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Sharp Sobolev Inequalities in Critical Dimensions

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1. Introduction

Let $K \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 2K+1)$ be a regular bounded domain in $\mathbb{R}^N$. We consider the semilinear polyharmonic problem

$$(-\Delta)^K u = \lambda u + |u|^{s-2}u \ \text{ in } \Omega, \tag{1}$$

where

$$s := \frac{2N}{N-2K}$$

denotes the critical Sobolev exponent. For $K = 1$, Brezis and Nirenberg [3] studied the existence of positive solutions of (1) with homogenous Dirichlet boundary conditions

$$u = 0 \ \text{ on } \partial\Omega. \tag{2}$$

They discovered the following remarkable phenomenon: the qualitative behavior of the set of solutions of (1) and (2) is highly sensitive to N, the dimension of the space. To state their result precisely, let us denote by $\lambda_1 > 0$ the first eigenvalue of $-\Delta$ in Ω. Brezis and Nirenberg showed for $K = 1$ that: (a) in dimension $N \geq 4$, there exists a positive solution of (1) and (2) if and only if $\lambda \in (0, \lambda_1)$; while (b) in dimension $N = 3$ and when $\Omega = B_1$ is the unit ball, there exists a positive solution of (1) and (2) if and only if $\lambda \in (\lambda_1/4, \lambda_1)$.

Pucci and Serrin [13] later considered the general polyharmonic problem (1) with $K \geq 1$ and with homogenous Dirichlet boundary conditions given by

$$D^k u = 0 \ \text{ on } \partial\Omega \quad \text{for } \ k = 0, \dots, K-1. \tag{3}$$

Here $D^k u$ denotes any derivative of order k of the function u. Pucci and Serrin were interested in the existence of nontrivial radial solutions of (1) subject to the boundary conditions (3) in the case $\Omega = B_1$. They introduced the notion of *critical dimensions* for (1) and (3) as the dimensions N for which radial solutions exist only for $\lambda > \lambda^*$, where $\lambda^* > 0$. Moreover, they conjectured that, given $K \geq 1$, the critical dimensions are given by $2K+1 \leq N \leq 4K-1$. It is shown in [7] that the dimensions $N \geq 4K$ are not critical, and the conjecture of Pucci and Serrin has been partially solved; see [1; 4; 8; 13; 14] and the references therein.

The critical dimensions are intimately related to the existence of sharp Sobolev inequalities. Indeed, motivated by the nonexistence results in [3], Brezis and Lieb

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[2] proved that, for any bounded set Ω in $\mathbb{R}^3$, there exists a constant $C > 0$ (depending on Ω) such that

$$S(\mathbb{R}^3)\|f\|^2_{L^6(\Omega)} + C\|f\|^2_{L^3_w(\Omega)} \le \|\nabla f\|^2_{L^2(\Omega)} \quad \forall f \in H^1_0(\Omega), \tag{4}$$

where:

$$S(\mathbb{R}^3) := \inf_{f\in H^1_0(\Omega)\setminus\{0\}} \frac{\|\nabla f\|^2_{L^2}}{\|f\|^2_{L^6}} = \inf_{f\in H^1_0(\mathbb{R}^3)\setminus\{0\}} \frac{\|\nabla f\|^2_{L^2}}{\|f\|^2_{L^6}}$$

is the best Sobolev constant for the embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$; $\|f\|_{L^q(\Omega)}$ is the L^q-norm; and $L^q_w(\Omega)$ denotes the weak L^q-norm, defined by

$$\|f\|_{L^q_w(\Omega)} := \sup_{A\subset\Omega,\, |A|>0} |A|^{(1-q)/q} \int_A |f|. \tag{5}$$

This result has more recently been generalized by Gazzola and Grunau [5] to any H^K_0 for $K \ge 1$. More precisely, for any bounded domain $\Omega \subset \mathbb{R}^N$ with $2K+1 \le N \le 4K-1$, they proved that there exists a constant $C > 0$ (depending on N, K, and Ω) such that

$$S_K(\mathbb{R}^N)\|f\|^2_{L^s(\Omega)} + C\|f\|^2_{L_w^{N/(N-2K)}(\Omega)} \le \|f\|^2_{K,2,\Omega} \quad \forall f \in H^K_0(\Omega). \tag{6}$$

In (6) we have, by definition,

$$\|f\|^2_{K,2,\Omega} := \begin{cases} \displaystyle\int_\Omega ((-\Delta)^M f)^2\,dx & \text{if } K = 2M \text{ is even,} \\ \displaystyle\int_\Omega |\nabla(-\Delta)^M f|^2\,dx & \text{if } K = 2M+1 \text{ is odd,} \end{cases} \tag{7}$$

and

$$S_K(\mathbb{R}^N) := \inf_{f\in H^K_0(\Omega)\setminus\{0\}} \frac{\|f\|^2_{K,2,\Omega}}{\|f\|^2_{L^s(\Omega)}} = \inf_{f\in H^K_0(\mathbb{R}^N)\setminus\{0\}} \frac{\|f\|^2_{K,2,\mathbb{R}^N}}{\|f\|^2_{L^s(\mathbb{R}^N)}} \tag{8}$$

is the best Sobolev constant for the embedding $H^K_0(\Omega) \hookrightarrow L^s(\Omega)$. Recall that the exponent s is defined by $s = \frac{2N}{N-2K}$.

In this paper, we pursue the study of these sharp Sobolev inequalities for some function spaces that are naturally associated to variational problems. More precisely, we consider the space

$$H^K_\theta(\Omega) = \left\{ v \in H^K(\Omega) \;\middle|\; (-\Delta)^i v = 0 \text{ on } \partial\Omega \ \forall 0 \le i < \left[\!\left[\frac{K+1}{2}\right]\!\right] \right\},$$

where $\left[\!\left[\frac{K+1}{2}\right]\!\right] = M+1$ if $K = 2M+1$ is odd and $\left[\!\left[\frac{K+1}{2}\right]\!\right] = M+1$ if $K = 2M+2$ is even. This definition is motivated by the fact that Navier conditions

$$(-\Delta)^i u = 0 \text{ on } \partial\Omega,$$

for all $0 \le i \le K-1$, are natural boundary conditions for critical points of variational problems involving higher powers of $-\Delta$. Granted this definition, we establish the following sharp Sobolev inequality.

THEOREM 1. *Let Ω be a regular bounded domain in $\mathbb{R}^N$ and let $2K+1 \le N \le 4K-1$. Then, for any $1 \le q < \frac{N}{N-2K}$, there exists a constant $C > 0$ (depending on Ω, N, K, and q) such that*

$$S_K(\mathbb{R}^N)\|f\|^2_{L^s(\Omega)} + C\|f\|^2_{L^q(\Omega)} \le \|f\|^2_{K,2,\Omega} \quad \forall f \in H^K_\theta(\Omega). \tag{9}$$

This result means that, on regular bounded domains, the classical Sobolev inequality associated to the embedding

$$H^K_\theta(\Omega) \hookrightarrow L^s(\Omega)$$

with optimal constant $S_K(\mathbb{R}^N)$ can be improved by adding a remainder term of L^2-norm precisely when $2K+1 \le N \le 4K-1$. Therefore, in some sense, this result describes the relation between the sharp inequalities for a Sobolev embedding and the critical dimensions conjectured by Pucci and Serrin [13].

This paper is organized as follows. In Section 2, we study the best constant involving the critical exponent in the Sobolev inequality. Section 3 is devoted to the proof of Theorem 1.

2. Best Constants for Sobolev Inequalities

In this section, we analyze the best Sobolev constants for functions defined either on the whole space or on the half space $\mathbb{R}^N_+ := \{x = (x_1, \dots, x_N) \mid x_1 > 0\}$. Using this, we obtain the best Sobolev constants for functions defined in bounded domains. By definition, the space $\mathcal{D}^{K,2}(\mathbb{R}^N)$ (resp. $\mathcal{D}^{K,2}(\mathbb{R}^N_+)$) is the completion of $C_0^\infty(\mathbb{R}^N)$ (resp. $C_0^\infty(\mathbb{R}^N_+)$) for the norm $\|\cdot\|_{K,2,\mathbb{R}^N}$ (resp. $\|\cdot\|_{K,2,\mathbb{R}^N_+}$). We also define

$$\mathcal{D}^{K,2}_\theta(\mathbb{R}^N_+) := \Big\{u|_{\mathbb{R}^N_+} \;\Big|\; u \in \mathcal{D}^{K,2}(\mathbb{R}^N) \text{ and } (-\Delta)^i u = 0 \text{ on } x_1 = 0 \quad \forall 0 \le i < \Big[\frac{K+1}{2}\Big]\Big\}.$$

Finally, we set

$$S_{K,0}(\mathbb{R}^N) := \inf\{\|u\|^2_{K,2,\mathbb{R}^N} \mid u \in \mathcal{D}^{K,2}(\mathbb{R}^N) \text{ and } \|u\|_{L^s(\mathbb{R}^N)} = 1\},$$

$$S_{K,0}(\mathbb{R}^N_+) := \inf\{\|u\|^2_{K,2,\mathbb{R}^N_+} \mid u \in \mathcal{D}^{K,2}(\mathbb{R}^N_+) \text{ and } \|u\|_{L^s(\mathbb{R}^N_+)} = 1\},$$

$$S_{K,\theta}(\mathbb{R}^N_+) := \inf\{\|u\|^2_{K,2,\mathbb{R}^N_+} \mid u \in \mathcal{D}^{K,2}_\theta(\mathbb{R}^N_+) \text{ and } \|u\|_{L^s(\mathbb{R}^N_+)} = 1\}.$$

Using the strategy developed in [18], we can show the following.

THEOREM 2. *Assume that $N \ge 2K+1$; then the following equalities hold*:

$$S_K(\mathbb{R}^N) = S_{K,0}(\mathbb{R}^N) = S_{K,0}(\mathbb{R}^N_+) = S_{K,\theta}(\mathbb{R}^N_+).$$

Moreover, $S_{K,0}(\mathbb{R}^N)$ is achieved by a family of functions given by

$$U_{K,\varepsilon,y}(x) := C_{N,K}\frac{\varepsilon^{(N-2K)/2}}{(\varepsilon^2 + |x-y|^2)^{(N-2K)/2}}, \tag{10}$$

where $y \in \mathbb{R}^N$, $\varepsilon > 0$, and the constant $C_{N,K}$ is chosen so that $\|U_{K,\varepsilon,y}\|_{L^s(\mathbb{R}^N)} = 1$.

The proof of this result is based on Talenti's comparison principle. Recall that, for any function ϕ, the Schwarz symmetrization of ϕ is defined by

$$\phi^*(x) := \inf\{y \geq 0 \mid \mu(y) < \sigma_N |x|^N\},$$

where $\mu(y) := \text{meas}\{x \in \Omega \mid |\phi(x)| > y\}$ and σ_N is the measure of the N-dimensional unit ball. Observe that, when the function ϕ is defined over a bounded set Ω, its Schwarz symmetrization ϕ^* is defined on the ball Ω^* chosen so that $\text{meas}(\Omega) = \text{meas}(\Omega^*)$. Granted these definitions, Talenti's comparison principle can be stated as follows.

PROPOSITION 1 [15]. *Assume that Ω is a regular domain in $\mathbb{R}^N$ and let u be a weak solution of the problem*

$$\begin{cases} -\Delta u = f \ \text{ in } \Omega, \\ \quad u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{11}$$

with $f \in L^{2N/(N+2)}(\Omega)$. Then

$$u^* \leq v \ \text{ a.e. in } \Omega^*, \tag{12}$$

where v is the weak solution of

$$\begin{cases} -\Delta v = f^* \ \text{ in } \Omega^*, \\ \quad v = 0 \quad \text{ on } \partial\Omega^*. \end{cases} \tag{13}$$

This result can be easily generalized to give our next proposition.

PROPOSITION 2. *Let $f \in C_0^\infty(\mathbb{R}^N_+)$ and $\alpha \in \mathbb{N}$. Assume that u is a weak solution of the problem*

$$\begin{cases} (-\Delta)^\alpha u = f \ \text{ in } \mathbb{R}^N_+, \\ u = (-\Delta)u = \cdots = (-\Delta)^{\alpha-1} u = 0 \ \text{ on } \partial\mathbb{R}^N_+. \end{cases} \tag{14}$$

Then

$$u^* \leq v \ \text{ a.e. in } \mathbb{R}^N, \tag{15}$$

where $v \in \mathcal{D}^{2\alpha,2}(\mathbb{R}^N)$ is the weak solution of

$$(-\Delta)^\alpha v = f^* \ \text{ in } \mathbb{R}^N. \tag{16}$$

Proof. The proof is by induction on α. When $\alpha = 1$, the result simply corresponds to Talenti's comparison principle. Now assume that the result is true for $\alpha = k$. For $\alpha = k+1$, using the above hypothesis together with the fact that $-\Delta v \in \mathcal{D}^{2k,2}(\mathbb{R}^N)$ yields

$$(-\Delta u)^* \leq -\Delta v \ \text{ a.e. in } \mathbb{R}^N. \tag{17}$$

Let w be the solution of the problem

$$-\Delta w = (-\Delta u)^* \ \text{ in } \mathbb{R}^N \tag{18}$$

that satisfies $w \to 0$ as $|x| \to \infty$. Since $-\Delta$ is a positive operator, the maximum principle yields

$$v \geq w \ \text{ a.e. in } \mathbb{R}^N. \tag{19}$$

Applying Talenti's comparison principle, we obtain

$$u^* \le w \quad \text{a.e. in } \mathbb{R}^N. \tag{20}$$

The result of the proposition follows at once from (19) and (20). □

Let us now recall another result of Talenti.

PROPOSITION 3 [16]. *Assume $u \in W^{1,q}(\mathbb{R}^N)$ for any $q \ge 1$. Then*

$$\|\nabla(u^*)\|_{L^q(\mathbb{R}^N)} \le \|\nabla u\|_{L^q(\mathbb{R}^N)}. \tag{21}$$

REMARK. Using a simple density argument, it should be clear that this inequality also holds when $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$.

Proof of Theorem 2. Given $u \in \mathcal{D}_\theta^{K,2}(\mathbb{R}_+^N)$, we define $f := (-\Delta)^{\left[\frac{K}{2}\right]}u$. Observe that $f \in L^2(\mathbb{R}_+^N)$ when K is even whereas $f \in \mathcal{D}^{1,2}(\mathbb{R}_+^N)$ when K is odd.

A classical density argument ensures the existence of a sequence of functions $(f_n)_n \subset C_0^\infty(\mathbb{R}_+^N)$ such that

$$f_n \to f \quad \text{in } L^t(\mathbb{R}_+^N),$$

where

$$t := \begin{cases} 2 & \text{when } K \text{ is even,} \\ \frac{2N}{N-2} & \text{when } K \text{ is odd.} \end{cases}$$

We set $M := \left[\frac{K}{2}\right]$ and consider the problem

$$\begin{cases} (-\Delta)^M u_n = f_n & \text{in } \mathbb{R}_+^N, \\ u_n = (-\Delta)u_n = \cdots = (-\Delta)^{M-1}u_n = 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \tag{22}$$

It is clear that $f_n \in L^p(\mathbb{R}_+^N)$ for any $p \ge 1$. Therefore, applying standard elliptic estimates, we see that for all $\alpha \in \mathbb{N}$ with $|\alpha| < M$ and for all $1 \le p$,

$$\|(-\Delta)^\alpha u_n\|_{L^q(\mathbb{R}_+^N)} \le C(p,\alpha)\|f_n\|_{L^p(\mathbb{R}_+^N)},$$

where the exponent q is defined by the identity

$$\frac{1}{q} = \frac{1}{p} - \frac{2(M-\alpha)}{N}$$

and where the constant $C(p,\alpha)$ is independent of n.

Now, let us define v_n to be the solution of the problem

$$\begin{cases} (-\Delta)^M v_n = f_n^* & \text{in } \mathbb{R}^N, \\ v_n \in \mathcal{D}^{2M,2}(\mathbb{R}^N). \end{cases} \tag{23}$$

Using the contraction property of the Schwarz symmetrization, we obtain

$$\|v_n - v_m\|_{L^{2N/(N-2K)}(\mathbb{R}^N)} \le C\|f_n^* - f_m^*\|_{L^t(\mathbb{R}^N)} \le C\|f_n - f_m\|_{L^t(\mathbb{R}_+^N)}.$$

As n and m tend to ∞, the right-hand side of this sequence of inequalities converges to 0. This implies that $(v_n)_n$ is a Cauchy sequence in $L^{2N/(N-2K)}(\mathbb{R}^N)$, which is complete. Hence, there exists some $v \in L^{2N/(N-2K)}(\mathbb{R}^N)$ such that

$$v_n \to v \text{ in } L^{2N/(N-2K)}(\mathbb{R}^N).$$

Moreover, we can also assume that (up to a subsequence) we have

$$v_n \to v \text{ a.e. in } \mathbb{R}^N.$$

On the other hand, observe that the sequence $(v_n)_n$ is bounded in $\mathcal{D}^{K,2}(\mathbb{R}^N)$ and, since $\mathcal{D}^{K,2}(\mathbb{R}^N)$ is reflexive, we can always assume that (up to a subsequence)

$$v_n \rightharpoonup v \text{ in } \mathcal{D}^{K,2}(\mathbb{R}^N).$$

Furthermore,

$$\|f_n^* - f^*\|_{L^t(\mathbb{R}^N)} \le \|f_n - f\|_{L^t(\mathbb{R}^N_+)},$$

which implies

$$f_n^* \to f^* \text{ in } L^t(\mathbb{R}^N).$$

Thus

$$(-\Delta)^M v = f^* \text{ in } \mathcal{D}'(\mathbb{R}^N).$$

Standard elliptic estimates imply that

$$u_n \to u \text{ in } \mathcal{D}^{K,2}(\mathbb{R}^N_+),$$
$$u_n \to u \text{ in } L^{2N/(N-2K)}(\mathbb{R}^N_+),$$

which in turn yield

$$u_n^* \to u^* \text{ in } L^{2N/(N-2K)}(\mathbb{R}^N_+),$$
$$u_n^* \to u^* \text{ a.e. in } \mathbb{R}^N.$$

We now use the result of Proposition 2, which gives

$$u_n^* \le v_n \text{ a.e. in } \mathbb{R}^N.$$

Letting n tend to infinity in this inequality, we conclude that

$$u^* \le v \text{ a.e. in } \mathbb{R}^N.$$

In the case where $K = 2M$ is even, we find

$$\|u\|_{L^s(\mathbb{R}^N_+)} = \|u^*\|_{L^s(\mathbb{R}^N)} \le \|v\|_{L^s(\mathbb{R}^N)}$$

and

$$\begin{aligned}\|u\|_{K,2,\mathbb{R}^N_+} = \|(-\Delta)^M u\|_{L^2(\mathbb{R}^N_+)} &= \|((-\Delta)^M u)^*\|_{L^2(\mathbb{R}^N)} \\ &= \|(-\Delta)^M v\|_{L^2(\mathbb{R}^N)} = \|v\|_{K,2,\mathbb{R}^N}.\end{aligned}$$

Similarly, in the case where $K = 2M+1$ is odd, we set

$$\overline{(-\Delta)^M u}(x) = \begin{cases} (-\Delta)^M u(x) & \text{if } x \in \mathbb{R}^N_+, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \mathbb{R}^N_+, \end{cases}$$

and we see that $\overline{(-\Delta)^M u} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. This time, it follows from Proposition 3 that

$$\|u\|_{L^s(\mathbb{R}^N_+)} = \|u^*\|_{L^s(\mathbb{R}^N)} \le \|v\|_{L^s(\mathbb{R}^N)}$$

and

$$\begin{aligned}\|u\|_{K,2,\mathbb{R}^N_+} &= \|\nabla\overline{(-\Delta)^M u}\|_{L^2(\mathbb{R}^N)} \geq \|\nabla(\overline{(-\Delta)^M u})^*\|_{L^2(\mathbb{R}^N)}\\ &= \|\nabla((-\Delta)^M u)^*\|_{L^2(\mathbb{R}^N)} = \|\nabla(-\Delta)^M v\|_{L^2(\mathbb{R}^N)} = \|v\|_{K,2,\mathbb{R}^N}.\end{aligned}$$

From these relations, we obtain

$$\frac{\|u\|^2_{K,2,\mathbb{R}^N_+}}{\|u\|^2_{L^s(\mathbb{R}^N_+)}} \geq \frac{\|v\|^2_{K,2,\mathbb{R}^N}}{\|v\|^2_{L^s(\mathbb{R}^N)}} \geq S_{K,0}(\mathbb{R}^N),$$

which already implies that

$$S_{K,\theta}(\mathbb{R}^N_+) \geq S_{K,0}(\mathbb{R}^N). \tag{24}$$

Granted that $\mathcal{D}^{K,2}(\mathbb{R}^N_+) \subset \mathcal{D}^{K,2}_\theta(\mathbb{R}^N_+)$, it should be clear that

$$S_{K,0}(\mathbb{R}^N_+) \geq S_{K,\theta}(\mathbb{R}^N_+). \tag{25}$$

Hence, in order to complete the proof of the theorem, it remains to show that

$$S_{K,0}(\mathbb{R}^N) \geq S_{K,0}(\mathbb{R}^N_+).$$

Toward this end, assume that $u \in C_0^\infty(\mathbb{R}^N)$ with $\operatorname{supp}(u) \subset B(0, R)$ for some $R > 0$; we set $v(x) = u(x + 2Re_1)$, where $e_1 = (1, 0, \dots, 0)$. Obviously, $v \in C_0^\infty(\mathbb{R}^N_+)$ and

$$\frac{\|u\|^2_{K,2,\mathbb{R}^N}}{\|u\|^2_{L^s(\mathbb{R}^N)}} = \frac{\|v\|^2_{K,2,\mathbb{R}^N_+}}{\|v\|^2_{L^s(\mathbb{R}^N_+)}} \geq S_{K,0}(\mathbb{R}^N_+).$$

Minimizing over all functions u on the left-hand side, we see that

$$S_{K,0}(\mathbb{R}^N) \geq S_{K,0}(\mathbb{R}^N_+). \tag{26}$$

Combining this inequality with (24) and (25), we conclude that

$$S_{K,0}(\mathbb{R}^N) = S_{K,0}(\mathbb{R}^N_+) = S_{K,\theta}(\mathbb{R}^N_+).$$

Observe that $C_0^\infty(\mathbb{R}^N) \subset H_0^K(\mathbb{R}^N) \subset \mathcal{D}^{K,2}(\mathbb{R}^N)$ and also that $C_0^\infty(\mathbb{R}^N)$ is dense in $\mathcal{D}^{K,2}(\mathbb{R}^N)$. Therefore, we also have

$$S_K(\mathbb{R}^N) = S_{K,0}(\mathbb{R}^N).$$

Finally, the last statement in Theorem 2 follows from [14] and [19]. $\square$

We now turn to the study of the best Sobolev constants for bounded domains. Let Ω be a regular bounded domain of $\mathbb{R}^N$. We define

$$S_{K,\theta}(\Omega) := \inf_{v \in H_\theta^K(\Omega)\setminus\{0\}} \frac{\|v\|_{K,2,\Omega}^2}{\|v\|_{L^s(\Omega)}^2}$$

and

$$S_{K,0}(\Omega) := \inf_{v \in H_0^K(\Omega)\setminus\{0\}} \frac{\|v\|_{K,2,\Omega}^2}{\|v\|_{L^s(\Omega)}^2}.$$

Granted these definitions, we will establish the following theorem.

THEOREM 3. *Assume that $N \geq 2K+1$. Then*

$$S_{K,0}(\Omega) = S_{K,\theta}(\Omega) = S_K(\mathbb{R}^N). \tag{27}$$

Moreover, the infimum $S_{K,\theta}(\Omega)$ (resp. $S_{K,0}(\Omega)$) is not achieved—that is, it is not achieved by any function $u \in H_\theta^K(\Omega)$ (resp. $u \in H_0^K(\Omega)$).

The proof of this result requires two technical lemmas, the first of which relies on Lions's concentration compactness principle.

LEMMA 1. *Assume that $N > 2K$ and that $S_{K,\theta}(\Omega) < S_K(\mathbb{R}^N)$. Then every minimizing sequence $(u_n)_n$ of $S_{K,\theta}(\Omega)$ is relatively compact in $H_\theta^K(\Omega)$.*

Proof. Let $(u_n)_n$ be a minimizing sequence for $S_{K,\theta}(\Omega)$ in $H_\theta^K(\Omega)$; that is, $\|u_n\|_{K,2,\Omega}^2 \to S_{K,\theta}(\Omega)$ as $n \to \infty$ and $\|u_n\|_{L^s(\Omega)} = 1$.

In particular, $(u_n)_n$ is a bounded sequence in $H_\theta^K(\Omega)$, which is reflexive. Therefore, we can extract a subsequence (still denoted $(u_n)_n$) such that

$$u_n \rightharpoonup u \ \text{ weakly in } H_\theta^K(\Omega) \quad \text{and} \quad u_n \rightharpoonup u \ \text{ weakly in } L^s(\Omega).$$

Let us denote by $\mathcal{M}(\mathbb{R}^N)$ the space of nonnegative Radon measures on $\mathbb{R}^N$ that have finite mass, and let ζ_Ω denote the indicatrix function of the set Ω. We define $\mu_n := \zeta_\Omega F_K(u_n)\,dx$ and $\nu_n := \zeta_\Omega |u_n|^s\,dx$, where

$$F_K(v) := \begin{cases} ((-\Delta)^M v)^2 & \text{if } K = 2M \text{ is even,} \\ |\nabla(-\Delta)^M v|^2 & \text{if } K = 2M+1 \text{ is odd.} \end{cases}$$

It is easy to see that the sequences of measures $(\mu_n)_n$ and $(\nu_n)_n$ are bounded in $\mathcal{M}(\mathbb{R}^N)$. Up to a subsequence, we may always assume that

$$\mu_n \rightharpoonup \mu \quad \text{and} \quad \nu_n \rightharpoonup \nu$$

weakly in the sense of measures for some bounded nonnegative measures μ and ν on $\mathbb{R}^N$.

It follows from the concentration compactness principle of Lions [11] (see also [6] for a simplified blow-up analysis in the case of bounded domains) that there exists a set J (at most countable) and a set of points $\{x_j \mid j \in J\} \subset \bar{\Omega}$ such that

$$\nu = \zeta_\Omega |u|^s\,dx + \sum_{i \in J} \nu_j \delta_{x_j}$$

and

$$\mu \geq \zeta_\Omega F_K(u)\,dx + \sum_{i\in J} \mu_j \delta_{x_j},$$

where $\mu_j \geq \nu_j^{(N-2K)/N} S_K(\mathbb{R}^N)$ if $x_j \in \Omega$ and $\mu_j \geq \nu_j^{(N-2K)/N} S_{K,\theta}(\mathbb{R}^N_+)$ if $x_j \in \partial\Omega$. According to Theorem 2, we find

$$\nu(\mathbb{R}^N) = 1 = \|u\|^s_{L^s(\Omega)} + \sum_{i\in J} \nu_j$$

and also

$$\begin{aligned}\mu(\mathbb{R}^N) = S_{K,\theta}(\Omega) &\geq \|u\|^2_{K,2,\Omega} + \sum_{i\in J} \mu_j \\ &\geq S_{K,\theta}(\Omega)(\|u\|^s_{L^s(\Omega)})^{(N-2K)/N} + S_K(\mathbb{R}^N) \sum_{i\in J} (\nu_j)^{(N-2K)/N}.\end{aligned}$$

Finally, the function $t \mapsto t^{(N-2K)/N}$ is concave and so we have

$$\|u\|_{L^s(\Omega)} = 1.$$

Hence, it follows at once from weak lower semicontinuity that

$$\|u\|^2_{K,2,\Omega} \leq \liminf_{n\to\infty} \|u_n\|^2_{K,2,\Omega} = S_{K,\theta}(\Omega).$$

This completes the proof of Lemma 1. □

The second technical lemma reads as follows.

LEMMA 2. *Let u be any weak solution of*

$$\begin{cases} (-\Delta)^K u = \lambda |u|^{q-2} u + |u|^{s-2} u & \text{in } \Omega, \\ u = (-\Delta)u = \cdots = (-\Delta)^{K-1} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where $2 \leq q < s$ and $\lambda \in \mathbb{R}$. Then u is smooth.

In [18], Vorst proved this lemma for the case $K = 2$. Since his proof can easily be adapted to treat the general case, we omit the details.

The next lemma is a consequence of the well-known Pohozaev identity.

LEMMA 3. *Let $\Omega = B_1$ be the unit ball in $\mathbb{R}^N$, and assume that $N \geq 2K + 1$ and $2 \leq q < s$. Then there exists $\lambda_0 := \lambda_0(N, q, K) > 0$ (depending only on N, q, and K) such that (28) has no positive radial solution $u \in C^{2K}(\bar{\Omega})$ that is a decreasing function of $r = |x|$, provided one of following conditions is satisfied.*

(1) $\lambda \leq 0$.
(2) $2K < N < 4K$, $2 \leq q \leq \frac{4K}{N-2K}$, *and* $\lambda \in (0, \lambda_0)$.

Proof. First we observe that

$$(-\Delta)^K u \geq 0 \ \text{ in } B_1.$$

Using the maximum principle for the operator $-\Delta$ inductively, we obtain for any $i = 0, \ldots, K$ that

$$(-\Delta)^i u \geq 0 \ \text{ in } B_1.$$

Applying the Hopf lemma, for any $i = 0, \ldots, K-1$ we have

$$-\frac{\partial}{\partial r}(-\Delta)^i u > 0 \ \text{ on } \partial B_1. \tag{29}$$

Now consider u, a radial solution of (28). Multiplying (28) by $r\partial_r u$ and integrating by parts, we obtain the following Pohozaev formulas (see [12]):

$$\begin{aligned}\Big(\frac{N}{q} - \frac{N-2K}{2}\Big)\lambda \int_{B_1} u^q(x)\,dx \\ = \sum_{k=0}^{M-1} \int_{\partial B_1} ((-\Delta)^k u)'(1)((-\Delta)^{K-k-1}u)'(1)\,d\sigma\end{aligned} \tag{30}$$

if $K = 2M$ is even, and

$$\begin{aligned}\Big(\frac{N}{q} - \frac{N-2K}{2}\Big)\lambda \int_{B_1} u^q(x)\,dx &= \sum_{k=0}^{M-1} \int_{\partial B_1} ((-\Delta)^k u)'(1)((-\Delta)^{K-k-1}u)'(1)\,d\sigma \\ &\quad + \frac{1}{2}\int_{\partial B_1} [((-\Delta)^M u)'(1)]^2\,d\sigma\end{aligned} \tag{31}$$

if $K = 2M+1$ is odd. Here and below, $'$ denotes the partial derivative with respect to r, namely ∂_r.

Assume that $\lambda \leq 0$. Then (30) and (31) imply that

$$-((-\Delta)^i u)'(1) = 0 \tag{32}$$

for all $i = 0, \ldots, K-1$, which clearly contradicts (29).

Having ruled out the case where $\lambda \leq 0$, we now assume for the balance of the proof that $\lambda > 0$. For any $i \in \mathbb{N}$, consider the problem

$$\begin{cases} (-\Delta)^i w_i = 1 \ \text{ in } B_1, \\ w_i = -\Delta w_i = \cdots = (-\Delta)^{i-1} w_i = 0 \ \text{ on } \partial B_1. \end{cases} \tag{33}$$

Obviously, w_i is radial, positive, and decreasing in r. Furthermore, the maximum principle and the Hopf lemma together imply that, for all $0 \leq j < i$, the function $(-\Delta)^j w_i$ is decreasing in $r \in (0,1)$ and also that $((-\Delta)^j w_i)'(1) < 0$. Similarly, the function u is a decreasing function of $r \in (0,1)$. Hence, it follows from (28) that $(-\Delta)^K u$ is also a decreasing function of $r \in (0,1)$. Letting $0 \leq i \leq K-1$, we have

$$\begin{aligned}&\int_{B_1} ((-\Delta)^K u) w_i(x)\,dx \\ &\quad = \int_{B_1} ((-\Delta)^{K-i} u)((-\Delta)^i w_i)(x)\,dx = \int_{B_1} (-\Delta)^{K-i} u(x)\,dx \\ &\quad = -\int_{\partial B_1} ((-\Delta)^{K-i-1}u)'(1)\,d\sigma = -\sigma_{N-1}((-\Delta)^{K-i-1}u)'(1),\end{aligned} \tag{34}$$

where σ_{N-1} denotes the measure of the unit sphere in $\mathbb{R}^N$. Collecting (30), (31), and (34), we obtain

$$\begin{aligned}
\left(\frac{N}{q}-\frac{N-2K}{2}\right)&\lambda\int_{B_1} u^q(x)\,dx\\
&\geq \int_{\partial B_1}((-\Delta)^{K-1}u)'(1)u'(1)\,d\sigma\\
&= \frac{1}{\sigma_{N-1}}\int_{\partial B_1}((-\Delta)^{K-1}u)'(1)\,d\sigma\int_{\partial B_1}u'(1)\,d\sigma\\
&= \frac{1}{\sigma_{N-1}}\int_{B_1}((-\Delta)^{K}u)(x)\,dx\int_{B_1}((-\Delta)^{K}u)(x)w_{K-1}(x)\,dx\\
&\geq \frac{1}{\sigma_{N-1}}\int_{B_1}((-\Delta)^{K}u)(x)\,dx\int_{|x|\leq 1/2}((-\Delta)^{K}u)(x)w_{K-1}(x)\,dx\\
&\geq \frac{w_{K-1}(1/2)}{\sigma_{N-1}}\int_{B_1}(-\Delta)^{K}u(x)\,dx\int_{|x|\leq 1/2}(-\Delta)^{K}u(x)\,dx\\
&\geq \frac{w_{K-1}(1/2)}{2^N\sigma_{N-1}}\left(\int_{B_1}(-\Delta)^{K}u(x)\,dx\right)^2\\
&= C\left(\int_{B_1}|(-\Delta)^{K}u(x)|\,dx\right)^2 \geq C\|u\|^2_{L_w^{N/(N-2K)}(B_1)}.
\end{aligned}\tag{35}$$

It follows from (28) that

$$(-\Delta)^K u \geq u^{s-1},$$

which in turn implies that

$$\lambda\int_{B_1}u^q(x)\,dx \geq C\left(\int_{B_1}|(-\Delta)^{K}u(x)|\,dx\right)^2 \geq C\|u\|^{2(s-1)}_{L^{s-1}(B_1)}.\tag{36}$$

First assume that $q=\frac{4K}{N-2K}$ and $2K<N<4K$. Using a classical interpolation inequality, we can write

$$\begin{aligned}
\|u\|^q_{L^q(B_1)} &\leq \left(\|u\|^{N(N-2K)/8K^2}_{L_w^{s_1}(B_1)}\|u\|^{(N+2K)(4K-N)/8K^2}_{L^{s-1}(B_1)}\right)^q\\
&= \|u\|^{N/2K}_{L_w^{s_1}(B_1)}\|u\|^{(N+2K)(4K-N)/2K(N-2K)}_{L^{s-1}(B_1)}\\
&\leq C(\lambda\|u\|^q_{L^q(B_1)})^{N/4K}(\lambda\|u\|^q_{L^q(B_1)})^{(4K-N)/4K}\\
&= C\lambda\|u\|^q_{L^q(B_1)},
\end{aligned}$$

where $s_1:=\frac{N}{N-2K}$. Choosing λ small enough yields $u=0$. The proof is therefore complete in this case.

In the general case, when $2\leq q<\frac{4K}{N-2K}$ and $2K<N<4K$, it follows from (35) that

$$\lambda\int_{B_1}u^q(x)\,dx \geq C\left(\int_{B_1}|(-\Delta)^{K}u(x)|\,dx\right)^2 \geq C\|u\|^{2}_{L^{t}(B_1)},\tag{37}$$

where $t=\frac{2s-q-2}{s-q}<\frac{N}{N-2K}$. On the other hand, a classical interpolation inequality gives

$$\|u\|^q_{L^q(B_1)} \le (\|u\|^{\tilde t}_{L^t(B_1)}\|u\|^{1-\tilde t}_{L^{s-1}(B_1)})^q, \tag{38}$$

where $\tilde t := \frac{2s-q-2}{qs-2q}$.

Finally we use (36), (37), and (38) to conclude that

$$\|u\|^q_{L^q(B_1)} \le C_1(\lambda\|u\|^q_{L^q(B_1)})^{\tilde t q/2}(\lambda\|u\|^q_{L^q(B_1)})^{(1-\tilde t)q/2(s-1)} = C_1\lambda\|u\|^q_{L^q(B_1)}.$$

Again we see that $u = 0$, provided λ is chosen small enough. The proof of Lemma 3 is thus complete. □

We are now in a position to prove Theorem 3.

Proof of Theorem 3. Assume that $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^N$ are two domains in $\mathbb{R}^N$. We claim that

$$S_{K,0}(\Omega_1) \ge S_{K,0}(\Omega_2). \tag{39}$$

Indeed, for any $v \in \mathcal{D}^{K,2}(\Omega_1)$, define

$$\bar v(x) = \begin{cases} v(x) & \text{if } x \in \Omega_1, \\ 0 & \text{if } x \in \Omega_2 \setminus \Omega_1. \end{cases}$$

The claim follows immediately from the fact that $\bar v(x) \in \mathcal{D}^{K,2}(\Omega_2)$.

Let $B(x, r)$ denote the ball of radius r, centered at x, in $\mathbb{R}^N$. For all $r_1, r_2 > 0$ and all $v \in \mathcal{D}^{K,2}(B(0, r_1))$, we define $\tilde v(x) := v(r_1x/r_2)$. A direct computation shows that

$$\frac{\|\tilde v\|^2_{K,2,B(0,r_2)}}{\|\tilde v\|^2_{L^s(B(0,r_2))}} = \frac{\|v\|^2_{K,2,B(0,r_1)}}{\|v\|^2_{L^s(B(0,r_1))}}.$$

Hence, $S_{K,0}(B(0, r_1)) = S_{K,0}(B(0, r_2))$. This, together with (39), implies that

$$S_{K,0}(B(0, r)) \ge S_{K,0}(\mathbb{R}^N) \tag{40}$$

for all $r > 0$.

Conversely, given any $\varepsilon > 0$, it follows from the definition of $S_{K,0}(\mathbb{R}_N)$ that there exists a $u \in C_0^\infty(\mathbb{R}^N)$ such that

$$\frac{\|u\|^2_{K,2,\mathbb{R}^N}}{\|u\|^2_{L^s(\mathbb{R}^N)}} \le S_{K,0}(\mathbb{R}^N) + \varepsilon.$$

Let $R > 0$ be chosen so that the support of the function u is included in $B(0, R)$. It should be clear that

$$\frac{\|u\|^2_{K,2,\mathbb{R}^N}}{\|u\|^2_{L^s(\mathbb{R}^N)}} = \frac{\|u\|^2_{K,2,B(0,R)}}{\|u\|^2_{L^s(B(0,R))}} \ge S_{K,0}(B(0, R)).$$

As a consequence,

$$S_{K,0}(B(0,1)) = S_{K,0}(B(0, R)) \le S_{K,0}(\mathbb{R}^N) + \varepsilon.$$

Letting $\varepsilon \to 0$, we obtain

$$S_{K,0}(B(0,1)) \le S_{K,0}(\mathbb{R}^N). \tag{41}$$

Therefore, we conclude so far that

$$S_{K,0}(B(0,r)) = S_{K,0}(B(0,1)) = S_{K,0}(\mathbb{R}^N) \quad \forall r > 0. \tag{42}$$

Now let Ω be any regular open bounded domain. We choose $x \in \Omega$ and $r_1 > r_2 > 0$ in such a way that $B(x, r_2) \subset \Omega \subset B(x, r_1)$. It readily follows from (39) that

$$\begin{aligned} S_{K,0}(B(0,r_2)) &= S_{K,0}(B(x,r_2)) \geq S_{K,0}(\Omega) \\ &\geq S_{K,0}(B(x,r_1)) = S_{K,0}(B(0,r_1)). \end{aligned} \tag{43}$$

This, together with (42), implies that

$$S_{K,0}(\Omega) = S_{K,0}(\mathbb{R}^N). \tag{44}$$

Observe that the inclusion $H_0^K(\Omega) \subset H_\theta^K(\Omega)$ implies that

$$S_{K,\theta}(\Omega) \leq S_{K,0}(\Omega) = S_K(\mathbb{R}^N). \tag{45}$$

For any $u \in H_\theta^K(\Omega)$, we consider v the solution of

$$\begin{cases} (-\Delta)^M v = ((-\Delta)^M u)^* & \text{in } \Omega^*, \\ v = -\Delta v = \cdots = (-\Delta)^{M-1} u = 0 & \text{on } \partial\Omega^*, \end{cases} \tag{46}$$

where $M := \left[\frac{K}{2}\right]$. As in Theorem 2, we can use Talenti's comparison principle to deduce that

$$\|u\|_{K,2,\Omega} \geq \|v\|_{K,2,\Omega^*} \quad \text{and} \quad \|u\|_{L^s(\Omega)} \leq \|v\|_{L^s(\Omega^*)},$$

which implies immediately that

$$\frac{\|u\|^2_{K,2,\Omega}}{\|u\|^2_{L^s(\Omega)}} \geq \frac{\|v\|^2_{K,2,\Omega^*}}{\|v\|^2_{L^s(\Omega^*)}}.$$

Therefore, we have obtained

$$S_{K,\theta}(\Omega) \geq S_{K,\theta}(\Omega^*). \tag{47}$$

It now remains to show that $S_{K,\theta}(\Omega^*) \geq S_K(\mathbb{R}^N)$. In order to prove this inequality, we argue by contradiction. In this case, we can apply the result of Lemma 1, which guarantees that we can minimize the functional

$$E_K(v, \Omega^*) = \frac{\|v\|^2_{K,2,\Omega^*}}{\|v\|^2_{L^s(\Omega^*)}}, \quad v \in H_\theta^K(\Omega^*).$$

Let $u \in H_\theta^K(\Omega^*)$ be a minimum of E_K. It is common knowledge that we can assume without loss of generality that u is positive, radial, and decreasing in $r = |x|$ (observe that we can consider the solution v of (46) to be the minimizing function u). And, after a suitable dilation, we see that u satisfies the Euler–Lagrange equation (28) with $\lambda = 0$. However, by virtue of Lemmas 2 and 3, this leads to a contradiction. Hence, we conclude that

$$S_{K,\theta}(\Omega^*) \geq S_K(\mathbb{R}^N). \tag{48}$$

Combining (45), (47), and (48), we have proved that

$$S_{K,\theta}(\Omega) = S_K(\mathbb{R}^N).$$

It remains to show that $S_{K,\theta}(\Omega)$ (resp. $S_{K,0}(\Omega)$) is not achieved. Again, we argue by contradiction and assume that the infimum is achieved by some function u. As before, we consider $v \in H_\theta^K(\Omega^*)$ to be the solution of (46) for such u. The function v is positive, radial, and decreasing in r; moreover, it minimizes E_K in $H_\theta^K(\Omega^*)$. After a suitable dilation, we see that v would be a radial and positive solution of (28) for $\lambda = 0$, which (again) would contradict Lemmas 2 and 3. This completes the proof of Theorem 3. $\square$

3. Critical Dimensions

In this section, we are interested in the critical behavior of certain dimensions for the semilinear polyharmonic problem. We will try to give further evidence toward the conjecture of Pucci and Serrin. As we will see, these critical dimensions play an important role in deriving sharp Sobolev inequalities—in the spirit of what can be done in the case of complete manifold with the negative curvature (see [9; 10]).

Given $s > q \geq 2$ and $\lambda > 0$, we consider the functional

$$E_{K,q,\lambda,\Omega}(u) := \frac{\|u\|_{K,2,\Omega}^2 - \lambda\|u\|_{L^q(\Omega)}^2}{\|u\|_{L^s(\Omega)}^2} \quad \text{for any } \ u \in H_\theta^K(\Omega).$$

Let us define

$$S_{K,q,\lambda,\theta}(\Omega) := \inf_{u \in H_\theta^K(\Omega)\setminus\{0\}} E_{K,q,\lambda,\Omega}(u).$$

Using a proof similar to that in [3], we show the following lemma.

LEMMA 4. *Let Ω be a regular bounded domain in $\mathbb{R}^N$, and assume that $S_{K,q,\lambda,\theta}(\Omega) < S_K(\mathbb{R}^N)$. Then $S_{K,q,\lambda,\theta}(\Omega)$ is achieved by a function $u \in H_\theta^K(\Omega)$.*

Proof. Let $(u_n)_n$ be a minimizing sequence for $E_{K,q,\lambda,\Omega}$ in $H_\theta^K(\Omega)$. That is,

$$\|u_n\|_{L^s(\Omega)} = 1 \quad \text{and} \quad \|u_n\|_{K,2,\Omega}^2 - \lambda\|u_n\|_{L^q(\Omega)}^2 = S_{K,q,\lambda,\theta}(\Omega) + o(1).$$

In particular, $(u_n)_n$ is a bounded sequence in $H_\theta^K(\Omega)$. Up to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u \ \text{ weakly in } H_\theta^K(\Omega), \\ u_n &\rightharpoonup u \ \text{ weakly in } L^s(\Omega), \\ u_n &\to u \ \text{ strongly in } L^q(\Omega), \end{aligned}$$

so that

$$\begin{aligned} \|u_n\|_{K,2,\Omega}^2 &= \|u_n - u\|_{K,2,\Omega}^2 + \|u\|_{K,2,\Omega}^2 + o(1), \\ \|u_n\|_{L^s(\Omega)}^s &= 1 = \|u_n - u\|_{L^s(\Omega)}^s + \|u\|_{L^s(\Omega)}^s + o(1), \\ \|u_n\|_{L^q(\Omega)} &= \|u\|_{L^q(\Omega)} + o(1). \end{aligned} \tag{49}$$

We set $v_n := u_n - u$. It should be clear that

$$\|u\|_{K,2,\Omega}^2 - \lambda \|u\|_{L^q(\Omega)}^2 \ge S_{K,q,\lambda,\theta}(\Omega)\|u\|_{L^s(\Omega)}^2. \tag{50}$$

Combining (49) and (50), we obtain

$$\begin{aligned}\|v_n\|_{K,2,\Omega}^2 &= \lambda\|u\|_{L^q(\Omega)}^2 - \|u\|_{K,2,\Omega}^2 + S_{K,q,\lambda,\theta}(\Omega) + o(1)\\ &\le S_{K,q,\lambda,\theta}(\Omega)(1 - \|u\|_{L^s(\Omega)}^2) + o(1).\end{aligned} \tag{51}$$

Assume that $S_{K,q,\lambda,\theta}(\Omega) \le 0$. Then (51) leads to

$$\lim_{n\to\infty} \|v_n\|_{K,2,\Omega}^2 = 0, \tag{52}$$

since $\|u\|_{L^s(\Omega)}^2 \le 1$. Therefore, u is a minimizer.

Assume that $S_{K,q,\lambda,\theta}(\Omega) > 0$. It follows from Theorem 3 that

$$\|v_n\|_{L^s(\Omega)}^2 \le \frac{\|v_n\|_{K,2,\Omega}^2}{S_K(\mathbb{R}^N)}. \tag{53}$$

Using the fact $1 \le t^{2/s} + (1-t)^{2/s}$ holds for all $0 \le t \le 1$, we get from (49), (51), and (53) that

$$\begin{aligned}\|v_n\|_{K,2,\Omega}^2 &\le S_{K,q,\lambda,\theta}(\Omega)\|v_n\|_{L^s(\Omega)}^2 + o(1)\\ &\le \frac{S_{K,q,\lambda,\theta}(\Omega)}{S_K(\mathbb{R}^N)}\|v_n\|_{K,2,\Omega}^2 + o(1),\end{aligned}$$

which also implies (52). Again, this proves that u is a minimizer. This completes the proof of Lemma 4. □

We now turn to the proof of Theorem 1.

Proof of Theorem 1. Assume that λ is chosen sufficiently small so that

$$\|v\|_{K,2,\Omega}^2 - \lambda\|v\|_{L^q(\Omega)}^2 \ge 0 \quad \forall v \in H_\theta^K(\Omega).$$

As in Theorem 3, we have

$$E_{K,q,\lambda,\Omega}(v) \ge E_{K,q,\lambda,\Omega^*}(v^*) \quad \forall v \in H_\theta^K(\Omega). \tag{54}$$

Therefore, without loss of generality, we can assume that Ω is a ball. Inequality (9) is equivalent to $S_{K,q,\lambda,\theta}(\Omega) = S_K(\mathbb{R}^N)$ for some positive λ. We argue by contradiction. Assume that

$$S_{K,q,\lambda,\theta}(\Omega) < S_K(\mathbb{R}^N)$$

for all $\lambda > 0$. According to Lemma 4, we see that $S_{K,q,\lambda,\theta}(\Omega)$ is achieved by some u_λ. It follows from (54) that we can assume u_λ is positive, radial, and decreasing in r for any $\lambda > 0$ sufficiently small. After a suitable dilation, we obtain a positive radial solution of

$$\begin{cases} (-\Delta)^K u_\lambda = u_\lambda^{s-1} + \dfrac{\lambda u_\lambda^{q-1}}{\|u_\lambda\|_{L^q(\Omega)}^{q-2}} & \text{in } \Omega,\\ u_\lambda > 0 & \text{in } \Omega,\\ u_\lambda \in H_\theta^K(\Omega). \end{cases} \tag{55}$$

However, we can apply Pohozaev's identity as in Lemma 3 to find that

$$\begin{aligned}\left(\frac{N}{q}-\frac{N-2K}{2}\right)\lambda\|u_\lambda\|^2_{L^q(\Omega)} &\geq C\left(\int_\Omega |(-\Delta)^k u_\lambda|\right)^2 \\ &\geq C\|u_\lambda\|^2_{L_w^{N/(N-2K)}(\Omega)} \geq C\|u_\lambda\|^2_{L^q(\Omega)}. \end{aligned} \tag{56}$$

Therefore, necessarily $\lambda \geq \lambda_0 > 0$, which is clearly in contradiction with the fact that we can choose λ as small as we want. This completes the proof of Theorem 1. □

In order to describe the relations between inequality (9) and critical dimensions, we introduce the following.

DEFINITION 1. We will say that the dimension N is *weakly critical* if, for any bounded regular domain Ω, there exists a $\Lambda > 0$ such that, for all $\lambda \in (0, \Lambda)$,

$$S_{K,2,\lambda,\theta}(\Omega) = S_K(\mathbb{R}^N).$$

Granted this definition, we prove the following theorem.

THEOREM 4. *Let Ω be a regular bounded domain in $\mathbb{R}^N$, and let*

$$\lambda_{K,\theta}(\Omega) := \min_{u\in H_\theta^K(\Omega)\setminus\{0\}} \frac{\|u\|^2_{K,2,\Omega}}{\|u\|^2_{L^2(\Omega)}}$$

be the first eigenvalue of the operator $(-\Delta)^K$ in $H_\theta^K(\Omega)$. Then the following alternatives hold.

(1) *If $N \geq 4K$, then*

$$S_{K,2,\lambda,\theta}(\Omega) < S_K(\mathbb{R}^N)$$

for all $\lambda \in (0, \lambda_{K,\theta}(\Omega))$. In addition, $S_{K,2,\lambda,\theta}(\Omega)$ is achieved by a function u in $H_\theta^K(\Omega)$ that does not change sign in Ω. Finally, u is radial if Ω is a ball.

(2) *If $2K < N < 4K$, then there exists a $\Lambda \in (0, \lambda_{K,\theta}(\Omega))$ such that*

$$S_{K,2,\lambda,\theta}(\Omega) = S_{K,\theta}(\Omega) = S_K(\mathbb{R}^N)$$

for all $\lambda \in (0, \Lambda)$. In addition, $S_{K,2,\lambda,\theta}(\Omega)$ is not achieved.

The proof of Theorem 4 relies on the following result of Troy.

LEMMA 5 [17]. *Let $\Omega := B_R$, the ball of $\mathbb{R}^N$ of radius R and centered at the origin. Let u_i $(i = 1, \dots, n)$ denote a $C^2(\bar{B}_R)$ solution of*

$$-\Delta u_i = f_i(u_1, \dots, u_n)$$

in B_R, where the functions f_i are C^1 functions that satisfy

$$\frac{\partial f_i}{\partial u_k}(u_1, \dots, u_n) \geq 0 \quad \text{for } k \neq i,\ 1 \leq i, k \leq n.$$

Assume that, for all $i = 1, \dots, n$,

$$u_i > 0 \text{ in } B_R \quad \text{and} \quad u_i = 0 \text{ on } \partial B_R.$$

Then, for each $i = 1, \ldots, n$, the function u_i is radially symmetric and $\frac{\partial u_i}{\partial r}(s) < 0$ for $0 < s < R$.

We now turn to the proof of Theorem 4.

Proof of Theorem 4. Assume that $B(x, r) \subset \Omega$. As in [7] we choose $\xi \in C_0^\infty(B(x, r))$, a fixed (radial) cutoff function that satisfies $0 \le \xi \le 1$ on $B(x, r/2)$ and $|\nabla\xi| \le 4/r$. We define $w_\varepsilon := \xi u_\varepsilon \in C_0^\infty(\Omega)$, where

$$u_\varepsilon(x) := \frac{C_{N,K}\,\varepsilon^{(N-2K)/2}}{(\varepsilon^2 + |x|^2)^{(N-2K)/2}}$$

and where the choice of $C_{N,K}$ is designed to ensure that $\int_{\mathbb{R}^N} |u_\varepsilon(x)|^s \, dx = 1$.

A direct computation (see [7] for further details) leads to

$$E_{K,2,\lambda,\Omega}(w_\varepsilon) \le S_K(\mathbb{R}^N) - \lambda c_1 \varepsilon^{2K} + O(\varepsilon^{N-2K}) \quad \text{if } \ N > 4K$$

and

$$E_{K,2,\lambda,\Omega}(w_\varepsilon) \le S_K(\mathbb{R}^N) - \lambda c_2 \varepsilon^{2K} |\log \varepsilon| + O(\varepsilon^{2K}) \quad \text{if } \ N = 4K,$$

where c_1, c_2 are positive constants. Fixing $\varepsilon > 0$ sufficiently small, we conclude that

$$E_{K,2,\lambda,\Omega}(w_\varepsilon) < S_K(\mathbb{R}^N),$$

which in turn implies that

$$S_{K,2,\lambda,\theta}(\Omega) < S_K(\mathbb{R}^N).$$

It follows from the result of Lemma 4 that $S_{K,2,\lambda,\theta}(\Omega)$ is achieved by a function u in $H_\theta^K(\Omega)$, provided $N \ge 4K$ and $\lambda \in (0, \lambda_{K,\theta}(\Omega))$. Let u be a such a minimizer. We consider v the solution of

$$\begin{cases} (-\Delta)^M v = |(-\Delta)^M u| & \text{in } \Omega, \\ v = -\Delta v = \cdots = (-\Delta)^{M-1} v = 0 & \text{on } \partial\Omega, \end{cases} \tag{57}$$

where $M := \left[\frac{K}{2}\right]$. Clearly, $v \in H_\theta^K(\Omega)$. Moreover,

$$(-\Delta)^M (v \pm u) = |(-\Delta)^M u| \pm (-\Delta)^M u \ge 0 \ \text{ in } \Omega$$

and

$$v \pm u = -\Delta(v \pm u) = \cdots = (-\Delta)^{M-1}(v \pm u) = 0 \ \text{ on } \partial\Omega.$$

Applying the maximum principle, we obtain

$$(-\Delta)^{M-1}(v \pm u) \ge 0 \ \text{ in } \Omega. \tag{58}$$

Iterating this procedure, we conclude that

$$v \pm u \ge 0 \ \text{ in } \Omega. \tag{59}$$

Hence, $v(x) \ge |u(x)|$ for all $x \in \Omega$.

On the other hand, we have

$$\|v\|_{K,2,\Omega}^2 = \|u\|_{K,2,\Omega}^2. \tag{60}$$

Hence, we deduce that

$$E_{K,2,\lambda,\Omega}(v) \leq E_{K,2,\lambda,\Omega}(u).$$

This means v is also a minimizer and thus $v = |u|$. It follows from the strong maximum principle that

$$v(x) > 0 \text{ in } \Omega.$$

Observe that $(-\Delta)(v-u) \geq 0$ in Ω and $v-u=0$ on $\partial\Omega$. Using once more the strong maximum principle, we conclude that either $v(x) \equiv u(x)$ in Ω or $v(x) > u(x)$ in Ω. In the first case, we find $u = |u| > 0$ is positive in Ω. In the latter case, we conclude that $v(x) = -u(x)$ in Ω, that is, u is negative in Ω.

Now assume that Ω is a ball in $\mathbb{R}^N$. Further assume that u is a positive minimizer (otherwise it suffices to replace u by $-u$). After a suitable dilation, the function u satisfies the Euler–Lagrange equation

$$(-\Delta)^K u = \lambda u + u^{s-1} \geq 0 \text{ in } \Omega. \tag{61}$$

Therefore, we can apply the strong maximum principle and conclude that

$$(-\Delta)^i u > 0 \text{ in } \Omega \quad \forall 0 \leq i \leq K-1.$$

Setting $v_i := (-\Delta)^i u$ for $0 \leq i \leq K-1$, we obtain a solution to the following system:

$$(-\Delta) v_i = f_i(v_0, \ldots, v_{K-1}) \text{ in } \Omega \quad \forall 0 \leq i \leq K-1, \tag{62}$$

where

$$f_i(v_0, \ldots, v_{K-1}) = \begin{cases} v_{i+1} & \text{if } i < K-1, \\ |v_0|^{s-2} v_0 + \lambda v_0 & \text{if } i = K-1. \end{cases}$$

Obviously, $\frac{\partial f_i}{\partial v_j} \geq 0$ for $0 \leq i, j \leq K-1$. Making use of the result of Lemma 5, we conclude that v_i is radial symmetric. In particular, $u = v_0$ is radial and decreasing in r. This completes the proof of (1) in the statement of Theorem 4.

The proof of (2) is a simple corollary of Theorem 1 and Lemmas 2, 3, and 4. We leave the details to the reader. □

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