent with the measure m and then checking that this measure is finite.

Our paper is organized as follows. In Section 2 we prove that, for every λ, the Hausdorff dimension of the set $J_{\mathrm{bd}}(f_\lambda) = \{z \in J(f_\lambda) : \{f^n(z)\}$ is bounded$\}$

Received July 20, 2001. Revision received April 16, 2003.

The research of the first author was supported in part by NSF Grant no. DMS 9801583. The research of the second author was supported in part by Polish KBN Grant no. 2 PO3A 009 17.

is larger than 1. This does not require any assumption about hyperbolicity. We need this fact (which seems interesting in its own right) in Sections 2 and 6 for the proof of the existence of a conformal measure and in Section 5 for the existence of a Borel probability F-invariant ergodic measure equivalent with the conformal measure. Notice that Theorem 2.1 was already proved in [Ka] for the case of an attracting fixed point with λ real. In Sections 3–5 we give detailed proofs of the result just described in the case when f_λ has an attracting fixed point and λ is real. In Section 6 we show how to modify our arguments to make them work in the general case of an attracting periodic orbit. In the Appendix (Section 7) we provide an alternative direct proof of the fact that the Hausdorff dimension of the set $J_r(f_\lambda)$ is less than 2 without using the concept of conformal measures.

2. Bounded Orbits

Let

$$f_\lambda(z) = \lambda e^z, \quad \lambda \neq 0.$$

We shall prove the following.

THEOREM 2.1. *If $J_{\mathrm{bd}}(f_\lambda)$ is the set of all points $z \in J(f)$ such that $\{f_\lambda^n(z)\}_{n\ge 0}$, the forward orbit of z, is bounded, then* $\mathrm{HD}(J_{\mathrm{bd}}(f_\lambda)) > 1$.

Proof. Let $\log\lambda$ be the logarithm of λ satisfying $\mathrm{Im}\log\lambda \in (-\pi, \pi]$. Fix $R > 0$ and consider the square

$$S_R = (R, 2R) \times (R, 2R).$$

Let $\Pi = \{z \in \mathbb{C} : 0 \le \mathrm{Arg}(z) \le \pi/2\}$ be the first quadrant. For every $k \in \mathbb{Z}$ consider $l_k : \Pi \to \mathbb{C}$, the holomorphic branch of the map inverse to the map $z \mapsto \lambda e^z$ given by the formula

$$l_k(z) = -\log\lambda + \log|z| + i\,\mathrm{Arg}(z) + 2\pi i k, \quad 0 \le \mathrm{Arg}(z) \le \pi/2.$$

If $R > e^{|\log\lambda|}$ and $k \ge 1$ then $l_k(S_R) \subset \Pi$ and, for every $j \in \mathbb{Z}$,

$$\begin{aligned}\mathrm{Re}(l_j(l_k(z))) &= \log|l_k(z)| - \log|\lambda| \\ &= \log|-\log\lambda + \log|z| + i\,\mathrm{Arg}(z) + 2\pi i k| - \log|\lambda|.\end{aligned}$$

Define the set I_R to be

$$I_R = \Big\{k \ge 1 : R < \log\big(-|\log\lambda| + \big|\log\big(\sqrt{2}R\big) + 2\pi i k\big|\big) - \log|\lambda| \\ < \log\left(|\log\lambda| + \left|\log\big(2\sqrt{2}R\big) + \frac{5\pi}{2} i k\right|\right) - \log|\lambda| < 2R\Big\}$$

and, for every $k \in I_R$, put

$$I_{R,k} = \{j \ge 1 : R + 2\pi \le 2\pi j < 2R - 2\pi\}.$$

Notice that for every $k \in I_R$, $j \in \mathbb{Z}$, and $z \in S_R$, we have $R < \mathrm{Re}(l_j(l_k(z))) < 2R$; if $j \in I_{R,k}$, then

$$\mathrm{cl}(l_j \circ l_k(S_R)) \subset S_R.$$

We have produced in this way the finite family of maps

$$G_R = \{l_j \circ l_k \colon S_R \to S_R\}_{k \in I_R,\, j \in I_{R,k}}.$$

Each map $g \in G_R$ maps S_R conformally onto some topological disk whose closure is contained in S_R. Moreover, there exists a neighborhood $V \supset S_R$ such that each map $g \in G_R$ extends conformally to V, and it is easy to see that

$$\mathrm{cl}((l_j \circ l_k)(S_R)) \cap \mathrm{cl}((l_{j'} \circ l_{k'})(S_R)) = \emptyset$$

if $(j,k) \neq (j',k')$. Indeed, applying (for various $k \in I_R$) l_k to S_R, we obtain a collection of topological disks each of which is an image of the other by a translation $z \mapsto z + 2m\pi i$ for some $m \in \mathbb{Z}$. Each of these disks is contained in some horizontal strip of height $\pi/2$. It is therefore obvious that they are disjoint and that there exists a neighborhood $V \supset S_R$ such that the l_k extend conformally to V and $l_k(V) \cap l_{k'}(V) = \emptyset$. The sets $l_j(l_k(V)) \cap l_{j'}(l_{k'}(V))$ are disjoint for $k \neq k'$ because $l_k(V)$ and $l_{k'}(V)$ were already disjoint. Also, $l_j(l_k(V)) \cap l_{j'}(l_k(V)) = \emptyset$ for $j \neq j'$ because l_j and $l_{j'}$ are different branches of f_λ^{-1}. We define the compact set J_R as follows:

$$J_R = \bigcap_{n \geq 0} \bigcup_{g^n} g^n(S_R),$$

where we take the union over all possible compositions

$$g^n = g_{i_1} \circ \cdots \circ g_{i_n}, \quad g_{i_1}, \ldots, g_{i_n} \in G_R.$$

The map $f_\lambda|_{J_R} \colon J_R \to J_R$ is a conformal expanding repellor. In addition, it is easy to see that J_R is a Cantor set. For every $t \in \mathbb{R}$ the topological pressure $P_R(t)$ of the potential $-t\log|f_\lambda'|$ with respect to the repellor $f_\lambda|_{J_R} \colon J_R \to J_R$ can be calculated as follows:

$$P_R(t) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{g^n} \|(g^n)'\|^t,$$

where once again we sum up over all possible compositions

$$g^n = g_{i_1} \circ \cdots \circ g_{i_n}, \quad g_{i_1}, \ldots, g_{i_n} \in G_R.$$

It is well known (see [PU]; cf. [Bo]) that the Hausdorff dimension $t = \mathrm{HD}(J_R)$ of J_R is determined as the unique $t \in \mathbb{R}$ for which $P_R(t) = 0$. Since the function $t \mapsto P_R(t)$ is strictly decreasing, in order to prove that $\mathrm{HD}(J_R) > 1$ it is enough to show that $P_R(1) > 0$. Indeed, for $z \in S_R$ and all $k \in I_R$, $j \in I_{R,k}$, we have

$$\begin{aligned}
|(l_j \circ l_k)'(z)| = \frac{1}{|l_k(z)| \cdot |z|} &\geq \frac{1}{2\sqrt{2}R|-\log\lambda + \log|z| + i\,\mathrm{Arg}(z) + 2k\pi i|} \\
&\geq \frac{1}{2\sqrt{2}R(|\log\lambda| + |\log|z| + i\,\mathrm{Arg}(z) + 2k\pi i|)} \\
&\geq \frac{1}{2\sqrt{2}R\big(|\log\lambda| + \big|\log|z| + \frac{5}{2}k\pi i\big|\big)}. \qquad (2.1)
\end{aligned}$$

Let $|(l_j \circ l_k)'| = \inf\{|(l_j \circ l_k)'(z)| : z \in S_R\}$. Fix $t \geq 0$. Then, by (2.1),

$$\begin{aligned}
P_R(t) &\geq \log \sum_{k\in I_R} \sum_{j\in I_{R,k}} |(l_j \circ l_k)'|^t \\
&\geq \log \sum_{k\in I_R} \left(\frac{1}{2\sqrt{2}R}\right)^t \# I_{R,k} \left| |\log\lambda| + \log\big(2\sqrt{2}R\big) + \frac{5}{2}\pi ik \right|^{-t} \\
&\geq t\log\left(\frac{1}{2\sqrt{2}}\right) - t\log R + \log\left(\frac{R}{4\pi}\right) \\
&\quad + \log \sum_{k\in I_R} \left(|\log\lambda| + \left| \log\big(2\sqrt{2}R\big) + \frac{5}{2}\pi k \right| \right)^{-t},
\end{aligned}$$

where we have used the inequality $\# I_{R,k} \geq R/4\pi$, which is true for all R large enough. It follows from the definition of I_R that $\big(|\log\lambda| + \big|\log\big(2\sqrt{2}R\big) + \frac{5}{2}\pi ik\big|\big) \leq 4\pi k$, $\min(I_R) \leq e^{5R/4}$, and $\max(I_R) \geq e^{3R/2}$ for all R sufficiently large. Hence

$$\begin{aligned}
&P_R(t) \\
&\geq t\log\left(\frac{1}{2\sqrt{2}}\right) - t\log R + \log R - \log(4\pi) + \log \sum_{k=e^{5R/4}}^{e^{3R/2}} (4\pi k)^{-t} \\
&= t\log\left(\frac{1}{2\sqrt{2}}\right) - \log(4\pi) + \log R - t\log R - t\log(4\pi) + \log \sum_{k=e^{5R/4}}^{e^{3R/2}} k^{-t}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
P_R(1) &\geq \log\left(\frac{1}{2\sqrt{2}}\right) - 2\log(4\pi) + \log \sum_{k=e^{5R/4}}^{e^{3R/2}} k^{-1} \\
&\geq \log\left(\frac{1}{2\sqrt{2}}\right) - 2\log(4\pi) + \log(\log e^{3R/2} - \log e^{5R/4} - C) \\
&= \log\left(\frac{1}{2\sqrt{2}}\right) - 2\log 4\pi + \log\left(\frac{1}{4}R - C\right),
\end{aligned}$$

where $C > 0$ is a universal constant. Thus $P_R(1) > 0$ for R large enough and, consequently, $\mathrm{HD}(J_R) > 1$. By the definition of the set J_R we have $J_R \subset \{z : f_\lambda^{2n}(z) \in S_R \text{ for all } n \geq 0\}$. Since $|e^z| = e^{\mathrm{Re}(z)}$, we conclude that the forward orbit of each point in J_R is bounded for every $R > 0$. Since J_R is contained in the closure of fixed points (which are necessarily contracting) of all compositions of maps forming the system G_R, it follows that J_R is also contained in the closure of repelling periodic points of f, which in turn is contained in $J(f)$. Hence $J_R \subset J(f)$ and so $\mathrm{HD}(J_{\mathrm{bd}}(f_\lambda)) > 1$. □

We should like to point out that this result overlaps with those proven in [Ka]. More precisely, it follows from Theorem 2 in [Ka] (even though it is not stated explicitly there) that for $\lambda \in (0, 1/e)$ we have $\mathrm{HD}(J_{\mathrm{bd}}(f_\lambda)) > 1$. Unlike [Ka], however, we do not assume that λ is real and belongs to $(0, 1/e)$ nor that there exists an attracting fixed point of f.

The following observation, which concludes this section, can be deduced from [Ka, Thm. 2].

COROLLARY 2.2. *If* $\lambda \in (0, \infty)$, *then*

$$\lim_{\lambda \to 0} \mathrm{HD}(J_{\mathrm{bd}}(f_\lambda)) = 1.$$

3. Existence of Conformal Measure

From now on until the last section we assume that $\lambda \in (0, 1/e)$. Then $f = f_\lambda$ has a unique attracting fixed point $0 \in A_\lambda$, the basin of its immediate attraction, and $f_\lambda|_{\mathbb{R}}$ has another (positive, repelling) fixed point, which we denote by $q = q_\lambda$. Standard straightforward calculations show that

$$\{z : \mathrm{Re}(z) < q_\lambda\} \subset A_\lambda.$$

Let

$$P = \{z \in \mathbb{C} : -\pi < \mathrm{Im}(z) \le \pi\}$$

and let

$$P_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \ge q \text{ and } \mathrm{Im}(z) \in (-\pi, \pi]\}.$$

Fix $M > q_\lambda$ and set

$$P_M = \{z \in P : q_\lambda \le \mathrm{Re}(z) \le M\}.$$

Let

$$\pi_0 : \mathbb{C} \to P$$

be the projection given by $\pi_0(z) = w$ if and only if $w \in P$ and $e^z = e^w$. We define the map $F = F_\lambda : P \to P$ that we intend to work with by the formula

$$F(z) = \pi_0(f(z)). \tag{3.1}$$

In this section we construct a conformal measure for the map $F : P \cap J(f) \to P \cap J(f)$. Recall that a Borel measure m is called *t-conformal* (with $t > 0$) if, for any Borel set $A \subset P$ on which F is injective, we have

$$m(F(A)) = \int_A |F'|^t \, dm.$$

We shall frequently use the following obvious fact without explicitly invoking it.

THEOREM 3.1. *For any conformal measure m for $F : J(F) \to J(F)$ and any nonempty open subset U of $J(F)$ (in the relative topology on $J(F)$), $m(U) > 0$.*

Here, instead of the rectangle P_M, we consider a slightly modified rectangle. Indeed, notice that there exists a $p < q$ so close to q that, for every $M > q$, the set

$$\tilde{P}_M = \left\{z \in P : -\tfrac{3}{4}\pi < \mathrm{Im}\, z < \tfrac{3}{4}\pi,\ p < \mathrm{Re}\, z < M\right\}$$

is disjoint from the forward orbit of 0 under iterates of f. Consider the preimage $F^{-1}(\tilde{P}_M)$. This set is a union of infinitely many topological disks Q_i contained in the strip $-\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}$ (recall that the points $z \in P$ such that $|\operatorname{Im} z| > \frac{\pi}{2}$ are mapped into the region $\operatorname{Re} z < 0$, thus outside $\tilde{P}_M$). Moreover,

$$\overline{Q_i} \cap \overline{Q_j} = \emptyset.$$

Now we consider the finite family of disks Q_i^M whose closures are contained in $\tilde{P}_M$. In this way we obtain the finite iterated function system

$$\phi_i\colon \tilde{P}_M \to Q_i^M,$$

where ϕ_i is an appropriate holomorphic branch of F^{-1}. Let J_M be the limit set of this system and let m_M be the unique conformal measure. In this case this is simply the normalized Hausdorff measure with the exponent h_M equal to the Hausdorff dimension of J_M.

REMARK 3.2. We have $J_M \subset J_{M+1}$ for all M large enough. In order to see this, take Q_i^M and let Q_i^{M+1} be the preimage of $\tilde{P}_{M+1}$ under the same holomorphic branch F_*^{-1} of F^{-1}. Then, obviously, $Q_i^{M+1} \supset Q_i^M$. Since $F(Q_i^{M+1} \setminus Q_i^M) \subset \{z \in \tilde{P}_{M+1} : M < \operatorname{Re} z \le M+1\}$ and since the derivative of F_*^{-1} on $\{z \in \tilde{P}_{M+1} : M < \operatorname{Re} z \le M+1\}$ is bounded from above by $C_1 M^{-1}$, we conclude that $\operatorname{diam}(Q_i^{M+1} \setminus Q_i^M) \le C_2 M^{-1}$ for some appropriate constants C_1 and C_2. Since $Q_i^M \subset \{\operatorname{Re} z \le M\}$, this implies that

$$Q_i^{M+1} \subset \{\operatorname{Re} z \le M+1\}$$

for all M large enough. Hence, each $Q_i^{M+1} \supset Q_i^M$ is (see the definition) used in the construction of J_{M+1}. Thus, the corresponding limit set J_{M+1} contains J_M.

REMARK 3.3. We have $J_{\mathrm{bd}}(f) \cap P = \bigcup_{N=[q]+1}^{\infty} J_N$ and so, reasoning as in Remark 3.2, it follows from that remark and Theorem 2.1 that there exist $h_0 > 1$ and M_0 such that, for every $M > M_0$, $h_M = \operatorname{HD}(J_M) > h_0$.

PROPOSITION 3.4. *The sequence of measures m_M $(M \in \mathbb{N})$ is tight; that is, for every $\varepsilon > 0$ there exists an M so large that, for every N,*

$$m_N(\{z \in P : \operatorname{Re} z > M\}) < \varepsilon.$$

Proof. Fix $\varepsilon > 0$, $M > 0$, and $N \ge q$. We shall estimate separately the measure m_N of two sets, which cover $\{z \in P : \operatorname{Re} z > M\}$. First, we have

$$\begin{aligned} m_N(&\{x \in J_N : \operatorname{Re} F(x) \ge M\}) \\ &= \sum_{k \in \mathbb{Z}} m_N(\{x \in J_N : f(x) \in [M, N] \times (-\pi, \pi] + 2k\pi i\}). \end{aligned}$$

If $x \in J_N$ and $f(x) \in [M, N] \times [-\pi, \pi] + 2k\pi i$, then

$$|F'(x)| = |f(x)| \ge \tfrac{1}{2}(M + \pi|k|) \ge \tfrac{1}{2}(M + |k|),$$

which gives

$$m_N(\{x : \operatorname{Re} F(x) \geq M\}) \leq 2\sum_{k=0}^{\infty} m_N(\{x : M \leq \operatorname{Re} x \leq N\}) \cdot \frac{2^{h_N}}{(M+k)^{h_N}}$$
$$\leq 2^{h_N+1}\sum_{k=0}^{\infty}\frac{1}{(M+k)^{h_N}}, \tag{3.2}$$

where, let us recall, h_N is the exponent of the measure m_N. By Remark 3.3 and Remark 3.2 there exists a $T > q$ such that $h_N \geq h_T > 1$ for all $N \geq T$. If $N \leq M$, then

$$m_N(\{z \in P : \operatorname{Re} z > M\}) = 0. \tag{3.3}$$

If $M \geq T$ and $N > M$, then it follows from (3.2) that

$$m_N(\{x : \operatorname{Re} F(x) \geq M\}) \leq \frac{2^3}{h_N - 1}M^{1-h_N} \leq \frac{2^3}{h_T - 1}M^{1-h_T}. \tag{3.4}$$

Keeping $M \geq T$ and $N > M$, we now estimate the measure of the second set:

$$m_N(\{x : M < \operatorname{Re} x < N \text{ and } \operatorname{Re} F(x) < M\}).$$

If $\operatorname{Re} x > M$, then $|f(x)| > \lambda e^M$ and therefore $|\operatorname{Im} f(x)| \geq \sqrt{\lambda^2 e^{2M} - M^2}$. Thus,

$$m_N(\{x : M < \operatorname{Re} x < N \text{ and } \operatorname{Re} F(x) < M\})$$
$$\leq \text{const.} \sum_{k \geq (2\pi)^{-1}\sqrt{\lambda^2 e^{2M} - M^2}}^{\infty} (2\pi k)^{-h_N}$$
$$\leq \text{const.} \cdot \frac{1}{h_N - 1} e^{M(1-h_N)}$$
$$\leq \frac{\text{const.}}{h_T - 1} e^{M(1-h_T)}. \tag{3.5}$$

Combining this with (3.3) and (3.4) yields

$$m_N(\{x : \operatorname{Re} x > M\}) < \varepsilon$$

for all N and all M large enough. □

Since the sequence m_N is tight, it follows from Prochorov's theorem that there exists an increasing-to-infinity sequence $\{N_i\}_{i=1}^{\infty}$ such that the sequence $\{m_{N_i}\}_{i=1}^{\infty}$ weakly converges to some limit probability measure m. This is the measure we are looking for. Put

$$J(F) = P \cap J(f).$$

We shall prove the following.

THEOREM 3.5. *The measure m is h-conformal, where $h = \lim_{i\to\infty} h_{N_i}$ and $m(J(F)) = 1$.*

Proof. Since $J_M \subset J(F)$, $J(F)$ is closed, and $m_M(J_M) = 1$ for every $M > p$, it immediately follows from the definition of the measure m that $m(J(F)) = 1$.

In view of Remark 3.2, the sequence $\{h_N\}$ is eventually nondecreasing and hence the limit $\lim_{N\to\infty} h_N$ exists. Notice that each measure m_N is h_N-conformal for $F|_{J_N}$ but not for F itself (the set J_N is not backward invariant). However, if N is large enough then, for every Borel set $A \subset \{z : \operatorname{Re} z < N-1\}$ such that $F|_A$ is one-to-one, we have

$$m_N(F(A)) = \int_A |F'|^{h_N}\, dm_N. \tag{3.6}$$

To verify this, first we claim that

$$F(A) \cap J_N = F(A \cap J_N). \tag{3.7}$$

Indeed, $F(A \cap J_N) \subset F(A) \cap F(J_N) \subset F(A) \cap J_N$. To see the opposite inclusion, let $x \in F(A) \cap J_N$. Take $y \in A$ such that $F(y) = x$. Let Q be the component of $F^{-1}(\tilde{P}_N)$ containing y. We claim that Q is entirely contained in $\tilde{P}_N$, in other words, that Q is one of components Q_i^N used in the construction of J_N. Suppose, to the contrary, that Q intersects the line $\operatorname{Re} z = N$. Then for some $z \in Q$ we have $|f(z)| = |F'(z)| = \lambda e^N$. This means that Q is contained in a component of $f^{-1}(P_+ + 2k\pi i)$, where $k \geq (2\pi)^{-1}\sqrt{\lambda^2 e^{2N} - N^2}$. If N is large, this implies that

$$\operatorname{diam}(Q) \leq C\frac{N}{\lambda e^N} < 1.$$

But Q contains a point $y \in A$ and $A \subset P_{N-1}$. This contradiction shows that Q is entirely contained in $\tilde{P}_N$, that is, Q is one of components Q_i^N used in the construction of J_N. Since $x = F(y) \in J_N$, this implies that $y \in J_N$. The formula (3.7) is proved. Using (3.7), we can write

$$\begin{aligned} m_N(F(A)) &= m_N(F(A) \cap J_N) = m_N(F(A \cap J_N)) \\ &= \int_{A\cap J_N} |F'|^{h_N}\, dm_N = \int_A |F'|^{h_N}\, dm_N. \end{aligned}$$

Since the sequence $\{m_{N_i}\}$ converges weakly to m, we have

$$m_{N_i}(A) \to m(A)$$

for every Borel set A such that $m(\partial A) = 0$. In particular, this holds for every bounded Borel A such that $m(\partial A) = 0$ and $m(\partial F(A)) = 0$. For these sets A, using (3.6) yields

$$\begin{aligned} m(F(A)) &= \lim_{i\to\infty} m_{N_i}(F(A)) = \lim_{i\to\infty} \int_A |F'|^{h_{N_i}}\, dm_{N_i} \\ &= \int_A |F'|^h\, dm_{N_i} + \int_A (|F'|^{h_{N_i}} - |F'|^h)\, dm_{N_i}. \end{aligned}$$

The first summand converges to $\int_A |F'|^h\, dm$. The second summand can be estimated by $\sup_A(|F'|^{h_{N_i}} - |F'|^h)$. This tends to zero, since $|F'|$ is bounded on A and $h_{N_i} \to h$. Hence

$$m(F(A)) = \int_A |F'|^h\, dm. \tag{3.8}$$

Now take an arbitrary Borel set A such that $F|_A$ is injective; we can assume that A is bounded. Since $J(F) \subset \{z : \pi/2 \le \operatorname{Im} z \le \pi/2\}$ and thus (in the terminology of [DU1]) $\operatorname{Sing}(F\colon J(F) \to J(F)) = \emptyset$, and since $m(J(F)) = 1$, in order to verify the equality $m(F(A)) = \int_A |F'|^h \, dm$ it is enough to invoke [DU1, Lemma 2.4] and then apply (3.8). □

The existence of a conformal measure leads to the following straightforward corollary.

COROLLARY 3.6. *There exists a σ-finite measure $\tilde{m}$, which is h-conformal for $f|_{J(f)}$.*

Proof. Define $\tilde{m}$ on each strip $P_k = P + 2k\pi i$ as $m \circ \pi$, where (we recall) π is the natural projection of P_k onto P. Checking that $\tilde{m}$ is f-conformal is straightforward. Indeed, assume first that $A \subset P_n$ for some $n \in \mathbb{Z}$ and that $f|_A$ is injective. Let $Z_k = f^{-1}(P_k) \cap P$ for every $k \in \mathbb{Z}$ and let $\tilde{A} = A - 2\pi i n$. Then

$$\begin{aligned}
\tilde{m}(f(A)) &= \tilde{m}(f(\tilde{A})) = \sum_{k\in\mathbb{Z}} \tilde{m}(f(\tilde{A} \cap Z_k)) = \sum_{k\in\mathbb{Z}} m(\pi \circ f(\tilde{A} \cap Z_k)) \\
&= \sum_{k\in\mathbb{Z}} m(F(\tilde{A} \cap Z_k)) = \sum_{k\in\mathbb{Z}} \int_{\tilde{A}\cap Z_k} |F'|^h \, dm = \sum_{k\in\mathbb{Z}} \int_{\tilde{A}\cap Z_k} |f'|^h \, dm \\
&= \int_{\tilde{A}} |f'|^h \, dm = \int_A |f'|^h \, d\tilde{m}.
\end{aligned}$$

Now let $A \subset \mathbb{C}$ be an arbitrary Borel set on which f is injective, and let $A_k = A \cap P_k$. Since $A_k \cap A_j = \emptyset$ for $k \ne j$, we obtain

$$\tilde{m}(f(A)) = \sum_{k\in\mathbb{Z}} \tilde{m}(f(A_k)) = \sum_{k\in\mathbb{Z}} \int_{A_k} |f'|^h \, d\tilde{m} = \int_{A\in\mathbb{Z}} |f'|^h \, d\tilde{m}.$$

This ends the proof. □

Let

$$I_\infty(F) = \left\{ z \in P : \lim_{n\to\infty} F^n(z) = \infty \right\},$$

that is, $I_\infty(F)$ is the set of points escaping to infinity under forward iterates of F. Analogously define

$$I_\infty(f) = \left\{ z \in P : \lim_{n\to\infty} f^n(z) = \infty \right\}.$$

Let

$$J_r(F) = J(F) \setminus I_\infty(f) \quad \text{and} \quad J_r(f) = J(f) \setminus I_\infty(F),$$

and notice that $I_\infty(f) \cap P = I_\infty(F)$.

Let m be the h-conformal measure constructed in Theorem 3.5. We shall prove the following.

PROPOSITION 3.7. *There exists an $M > 0$ such that, for m-a.e. x,*

$$\liminf_{n\to\infty} \operatorname{Re} F^n(x) \le M.$$

In particular, $m(I_\infty(F)) = 0$ or equivalently $m(J_r(F)) = 1$.

Proof. Put

$$Y_M = \{z \in P : \operatorname{Re} z > M\}$$

and let $B \subset Y_M$ be an arbitrary Borel set. We shall estimate from above the measure $m(B \cap F^{-1}(B))$. We have

$$m(B \cap F^{-1}(B)) \le m(F^{-1}(B)) = \sum_{k\in\mathbb{Z}} m(x : f(x) \in B + 2k\pi i)$$

If $f(x) \in B + 2k\pi i$, then

$$|F'(x)| = |f'(x)| = |f(x)| > (M^2 + k^2)^{1/2}.$$

Therefore,

$$m(\{x : F(x) \in B\}) < 2\sum_{k=0}^{\infty} m(B) \cdot \frac{1}{(M^2 + k^2)^{h/2}} < \text{const.} \cdot m(B)M^{1-h}.$$

We thus obtain, in particular, that

$$m(B \cap F^{-1}(B)) < \frac{C}{M^{h-1}} m(B) \tag{3.9}$$

for every Borel set $B \subset Y_M$ and for some constant C independent of M and B. Since $B \cap F^{-1}(B) \subset Y_M$, one can now use the estimate (3.9) to get inductively

$$m(B \cap F^{-1}(B) \cap \cdots \cap F^{-n}(B)) < (CM^{1-h})^n m(B).$$

This implies that, for all M large enough,

$$m\left(\bigcap_{n=0}^{\infty} F^{-n}(Y_M)\right) = 0$$

and consequently

$$m\left(\bigcup_{k=0}^{\infty} F^{-k}\left(\bigcap_{n=0}^{\infty} F^{-n}(Y_M)\right)\right) = 0.$$

The proof is finished. $\square$

Let us now show that the estimates used in Proposition 3.7 and Proposition 3.4 lead to the following.

COROLLARY 3.8.

$$m(Y_M) < Ce^{(1-h)M}$$

for some constant C and all $M \ge 0$ large enough.

Proof. It follows from the proof of Proposition 3.7 that

$$m(\{x \in Y_M : F(x) \in Y_M\}) < m(Y_M) C M^{1-h},$$

and by the proof of Proposition 3.4 (formula (3.5)), with m_N replaced by m, we have that

$$m(\{x \in Y_M : \operatorname{Re} F(x) \leq M\}) < C e^{(1-h)M}.$$

These two sets cover the entire set Y_M. The first inequality says that (for all M sufficiently large) the first set covers less than, say, half the measure of Y_M. Thus,

$$m(Y_M) \leq 2m(\{x \in Y_M : \operatorname{Re} F(x) \leq M\}) < 2Ce^{(1-h)M}$$

and the proof is complete. □

4. Conformal, Hausdorff, and Packing Measures; Hausdorff Dimension

Let again $f = f_\lambda$, $q = q_\lambda$, and $F = F_\lambda$. Recall that

$$J(F) = J(f) \cap ([q, \infty) \times [-\pi, \pi]) = J(f) \cap ([q, \infty) \times [-\pi/2, \pi/2]).$$

Recall also that

$$P_+ = \{z \in \mathbb{C} : \operatorname{Re}(z) \geq q \text{ and } \operatorname{Im}(z) \in (-\pi, \pi]\}.$$

Fix some $R > q$. Consider a countable partition $\alpha = \{A_n : n \geq 0\}$ of P_+ defined as follows:

$$\begin{aligned} A_0 &= \{z \in P_+ : \operatorname{Re} z \leq R\}, \\ A_1 &= \{z \in P_+ : R < \operatorname{Re} z \leq R+1\}, \\ A_n &= \{z \in P_+ : R+n-1 < \operatorname{Re} z \leq R+n\} \quad \text{for } n \geq 1. \end{aligned}$$

We start this section with two technical lemmas.

LEMMA 4.1. *If the constant R is large enough* (*depending on* λ), *then for every $k \geq 0$ we have*

$$F(A_k) \supset A_0 \cup A_1 \cup \cdots \cup A_{k+1}.$$

Proof. Let $k \geq 1$. Then $f(A_k)$ is an annulus centered at 0 and bounded by two circles of radii λe^{R+k-1} and λe^{R+k}.

Let z_0 be the point in the outer circle such that $\operatorname{Re} z_0 = \lambda e^{R+k-1}$ and $\operatorname{Im} z_0 > 0$. A straightforward geometrical argument shows that if $R > 0$ is taken so large that

$$\lambda e^{R+k-1}\big(\sqrt{e^2-1} - 1\big) > 4\pi \quad \text{for all } k \geq 1,$$

then $f(A_k)$ contains some rectangle

$$0 < \operatorname{Re} z < \operatorname{Re} z_0, \qquad \operatorname{Im} z_0 - 4\pi < \operatorname{Im} z < \operatorname{Im} z_0.$$

If, moreover, $R > 0$ is taken so large that $\operatorname{Re} z_0 = \lambda e^{R+k-1} > k+1+R$, then this rectangle contains some component of the set $\pi_0^{-1}(A_0 \cup \cdots \cup A_{k+1})$. So, by definition,

$$F(A_k) \supset A_0 \cup \cdots \cup A_{k+1}.$$

It remains to check the case when $k = 0$. But $f(A_0)$ is the annulus of inner radius q and outer radius λe^R. If R is large, then this set contains $A_0 \cup A_1 = \{z \in P : q \le \operatorname{Re} z < R+1\}$. □

From now on in this section, fix the partition α satisfying the statement of Lemma 4.1. As an immediate consequence of this lemma we have the following.

COROLLARY 4.2. *For every $k \ge 0$,*

$$\lim_{n\to\infty} m(F^n(A_k)) = 1.$$

LEMMA 4.3. *For every $x \in J(F)$ and every $r > 0$,*

$$\lim_{n\to\infty} m(F^n(B(x,r))) = 1.$$

Proof. For every $k \ge 0$, let $A_k(x)$ be the element of partition α containing $F^k(x)$. Denote by $B_k(x)$ the component of $F^{-k}(A_k(x))$ containing x. Since diameters of A_k are bounded and since F is expanding on its Julia set, it follows that $\operatorname{diam}(B_k(x)) \to 0$ as $k \to \infty$. Hence for some $k \in \mathbb{N}$ we have $B(x,r) \supset B_k(x)$. Thus, for every $n \ge 0$,

$$F^{n+k}(B(x,r)) \supset F^{n+k}(B_k(x)) \supset F^n(A_k),$$

and the lemma follows from Corollary 4.2. □

Let us now prove the following.

THEOREM 4.4. *The h-conformal measure m is a unique t-conformal measure for F with $t > 1$. In addition, it is conservative and ergodic.*

Proof. Suppose that ν is a t-conformal measure for F with some $t > 1$. The same proof as in the case of the measure m shows that $\nu(I_\infty(F)) = 0$. Let $J_{r,N}(F)$ be the subset of $J_r(F)$ defined as follows: $z \in J_{r,N}(F)$ if the trajectory of z under F has an accumulation point in $\{\operatorname{Re} z < N\}$. Obviously, $\bigcup_N J_{r,N}(F) = J_r(F)$ and, by Proposition 3.7, there exists an $M > 0$ such that $\nu(J_{r,M}(F)) = m(J_{r,M}(F)) = 1$. Fix $z \in J_{r,N}(F)$. Then there exist $y \in J(F)$ such that $\operatorname{Re} y < N$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $y = \lim_{k\to\infty} F^{n_k}(z)$. Now consider (for k large enough) the sets $F_z^{-n_k}(B(y,\pi/4))$ and $F_z^{-n_k}(B(y,\pi/(4K)))$, where $F_z^{-n_k}$ is the holomorphic inverse branch of F^{n_k} defined on $B(y,\pi/2)$ and sending $F^{n_k}(z)$ to z; then, using conformality of measures m and ν along with Koebe's distortion theorem, we easily deduce that

$$B_N(\nu)^{-1}|(F^{n_k})'(z)|^{-t} \le \nu(B(z, c|(F^{n_k})'(z)|^{-1})) \le B_N(\nu)|(F^{n_k})'(z)|^{-t} \tag{4.1}$$

and

$$\begin{aligned} B_N(m)^{-1}|(F^{n_k})'(z)|^{-h} &\le m(B(z, c|(F^{n_k})'(z)|^{-1})) \\ &\le B_N(m)|(F^{n_k})'(z)|^{-h} \end{aligned} \tag{4.2}$$

for all $k \geq 1$ large enough, where $K = 16$ is the constant appearing in the Koebe distortion theorem and ascribed to the scale $1/2$ and where $B_N(\nu)$ is some constant depending on ν and N. Let M be fixed as before. Fix now E, an arbitrary bounded Borel set contained in $J_r(F)$, and let $E' = E \cap J_{r,M}(F)$. Since m is regular, for every $x \in E'$ there exists a radius $r(x) > 0$ of the form from (4.1) such that

$$m\left(\bigcup_{x\in E'} B(x, r(x)) \setminus E\right) < \varepsilon. \tag{4.3}$$

By the Besicovič theorem (see [G]) we can now choose a countable subcover $\{B(x_i, r(x_i))\}_{i=1}^{\infty}$, $r(x_i) \leq \varepsilon$, from the cover $\{B(x, r(x))\}_{x\in E'}$ of E of multiplicity bounded by some constant $C \geq 1$ that is independent of the cover. Hence, by (4.1), (4.2), and (4.3) we obtain

$$\begin{aligned}
\nu(E') = \nu(E) &\leq \sum_{i=1}^{\infty} \nu(B(x_i, r(x_i))) \leq B_M(\nu) \sum_{i=1}^{\infty} r(x_i)^t \\
&\leq B_M(\nu) B_M(m) \sum_{i=1}^{\infty} r(x_i)^{t-h} m(B(x_i, r(x_i))) \\
&\leq B_M(\nu) B_M(m) C \varepsilon^{t-h} m\left(\bigcup_{i=1}^{\infty} B(x_i, r(x_i))\right) \\
&\leq C B_M(\nu) B_M(m) \varepsilon^{t-h} (\varepsilon + m(E')) \\
&= C B_M(\nu) B_M(m) \varepsilon^{t-h} (\varepsilon + m(E)).
\end{aligned} \tag{4.4}$$

In the case when $t > h$, letting $\varepsilon \searrow 0$ yields $\nu(E) = 0$ and consequently $\nu(J(F)) = 0$, which is a contradiction. We obtain a similar contradiction assuming that $t < h$ and switching in (4.4) the roles of m and ν. Thus $t = h$ and, letting $\varepsilon \searrow 0$, we obtain from (4.4) that $\nu(E) \leq C B_M(\nu) B_M(m) m(E)$. Exchanging m and ν, we obtain $m(E) \leq C B_M(\nu) B_M(m) \nu(E)$. These two conclusions, along with the already mentioned fact that $m(J_r(F)) = \nu(J_r(F)) = 1$, imply that the measures m and ν are equivalent with Radon–Nikodym derivatives bounded away from zero and infinity.

Let us now prove that any h-conformal measure ν is ergodic. Indeed, suppose to the contrary that $F^{-1}(G) = G$ for some Borel set $G \subset J(F)$ with $0 < m(G) < 1$. But then the two conditional measures

$$\nu_G(B) = \frac{\nu(B \cap G)}{\nu(G)} \quad \text{and} \quad \nu_{J(F)\setminus G}(B) = \frac{\nu(B \cap J(F) \setminus G)}{\nu(J(F) \setminus G)}$$

would be h-conformal and mutually singular; a contradiction.

If now ν is again an arbitrary h-conformal measure, then by a simple computation (based on the definition of conformal measures) we see that the Radon–Nikodyn derivative $\phi = d\nu/dm$ is constant on grand orbits of F. Therefore, by ergodicity of m, we conclude that ϕ is constant m-almost everywhere. Since both m and ν are probability measures, this implies that $\phi = 1$ a.e. and hence $\nu = m$.

It remains to show that m is conservative. We shall prove first that every forward invariant ($F(E) \subset E$) subset E of $J(F)$ is either of measure 0 or 1. Indeed, suppose to the contrary that $0 < m(E) < 1$. Since $m(I_\infty(F)) = 0$, it suffices to show that

$$m(E \setminus I_\infty(F)) = 0.$$

Denote by Z the set of all points $z \in E \setminus I_\infty(F)$ such that

$$\lim_{r \to 0} \frac{m(B(z,r) \cap (E \setminus I_\infty(F)))}{m(B(z,r))} = 1. \tag{4.5}$$

In view of the Lebesgue density theorem (see e.g. [Fe, Thm. 2.9.11]), $m(Z) = m(E)$. Since $m(E) > 0$ we find at least one point $z \in Z$. Since $z \in J(F) \setminus I_\infty(F)$, there exist $x \in J(F)$ and an increasing sequence $\{n_k\}_{k=1}^\infty$ such that $x = \lim_{k\to\infty} F^{n_k}(z)$. Let

$$\delta = \min\{\pi/8, q/4\}.$$

Suppose that $m(B(x,\delta) \setminus E) = 0$. By conformality of m, $m(F(Y)) = 0$ for all Borel sets Y such that $m(Y) = 0$. Hence,

$$\begin{aligned} 0 &= m(F^n(B(x,\delta) \setminus E)) \geq m(F^n(B(x,\delta)) \setminus F^n(E)) \\ &\geq m(F^n(B(x,\delta)) \setminus E) \geq m(F^n(B(x,\delta)) - m(E) \end{aligned} \tag{4.6}$$

for all $n \geq 0$. By Lemma 4.3, $\lim_{n\to\infty} m(F^n(B(x,\delta)) = 1$. Then (4.6) implies that $0 \geq 1 - m(E)$, which is a contradiction. Consequently $m(B(x,\delta) \setminus E) > 0$. Hence, for every $j \geq 1$ large enough, $m(B(F^{n_j}(z), 2\delta) \setminus E) \geq m(B(x,\delta) \setminus E) > 0$. Therefore, since $F^{-1}(J(F) \setminus E) \subset J(F) \setminus E$, a standard application of Koebe's distortion theorem shows that

$$\limsup_{r \to 0} \frac{m(B(z,r) \setminus E)}{m(B(z,r))} > 0,$$

which contradicts (4.5). Thus either $m(E) = 0$ or $m(E) = 1$.

Conservativity is now straightforward. We need to show that, for every Borel set $B \subset J(F)$ with $m(B) > 0$, we have $m(G) = 0$, where

$$G = \left\{ x \in J(F) : \sum_{n \geq 0} \chi_B(F^n(x)) < +\infty \right\}.$$

Indeed, suppose that $m(G) > 0$ and, for all $n \geq 0$, let

$$\begin{aligned} G_n &= \left\{ x \in J(F) : \sum_{k \geq n} \chi_B(F^n(x)) = 0 \right\} \\ &= \{x \in J(F) : F^k(x) \notin B \text{ for all } k \geq n\}. \end{aligned}$$

Since $G = \bigcup_{n \geq 0} G_n$, there exists a $k \geq 0$ such that $m(G_k) > 0$. Since all the sets G_n are forward invariant, we conclude that $m(G_k) = 1$. But on the other hand, all the sets $F^{-n}(B)$, $n \geq k$, are of positive measure and are disjoint from G_k. This contradiction finishes the proof. □

In the proof of the following theorem (as well as in the proofs of Proposition 4.8 and Theorem 4.9) we use various forms of the converse Frostman's type lemmas (see e.g. [DU3; PU, Chap. 6]).

THEOREM 4.5. *If $\lambda \in (0, 1/e)$, then the h-dimensional Hausdorff measure H^h of $J_r(F)$ is finite, the measure H^h of $J_r(f_\lambda)$ is σ-finite, and*

$$h_\lambda = \mathrm{HD}(J_{\mathrm{bd}}(f_\lambda)) = \mathrm{HD}(J_r(f_\lambda)) < 2,$$

where h_λ is the exponent of the conformal measure $m = m_\lambda$ (see Theorem 3.5 and Theorem 4.4).

Proof. Fix $\lambda \in (0, 1/e)$. Put $f = f_\lambda$ and $h = h_\lambda$. By the definition of the numbers h_N (see the beginning of Section 4) and Theorem 3.5, $h \le \mathrm{HD}(J_{\mathrm{bd}}(f))$. It follows from (4.1) applied with the measure m that the h-dimensional Hausdorff measure $\mathrm{H}^h(J_{r,M}(F))$ is finite. Since $m(J_{N,r}(F) \setminus J_{r,M}(F)) = 0$, we deduce in a similar way (using again (4.1)) that $\mathrm{H}^h(J_{r,N}(F) \setminus J_{r,M}(F)) = 0$ for all $N > M$. Since $\bigcup_{N \ge M} J_{r,N}(F) = J_r(F)$, we thus conclude that $\mathrm{H}^h(J_r(F)) = \mathrm{H}^h(J_{r,M}(F)) < \infty$ and consequently $\mathrm{HD}(J_r(F)) \le h$.

Since $J_r(f) = \bigcup_{n \in \mathbb{Z}} (J_r(F) + 2\pi i n)$, we thus conclude that $\mathrm{H}^h|_{J_r(f)}$ is σ-finite and that $\mathrm{HD}(J_r(f)) \le h$. It therefore remains to demonstrate that $\mathrm{HD}(J_r(F)) < 2$. For otherwise, it would follow from (4.1) and (4.4), with the measure ν replaced by m and m replaced by planar Lebesgue measure, that the planar Lebesgue measure of $J_r(F)$ is positive. This would, however, contradict McMullen's result [Mc], which finishes the proof. □

An alternative direct proof—not using the concept of conformal measures—of the fact that $\mathrm{HD}(J_r(f_\lambda)) < 2$ is provided in Corollary 7.3. Recall that in [DU2] (cf. [PU]) the dynamical dimension, proven in [PU] to be equal to the hyperbolic dimension, was defined as the supremum of Hausdorff dimensions of all probability-invariant ergodic measures with positive entropy. It has been conjectured that, in the case of rational functions, the dynamical dimension and the Hausdorff dimension of the Julia set coincide. Since each Borel probability f_λ-invariant measure is (by Poincaré's recurrence theorem) supported on $J_r(f)$, as an immediate consequence of Theorem 4.5 we get the following corollary, which disproves this conjecture in the case of transcendental entire functions.

COROLLARY 4.6. *If $\lambda \in (0, 1/e)$, then the supremum of Hausdorff dimensions of all probability f_λ-invariant ergodic measures is less than the Hausdorff dimension of the Julia set of f_λ.*

THEOREM 4.7. *The function $\lambda \mapsto \mathrm{HD}(J_r(f_\lambda))$ is continuous in the interval $(0, 1/e)$.*

Proof. Fix $\lambda \in (0, 1/e)$ and a sequnce $\lambda_n \in (0, 1/e)$ converging to λ. Since there exist quasiconformal conjugacies between the maps f_{λ_n} and f_λ with dilation constants converging to 1 when $n \to \infty$, the required fact follows. □

Let P^h be the h-dimensional packing measure (see [TT]; cf. e.g. [PU] for its definition and some basic properties). The last three results of this section provide (in a sense) a complete description of the geometrical structure of the sets $J_r(F)$ and $J_r(f)$, and they also exhibit the geometrical meaning of the h-conformal measure m.

PROPOSITION 4.8. *We have* $\mathrm{P}^h(J_r(f)) = \infty$; *in fact,* $\mathrm{P}^h(G) = \infty$ *for every open nonempty subset of* $J_r(f)$.

Proof. Since $m(J_r(F) \cap (P \setminus P_M)) > 0$ for every $M \in \mathbb{R}$, it follows from Birkhoff's ergodic theorem and Theorem 5.2 (whose proof is obviously independent of the results proven in the remainder of this section) that there exists a set $E \subset J_r(F)$ such that $m(E) = 1$ and

$$\limsup_{n \to \infty} \operatorname{Re} F^n(z) = \infty \tag{4.7}$$

for every $z \in E$. Fix $z \in E$ and $n \geq 1$, and consider the ball $B(z, K^{-1}|(F^n)'(z)|^{-1})$, where $K = 16$ is the Koebe constant corresponding to the scale $1/2$. Then

$$B(z, K^{-1}|(F^n)'|(z)|^{-1}) \subset F_z^{-n}(B(F^n(z), 1)),$$

where $F_z^{-n} \colon B(F^n(z), 1) \to \mathbb{C}$ is the analytic inverse branch of F^n mapping $F^n(z)$ to z. Applying Koebe's distortion theorem, conformality of the measure m, and Corollary 3.8, we obtain

$$\begin{aligned}
&m(B(z, K^{-1}|(F^n)'(z)|^{-1}) \\
&\quad \leq K^h |(F^n)'(z)|^{-h} m(B(F^n(z), 1)) \\
&\quad \leq K^{2h} (K^{-1}|(F^n)'(z)|^{-1})^h m(Y_{\operatorname{Re} F^n(z) - 1}) \\
&\quad \leq K^{2h} C \exp((1-h)(\operatorname{Re} F^n(z) - 1))(K^{-1}|(F^n)'(z)|^{-1})^h.
\end{aligned}$$

Hence, using (4.7), we conclude that

$$\liminf_{r \to 0} \frac{m(B(z, r))}{r^h} = 0.$$

Since $m(G \cap J_r(F)) > 0$ for every nonempty open subset of $J_r(F)$, this implies (see an appropriate converse Frostman's type lemma in [DU3] or [PU]) that $\mathrm{P}^h(G) = \infty$. Since $J_r(f) = \bigcup_{k \in \mathbb{Z}} (J_r(F) + 2\pi i k)$, we are therefore done. □

THEOREM 4.9. $0 < \mathrm{H}^h(J_r(F)) < \infty$.

Proof. We know from Theorem 4.5 that $\mathrm{H}^h(J_r(F)) < \infty$, so we need only show that $\mathrm{H}^h(J_r(F)) > 0$. We have $m(J_r(F)) = 1$ and thus it suffices to demonstrate that, for every $z \in J_r(F)$ and all $r > 0$ sufficiently small (depending on z),

$$m(B(z, r)) \leq C r^h$$

for some constant $0 \leq C < \infty$ independent of z and r. Indeed, put

$$\theta = \min\{\pi, \operatorname{dist}(J(F), \{f^k(0) : k \geq 0\})\}.$$

Fix $z \in J_r(F)$, $0 < r \le \theta(32|f'(z)|)^{-1}$. Since $F\colon J(F) \to J(F)$ is an expanding map, there exists a largest $n \ge 1$ such that

$$r|(f^n)'(z)| \le \frac{\theta}{32}. \tag{4.8}$$

Thus

$$r|(f^{n+1})'(z)| > \frac{\theta}{32}. \tag{4.9}$$

It follows from the definition of θ that the holomorphic inverse branch $f_z^{-n}\colon B(f^n(z),\theta) \to \mathbb{C}$ of f^n, sending $f^n(z)$ to z, is well-defined. Since $f|_{B(f^n(z),\theta)}$ is one-to-one and since, by Koebe's $\frac{1}{4}$-theorem, $f(B(f^n(z),\theta)) \supset B\big(f^{n+1}(z), \frac{1}{4}\theta|f'(f^n(z))|\big)$, we conclude that the holomorphic inverse branch $f_z^{-(n+1)}$: $B\big(f^{n+1}(z), \frac{1}{4}\theta|f'(f^n(z))|\big) \to \mathbb{C}$ of f^{n+1}, mapping $f^{n+1}(z)$ to z, is well-defined. Since

$$4r|(f^{n+1})'(z)| = 4r|(f^n)'(z)| \cdot |f'(f^n(z))| = \theta\big(\tfrac{32}{\theta}r|(f^n)'(z)|\big) \cdot \tfrac{1}{8}|f'(f^n(z))|$$

and since, by (4.8), $\frac{32}{\theta}r|(f^n)'(z)| \le 1$, we conclude that

$$4r|(F^{n+1})'(z)| \le \tfrac{1}{8}\theta|f'(f^n(z))|.$$

Applying Koebe's $\frac{1}{4}$-theorem again, we see that

$$\begin{aligned} f_z^{-(n+1)}(B(f^{n+1}(z), 4r|(f^{n+1})'(z)|)) &\supset B(z, |(f^{n+1})'(z)|^{-1} r|(f^{n+1})'(z)|) \\ &= B(z, r). \end{aligned}$$

The ball $B(f^{n+1}(z), 4r|(f^{n+1})'(z)|)$ intersects at most $\frac{1}{2\pi}4r|(f^{n+1})'(z)| + 1 \le r|(f^{n+1})'(z)|$ horizontal strips of the form $2\pi ik + P$ ($k \in \mathbb{Z}$); therefore, using Koebe's distortion theorem, h-conformality of the measure $\tilde{m}$, and (4.9), we obtain

$$\begin{aligned} &r^{-h}(m(B(z,r)) \\ &\quad \le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|)m(\pi_0(B(f^{n+1}(z), 4r|(f^{n+1})'(z)|))) \\ &\quad \le r^{-h}K^h|(f^{n+1})'(z)|^{-h}(r|(f^{n+1})'(z)|) \\ &\quad = K^h(r|(f^{n+1})'(z)|)^{1-h} \le K^h\big(\tfrac{32}{\theta}\big)^{h-1}, \end{aligned}$$

where $K = 16$ is the Koebe constant corresponding to the scale $1/2$. We are done by applying an appropriate converse Frostman's type lemma. □

As an immediate consequence of this theorem we obtain the following.

COROLLARY 4.10. *The h-dimensional Hausdorff measure of the set J_r is positive.*

5. Invariant Measures

In order to prove Theorem 5.2, we must apply a general sufficient condition (proven in [Ma]) for the existence of σ-finite absolutely continuous invariant measure. In order to formulate this condition, suppose X is a σ-compact metric space, ν a Borel probability measure on X that is positive on open sets, and that a measurable map $T\colon X \to X$ is given with respect to which the measure ν is quasi-invariant, that is, $\nu \circ T^{-1} \ll \nu$. Moreover, we assume the existence of a countable partition $\alpha = \{A_n : n \geq 0\}$ of subsets of X that are all σ-compact and of positive measure ν. We also assume that $\nu\big(X \setminus \bigcup_{n\geq 0} A_n\big) = 0$, and if there exists a $k \geq 0$ such that

$$\nu(T^{-k}(A_m) \cap A_n) > 0 \quad \text{for all } \ m, n \geq 1,$$

then the partition α is called *irreducible.* Martens's result reads as follows.

THEOREM 5.1 [Ma, Prop. 2.6, Thm. 2.9]. *Suppose that $\alpha = \{A_n : n \geq 0\}$ is an irreducible partition for $T\colon X \to X$, and suppose that T is conservative and ergodic with respect to the measure ν. If for every $n \geq 1$ there exists $K_n \geq 1$ such that for all $k \geq 0$ and all Borel subsets A of A_n we have*

$$K_n^{-1}\frac{\nu(A)}{\nu(A_n)} \leq \frac{\nu(T^{-k}(A))}{\nu(T^{-k}(A_n))} \leq K_n\frac{\nu(A)}{\nu(A_n)},$$

then T has a σ-finite T-invariant measure μ that is absolutely continuous with respect to ν. In addition, μ is equivalent with ν, conservative and ergodic, and unique up to a multiplicative constant. Moreover, for every Borel set $A \subset X$,

$$\mu(A) = \lim_{n\to\infty} \frac{\sum_{k=0}^{n} \nu(T^{-k}(A))}{\sum_{k=0}^{n} \nu(T^{-k}(A_0))}.$$

The main result of this section is the following.

THEOREM 5.2. *There exists a probability F-invariant measure μ that is absolutely continuous with respect to h-conformal measure m. In addition, μ is equivalent with m and ergodic.*

Proof. Let us first prove that there exists a σ-finite ergodic F-invariant measure μ that is equivalent with m. Let α be the partition constructed at the beginning of Section 4 with the constant $R > 0$ sufficiently large (as required in Lemma 4.1). In view of Koebe's distortion theorem, there exists a constant $K \geq 1$ such that, if $F_*^{-n}\colon P \to P$ is a holomorphic branch of F^{-n}, then for every $k \geq 0$ and all $x, y \in A_k$ we have

$$\frac{|(F_*^{-n})'(y)|}{|(F_*^{-n})'(x)|} \leq K. \tag{5.1}$$

We thus obtain, for all Borel sets $A, B \subset A_k$ with $m(B) > 0$ and all $n \geq 0$, that

$$\frac{m(F_*^{-n}(A))}{m(F_*^{-n}(B))} = \frac{\int_A |(F_*^{-n})'|^h \, dm}{\int_A |(F_*^{-n})'|^h \, dm} \leq \frac{\sup_{A_k}\{|(F_*^{-n})'|^h\} m(A)}{\inf_{A_k}\{|(F_*^{-n})'|^h\} m(B)} \leq K^h \frac{m(A)}{m(B)}.$$

Therefore,

$$\begin{aligned} m(F^{-n}(A)) &= \sum_* m(F_*^{-n}(A)) \leq \sum_* K^h m(F_*^{-n}(B)) \frac{m(A)}{m(B)} \\ &= K^h m(F^{-n}(B)) \frac{m(A)}{m(B)}, \end{aligned} \tag{5.2}$$

where the summation is taken over all holomorphic inverse branches of F^n. In view of Lemma 4.3, for every $k \geq 0$ and every $l \geq 0$ there exist $n_{k,l} \geq 0$ such that

$$F^{n_{k,l}}(A_k) \supset A_l. \tag{5.3}$$

Applying now (5.2) and (5.3) along with Theorem 4.4 and Theorem 5.1 concludes the proof of the existence of the required σ-finite measure μ.

It only remains to show that μ is finite. And indeed, fix $0 < p < q$ with the same requirements as in the definition of $\tilde{P}_M$ in the beginning of Section 4. Each holomorphic branch $F_*^{-j} : P \to P$ of F^{-j} restricted to the set $A_0 \cup A_1 \cdots \cup A_n$ extends in a holomorphically univalent fashion to the set $\{z \in \mathbb{C} : p < \operatorname{Re} z < R + n + 1$ and $-2n\pi \leq \operatorname{Im} z \leq 2n\pi\}$; hence it follows from Koebe's distortion theorem that there exists a constant $C_1 \geq 1$ such that, for every $n \geq 0$, all $x \in A_0$, and all $y \in A_n$, we have

$$\frac{|(F_*^{-j})'(y)|}{|(F_*^{-j})'(x)|} \leq C_1(Rn)^3.$$

Therefore, using Lemma 3.8, we obtain

$$\frac{m(F_*^{-j}(A_n))}{m(F_*^{-j}(A_0))} \leq C_1(Rn)^3 \frac{m(A_n)}{m(A_0)} \leq C_1(Rn)^3 C m(A_0)^{-1} e^{(1-h)Rn}.$$

Hence

$$\frac{m(F^{-j}(A_n))}{m(F^{-j}(A_0))} \leq C_1(Rn)^3 C m(A_0)^{-1} e^{(1-h)(R+n-1)}$$

and consequently, for every $k \geq 0$,

$$\frac{\sum_{j=0}^{k} m(F^{-j}(A_n))}{\sum_{j=0}^{k} m(F^{-j}(A_0))} \leq C_1(Rn)^3 C m(A_0)^{-1} e^{(1-h)(R+n-1)}.$$

Thus, applying Theorem 5.1 yields

$$\mu(A_n) = \lim_{k\to\infty} \frac{\sum_{j=0}^{k} m(F^{-j}(A_n))}{\sum_{j=0}^{k} m(F^{-j}(A_0))} \leq C_1(Rn)^3 C m(A_0)^{-1} e^{(1-h)(R+n-1)}.$$

Since $R > 0$, we finally get $\mu(J(F)) = \sum_{n\geq 0} \mu(A_n) < \infty$. We are done. $\square$

6. General Hyperbolic Case

In this section we outline the argument showing that the phenomenon described previously holds also for every map $f_\lambda = \lambda e^z$ such that f_λ has an attracting periodic orbit.

We decided to write the details of the proof for the particular case of the attracting fixed point because the dynamics is very simple in this case. On the other hand, the extension of the arguments for the general hyperbolic case is rather straightforward, but it requires some extra information about the structure of the Julia set (see [BD]). So, in what follows we rely on the description given in [BD] as well as the notation of that paper. We recall it briefly: $z_0, \dots, z_n = z_0$ is an attracting cycle of f. Assume that the singular value 0 is contained in the domain A_1, the immediate basin of attraction of z_1. The topological disk B_{n+1} containing z_1 is chosen so that $0 \in B_{n+1}$ and $f^n(B_{n+1}) \subset B_{n+1}$. Then B_n is defined as $B_n = f^{-1}(B_{n+1})$. The set B_n contains some half-plane $\operatorname{Re} z < -M$ and $z_0 \in B_n$.

For $j = 1, \dots, n$, let B_{n-j} be the connected component of $f^{-1}(B_{n-j+1})$ containing z_{n-j}. Observe that B_1 is contained in the immediate basin of attraction of z_1 and that $B_{n+1} \subset B_1$. The set B_0 contains B_n, and $f^n(B_0) = B_n$.

For $i < n$, B_{n-i} is a simply connected unbounded set that is bounded by a simple curve—a "finger" in the terminology of [BD]. The set B_0 is a complement of a union of infinitely many such fingers F_i. In order to build an appropriate dynamics, we fix one component (finger) F_0 of the complement of B_0 (obviously, $F_i = F_0 + 2k\pi i$; see [BD, Fig. 3]). Let

$$P = F_0 \setminus \pi^{-1}\left(\bigcup_{i=1}^{n-1} B_i\right),$$

where π is the natural projection $\pi \colon \bigcup F_i \to F_0$. Then

$$f(P) \supset \bigcup_k (P + 2k\pi i)$$

and, modifying the set P slightly, we can actually require that

$$f(P) \supset \overline{\bigcup_k (P + 2k\pi i)}.$$

Now, $F \colon P \cap f^{-1}(\pi^{-1}(P)) \to P$ is defined as $F = \pi \circ f$.

Let

$$J(F) = \{z \in P : F^n \text{ is defined for all } n \geq 0\}.$$

One can easily see that

$$J(f) \cap P = J(F).$$

The whole construction given in previous sections can now be repeated. We omit the details and summarize the results as follows.

THEOREM 6.1. *Assume that the map $f(z) = \lambda e^z$ has an attracting periodic orbit. Denote by*

$$J_r = \{z \in J(f) : f^n(z) \text{ does not tend to } \infty\}.$$

Then $h = \mathrm{HD}(J_r) < 2$. Moreover, there exists a h-conformal measure m for the map $F \colon J(F) \to J(F)$ and a σ-finite conformal measure $\tilde{m}$ for $f \colon J(f) \to J(f)$ satisfying $\tilde{m}(I_\infty(f)) = 0$. The h-dimensional Hausdorff measure of $J(F)$ is finite, whereas the h-dimensional packing measure is infinite. There exists a probability ergodic F-invariant measure μ that is equivalent to m.

7. Appendix

Our main goal in this appendix is to provide an alternative direct proof—without using the concept of conformal measures—of the fact that the Hausdorff dimension of the set $J_r(f_\lambda)$ is less than 2. Let

$$J_{ru}(f_\lambda) = \left\{ z \in J(f_\lambda) : \liminf_{n\to\infty} |f_\lambda^n(z)| < \infty \text{ and } \limsup_{n\to\infty} |f_\lambda^n(z)| = \infty \right\}.$$

We start with the following lemma.

LEMMA 7.1. *If* $\lambda \in (0,\infty)$, *then*

$$\limsup_{\lambda\to 0} \mathrm{HD}(J_{ru}(f_\lambda)) \le 1.$$

Proof. Fix $\lambda \in (0, 1/e)$. Given an integer $k \ge 2$, consider the set

$$J_k(M) = \{z \in P_M \cap J(f_\lambda) : \mathrm{Re}(f^k(z)) \le M \text{ and } \mathrm{Re}(f^j(z)) > M \text{ for all } j = 1, \dots, k-1\}$$

and define the map $F_k\colon J_k(M) \to P_M$ by the formula

$$F_k(z) = \pi_0(f^k(z)) = F^k(z).$$

If $z \in J_k(M)$, then $\mathrm{Re}(f^{k-1}(z)) > M$ and therefore $|f^k(z)| > \lambda e^M$. Since $\mathrm{Re}(f^k(z)) \le M$, this implies that $|\mathrm{Im}(f^k(z))| > \sqrt{\lambda^2 e^{2M} - M} \ge \lambda e^M/2$ for all M large enough. Since also $\mathrm{Re}(f^j(z)) > M$ for every $z \in J_k(M)$ and all $j = 0, 1, \dots, k-1$, we may conclude that, for every $w \in F_k(J_k(M))$ and every $t > 1$,

$$\begin{aligned}\sum_{z\in F_k^{-1}(w)} |F_k'(z)|^{-t} &\le \sum_{|n|\ge \lambda e^M/4\pi} \left(\frac{1}{2\pi|n|}\right)^t \left(\sum_{n=-\infty}^{+\infty} \frac{1}{(M^2 + (2\pi n)^2)^{t/2}}\right)^{k-1} \\ &\le \frac{(4\pi)^{t-1}}{t-1} \lambda^{1-t} (M^{1-t}\Sigma_t)^{k-1}, \qquad (7.1)\end{aligned}$$

where

$$\Sigma_t = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{(1+u^2)^{t/2}}\, du.$$

Since all the sets $J_k(M)$, $k \ge 2$, are mutually disjoint, putting $J_\infty(M) = \bigcup_{k\ge 2} J_k(M)$ allows us to define the map $F_\infty\colon J_\infty(M) \to P_M$ by the requirement that $F_\infty|_{J_k(M)} = F_k$. It then follows from (7.1) that, for every $w \in J(f) \cap P_M$,

$$\begin{aligned}\sum_{z\in F_\infty^{-1}(w)} |F_\infty'(z)|^{-t} &\le \frac{(4\pi)^{t-1}}{t-1} (\lambda e^M)^{1-t} \sum_{j=1}^{\infty} (\Sigma_t M^{1-t})^j \\ &= \frac{(4\pi)^{t-1}}{t-1} (\lambda e^M)^{1-t} \Sigma_t M^{1-t} \frac{1}{1 - \Sigma_t M^{1-t}} \\ &\le 2\Sigma_t \frac{(4\pi)^{t-1}}{t-1} (\lambda e^M)^{1-t} M^{1-t} \qquad (7.2)\end{aligned}$$

for all M sufficiently large. Fix now $k \geq 1$ and define

$$E_k(M) = \{z \in P_M \cap J(f) : \mathrm{Re}(f^j(z)) \leq M \text{ for all } j = 0, 1, \ldots, k-1 \\ \text{and } F^{k-1}(z) \in J_\infty(M)\}.$$

Put $E_\infty(M) = \bigcup_{k\geq 1} E_k(M)$. Since the sets $E_k(M)$, $k \geq 1$, are mutually disjoint, we can define the map $G\colon E_\infty(M) \to P_M$ by setting

$$G(z) = F_\infty(F^{k-1}(z))$$

if $z \in E_k(M)$.

Note that $E_1(M) = J_\infty(M)$ and $G|_{E_1(M)} = F$. Since $\mathrm{Re}(f^j(z)) \geq q_\lambda$ for all $z \in J(f_\lambda)$ and all $j \geq 0$, it follows that for all $w \in P_M \cap J(f_\lambda)$ we have

$$\begin{aligned}\sum_{z\in G^{-1}(w)} &|G'(z)|^{-t} \\ &\leq \left(2\Sigma_t \frac{(4\pi)^{t-1}}{t-1}(\lambda e^M)^{1-t} M^{1-t}\right) \sum_{k=1}^{\infty}\left(\sum_{n=-\infty}^{+\infty} \frac{1}{(q_\lambda^2 + (2\pi n)^2)^{t/2}}\right)^{k-1} \\ &\leq 2\Sigma_t \frac{(4\pi)^{t-1}}{t-1}(\lambda e^M)^{1-t} M^{1-t} \sum_{k=0}^{\infty} (q_\lambda^{1-t}\Sigma_t)^k.\end{aligned}$$

Fix now $\lambda > 0$ so small that q_λ is so large that $q_\lambda^{1-t}\Sigma_t < 1/2$. Then, for all $w \in P_M \cap J(f_\lambda)$, we obtain

$$\sum_{z\in G^{-1}(w)} |G'(z)|^{-t} \leq C_t(Me^M)^{1-t} \tag{7.3}$$

for some constant C_t depending on t and independent of M. Now there exist $0 < p_\lambda < q_\lambda$ such that $\{z \in \mathbb{C} : \mathrm{Re}(z) > p_\lambda\} \cap \overline{\{f_\lambda^n(0) : n \geq 0\}} = \emptyset$. Cover the set $Q_M = \{z \in \mathbb{C} : p_\lambda \leq \mathrm{Re}(z) \leq M+1\}$ by the family $\Re_M$ of nonoverlapping rectangles intersecting $G(E_\infty(M))$ of the form $\Delta \times \left[-\frac{3}{2}\pi, \frac{3}{2}\pi\right]$ with the lengths of Δ equal to 1. For every element $R \in \Re_M$, fix one element $w_R \in R \cap G(E_\infty(M))$. Then the family $\{G_z^{-1}(R) : R \in \Re_M, z \in G^{-1}(w_R)\}$ covers $E_\infty(M)$, where $G_z^{-1}\colon Q_M \to \mathbb{C}$ is the holomorphic branch of G sending w to z. It follows from Koebe's distortion theorem and (7.3) that, if $R \in \Re_M$ and $v \in R$, then $\sum_{z\in G^{-1}(w_R)} |(G_z^{-1})'(v)|^t \leq C_t'(Me^M)^{1-t}$ for some constant C_t' that is independent of M. Consequently,

$$\begin{aligned}\sum_{R\in\Re_M}\sum_{z\in G^{-1}(w_R)} \mathrm{diam}^t(G_z^{-1}(R))) &\leq \sum_{R\in\Re_M}\sum_{z\in G^{-1}(w_R)} |(G_z^{-1})'(v_z)|^t \,\mathrm{diam}^t(R) \\ &\leq (3\pi+1)^t C_t' \sum_{R\in\Re_M} (Me^M)^{1-t} \\ &\leq (3\pi+1)^t C_t'(M+1)(Me^M)^{1-t},\end{aligned}$$

where $v_z \in R$ is chosen so that $|(G_z^{-1})'(v_z)| = \sup_{v\in R}\{|(G_z^{-1})'(v)|\}$. Since

$$J_{ru}(f_\lambda) \cap \{z \in \mathbb{C} : -\pi \le \mathrm{Im}(z) \le \pi\} \subset \bigcup_{M\ge N} E_\infty(M) \quad \text{for all } N \ge 1,$$

$$\sum_{M\ge N} \sum_{R\in\Re_M} \sum_{z\in G^{-1}(w_R)} \mathrm{diam}^t(G_z^{-1}(R)) \le (3\pi+1)^t C_t' \sum_{M=N}^{\infty} (M+1)(Me^M)^{1-t},$$

and $\lim_{N\to\infty}\big((3\pi+1)^t C_t' \sum_{M=N}^{\infty}(M+1)(Me^M)^{1-t}\big) = 0$, we conclude that

$$\mathrm{HD}(J_{ru}(f_\lambda) \cap \{z \in \mathbb{C} : -\pi \le \mathrm{Im}(z) \le \pi\}) \le t.$$

Since

$$J_{ru}(f_\lambda) = \bigcup_{n\in\mathbb{Z}} (J_{ru}(f_\lambda) \cap \{z \in \mathbb{C} : -\pi \le \mathrm{Im}(z) \le \pi\} + 2\pi i n),$$

we conclude that $\mathrm{HD}(J_{ru}(f_\lambda)) \le t$. The proof is finished. □

Let

$$J_r(f_\lambda) = \Big\{z \in J(f_\lambda) : \liminf_{n\to\infty} |f_\lambda^n(z)| < \infty\Big\}.$$

Since $J_r(f_\lambda) = J_{\mathrm{bd}}(f_\lambda) \cup J_{ru}(f_\lambda)$, combining Lemma 7.1 and Corollary 2.2 yields the following theorem.

THEOREM 7.2. *If* $\lambda \in (0, \infty)$, *then*

$$\lim_{\lambda\to 0} \mathrm{HD}(J_r(f_\lambda)) = 1.$$

COROLLARY 7.3. *If* $|\lambda| < 1/e$ *and* $\lambda \ne 0$, *then* $\mathrm{HD}(J_r(f_\lambda)) < 2$.

Proof. We use the following theorem, proven in [As, Cor. 1.3] (cf. [GL, Thm. 5, p. 13]).

THEOREM 7.4. *If* $f : \Omega \to \Omega'$ *is a* K*-quasiconformal homeomorphism and* $E \subset \Omega$ *is a compact set, then*

$$\mathrm{HD}(f(E)) \le \frac{2K\,\mathrm{HD}(E)}{2 + (K-1)\,\mathrm{HD}(E)}.$$

Although Astala's result is stated for compact sets E only, it actually holds for all subsets E of Ω. Indeed, assuming first that $\bar{E} \subset G$ and that the closure $\bar{E}$ is compact, we see that [LV, Thm. II.8.1] applies and so Astala's proof goes through step by step. Now, it suffices to observe that the Hausdorff dimension is σ-stable and that each subset of Ω is a countable union of sets whose closures are compact subsets of Ω. In particular, quasiconformal maps send sets whose Hausdorff dimension is less than 2 into sets with Hausdorff dimension less than 2. Since all the maps f_λ with $|\lambda| < 1/e$ and $\lambda \ne 0$ are mutually quasiconformally conjugate, combining Theorem 7.2 and Theorem 7.4 yields our corollary. □

References

[As] K. Astala, *Area distortion of quasiconformal mappings,* Acta Math. 173 (1994), 37–60.

[BD] R. Bhattacharjee and R. Devaney, *Tying hairs for structurally stable exponentials,* Ergodic Theory Dynam. Systems 20 (2000), 1603–1617.

[Bo] R. Bowen, *Hausdorff dimension of quasi-circles,* Inst. Hautes Études Sci. Publ. Math. 50 (1979), 11–25.

[DU1] M. Denker and M. Urbański, *On the existence of conformal measures,* Trans. Amer. Math. Soc. 328 (1991), 563–587.

[DU2] ———, *On Sullivan's conformal measures for rational maps of the Riemann sphere,* Nonlinearity 4 (1991), 365–384.

[DU3] ———, *Geometric measures for parabolic rational maps,* Ergodic Theory Dynam. Systems 12 (1992), 53–66.

[Fe] H. F. Federer, *Geometric measure theory,* Springer-Verlag, New York, 1969.

[GL] F. P. Gardiner and N. Lakic, *Quasiconformal Teichmüller theory,* Amer. Math. Soc., Providence, RI, 2000.

[Gu] M. Guzmán, *Differentiation of integrals in* $\mathbb{R}^n$, Measure theory (Proc. Conf., Oberwolfach, 1975), Lecture Notes in Math., 541, pp. 181–185, Springer-Verlag, Berlin, 1976.

[Ka] B. Karpińska, *Area and Hausdorff dimension of the set of accessible points of the Julia sets of* λe^z *and* $\lambda \sin z$, Fund. Math. 159 (1999), 269–287.

[LV] O. Lehto and K. I. Virtanen, *Quasiconformal mappings in the plane,* 2nd ed., Springer-Verlag, Berlin, 1973.

[Ma] M. Martens, *The existence of* σ*-finite invariant measures: Applications to real one-dimensional dynamics,* preprint.

[Mc] C. McMullen, *Area and Hausdorff dimension of Julia sets of entire functions,* Trans. Amer. Math. Soc. 300 (1987), 329–342.

[PU] F. Przytycki and M. Urbański, *Fractals in the plane—the ergodic theory methods,* to appear [available at ⟨http://www.math.unt.edu/~urbanski⟩].

[TT] S. J. Taylor and C. Tricot, *Packing measure, and its evaluation for a Brownian path,* Trans. Amer. Math. Soc. 288 (1985), 679–699.

M. Urbański
Department of Mathematics
University of North Texas
Denton, TX 76203-1430

urbanski@unt.edu

A. Zdunik
Institute of Mathematics
Warsaw University
02-097 Warszawa
Poland

aniazd@mimuw.edu.pl