Weighted L^2-Cohomology of Bounded Domains with Smooth Compact Quotients

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1. Introduction

Let Ω be a bounded domain in $\mathbb{C}^n$. The Bergman metric on Ω is a Kähler metric invariant under the group $\mathrm{Aut}(\Omega)$ of biholomorphic automorphisms of Ω. Denote the Bergman metric on Ω by ds^2_Ω, and denote its Kähler form by ω. For $0 \le p, q \le n$ we denote by $\mathcal{H}^{p,q}_2(\Omega)$ the space of square integrable harmonic (p, q)-forms on Ω with respect to ds^2_Ω. When the boundary of Ω is smooth, Donnelly and Fefferman proved the following result.

THEOREM [DF]. *If Ω is a strictly pseudoconvex domain in $\mathbb{C}^n$, then*

$$\dim \mathcal{H}^{p,q}_2(\Omega) = \begin{cases} 0 & \text{if } p+q \neq n, \\ \infty & \text{if } p+q = n. \end{cases} \tag{1.1}$$

See also [D], where Donnelly gave an alternative proof of this theorem using a criterion of Gromov [Gro].

It is known that (1.1) also holds for bounded symmetric domains whose boundaries are not smooth in general (see [Gro] and [Ka]). It is thus natural to ask: Does (1.1) hold for bounded domains in C^n without any conditions on the boundary? An important class of bounded domains are those that cover compact manifolds, and they have been extensively studied (see e.g. [Ca; Fr; Kob; Si; V]). In this article, we consider the spaces of harmonic forms on such domains that are square integrable with respect to certain weight functions. Our result can be regarded as a partial affirmative answer to the above question for such domains.

For $z \in \Omega$, we denote by $d(z) = \mathrm{dist}(z; \partial\Omega)$ the Euclidean distance between z and the boundary $\partial\Omega$ of Ω. For $s \in \mathbb{R}$ we define

$$\mathcal{H}^{p,q}_{2,s}(\Omega) := \left\{ \phi \in \mathcal{A}^{p,q}(\Omega) \;\middle|\; \Box\phi = 0 \text{ and } \int_\Omega \|\phi(z)\|^2 \frac{1}{d(z)^s} \frac{\omega^n}{n!} < \infty \right\}. \tag{1.2}$$

Here $\Box$ and $\|\cdot\|$ denote (respectively) the Laplacian and the pointwise norm with respect to ds^2_Ω. It is easy to see that, for $s > 0$, each $\mathcal{H}^{p,q}_{2,s}(\Omega)$ forms a vector subspace of $\mathcal{H}^{p,q}_2(\Omega)$ and

$$\mathcal{H}^{p,q}_{2,s}(\Omega) \subset \mathcal{H}^{p,q}_{2,s'}(\Omega) \quad \text{if } s \ge s'. \tag{1.3}$$

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Let $K(z, w) \in C^\infty(\Omega \times \Omega)$ denote the Bergman kernel function on Ω. We define two constants $r_i = r_i(\Omega)$, $i = 1, 2$, given by

$$\begin{aligned} r_1 &:= \sup\{\, r \in \mathbb{R} \mid K(z,z) \geq C_1/d(z)^r \\ &\qquad\qquad \text{for some } C_1 > 0 \text{ and for all } z \in \Omega\,\}, \\ r_2 &:= \inf\{\, r \in \mathbb{R} \mid K(z,z) \leq C_2/d(z)^r \\ &\qquad\qquad \text{for some } C_2 > 0 \text{ and for all } z \in \Omega\,\}. \end{aligned} \tag{1.4}$$

Obviously one has $r_1 \leq r_2$ for any Ω, and one can easily construct examples for which $r_1 < r_2$. However, for a bounded strictly pseudoconvex domain, one always has $r_1 = r_2 = n + 1$ (see e.g. [Fe]). Our main result in this article is as follows.

MAIN THEOREM. *Let Ω be a bounded domain in $\mathbb{C}^n$, and let r_1, r_2 be as in* (1.4). *Suppose that there exists a discrete torsion-free subgroup $\Gamma \subset \mathrm{Aut}(\Omega)$ such that $\Gamma\backslash\Omega$ is compact. Then*

$$\dim \mathcal{H}^{p,q}_{2,s}(\Omega) = 0 \quad \textit{for any } s > \frac{r_2}{r_1}, \quad p + q \neq n. \tag{1.5}$$

In particular, one has

$$\dim \mathcal{H}^{p,q}_{2,s}(\Omega) = 0 \quad \textit{for any } s > n, \quad p + q \neq n. \tag{1.6}$$

Under the hypothesis of the Main Theorem, one necessarily has $2 \leq r_1 \leq r_2 \leq 2n$ (see Proposition 3.1 and Proposition 4.2). For each n, it is easy to verify that one indeed has $r_1 = 2$ and $r_2 = 2n$ when Ω is the unit polydisc $\triangle^n$ in $\mathbb{C}^n$. Also, the Bergman metric on Ω is necessarily complete, since it descends to a Kähler metric on the compact manifold $\Gamma\backslash\Omega$. Thus we have excluded those domains, such as the punctured unit disc in $\mathbb{C}$, whose Bergman metrics are incomplete.

We remark that there are bounded domains that admit smooth compact quotients but are not bounded symmetric domains. Such examples can be given by the universal covers of the Kodaira surfaces constructed in [Kod] (see [Fr, Remark 2.3] and [Gri, Lemma 6.2]).

The author does not know of any examples of bounded domains in $\mathbb{C}^n$ for which (1.1) or (1.5) fails to hold. Thus it would be interesting to know whether (1.5) can be improved or not. Gromov has even asked whether the stronger statement (1.1), which corresponds to the case $s = 0$, holds for all bounded domains of holomorphy. Our method does not seem to generalize directly to such cases.

The author learned about the problem of L^2-cohomology on bounded domains from Professor M. Gromov. The author would like to take this opportunity to express his thanks to Professor Gromov and Professor N. Mok for their enlightening conversations and valuable suggestions.

2. The Bergman Metric

Let Ω be a bounded domain in $\mathbb{C}^n$, and let $K(z, w) \in C^\infty(\Omega \times \Omega)$ denote the Bergman kernel function on Ω. The Bergman metric ds^2_Ω on Ω is a Kähler metric whose Kähler form ω is given by

$$\omega = \sqrt{-1}\partial\bar{\partial}\log K(z,z) = d\eta \quad \text{where } \eta := \sqrt{-1}K(z,z)^{-1}\partial K(z,z). \tag{2.1}$$

It is well known that $K(z,z)$ satisfies the following transformation rule:

$$K(z,z) = |\det(\partial\gamma)(z)|^2 \cdot K(\gamma(z),\gamma(z)) \quad \text{for all } z\in\Omega,\ \gamma\in \mathrm{Aut}(\Omega). \tag{2.2}$$

Let Γ be as in the Main Theorem, so that $\Gamma\backslash\Omega$ is a compact manifold. It is well known that ds^2_Ω is invariant under $\mathrm{Aut}(\Omega)$ and thus it descends to a Kähler metric on $\Gamma\backslash\Omega$. However, the $(1,0)$-form η is not invariant under Γ and thus the L^∞-norm of η may not be finite. Denote the projectivized tangent bundle of Ω by $\mathbb{P}T\Omega$, and define a function $\lambda\colon \mathbb{P}T\Omega\to\mathbb{R}$ as follows:

$$\lambda([X_z]) := \sup_f\left\{ |f(z)|^2 \;\middle|\; f\in\mathcal{O}(\Omega),\ X_z(f)=0, \text{ and } \int_\Omega |f(z)|^2\,d\mu \le 1\right\} \tag{2.3}$$

for $z\in\Omega$ and nonzero $X_z\in T_z\Omega$. Here $\mathcal{O}(\Omega)$ denotes the space of holomorphic functions on Ω, $[X_z]$ denotes the equivalence class of X_z in $\mathbb{P}T_z\Omega$, and $d\mu = (i/2)^n dz^1\wedge dz^2\wedge\cdots\wedge dz^n\wedge d\bar{z}^1\wedge d\bar{z}^2\wedge\cdots\wedge d\bar{z}^n$ denotes the Euclidean volume form. As in (1.2), we denote by $\|X_z\|$ the norm of X_z with respect to ds^2_Ω. We shall need the following observation of Donnelly, which is implicit in [D].

PROPOSITION 2.1. *For $z\in\Omega$ and nonzero $X_z\in T_z\Omega$, one has*

$$\frac{|\eta(X_z)|^2}{\|X_z\|^2} = \frac{K(z,z)}{\lambda([X_z])} - 1. \tag{2.4}$$

In particular, λ is a positive continuous function on $\mathbb{P}T\Omega$.

Proof. The formula in (2.4) is essentially [D, Prop. 3.1] stated in a precise manner, and it follows readily from the discussion in [D, p. 436]. By (2.4), one sees that the continuity of λ follows from that of η. □

3. The Kähler–Einstein Metric

We use the same notation as in Sections 1 and 2. Let Ω and Γ be as in the Main Theorem. Since Ω is a bounded domain in $\mathbb{C}^n$ admitting a smooth compact quotient $\Gamma\backslash\Omega$, it follows from a classical result of Siegel [Si] that Ω is necessarily a domain of holomorphy. Then, by a result of Mok and Yau [MY], there exists a complete Kähler–Einstein metric of negative Ricci curvature on Ω that is unique up to a constant multiple and invariant under $\mathrm{Aut}(\Omega)$. We shall denote this Kähler–Einstein metric by ds^2_{KE}. In this section, we study some consequences on Ω arising from ds^2_{KE}. Denote the volume form on Ω associated to ds^2_{KE} by dV_{KE}. In terms of Euclidean coordinates, we write

$$dV_{\mathrm{KE}} = V_{\mathrm{KE}}(z)(i/2)^n dz^1\wedge dz^2\wedge\cdots\wedge dz^n\wedge d\bar{z}^1\wedge d\bar{z}^2\wedge\cdots\wedge d\bar{z}^n. \tag{3.1}$$

Denote also the distance functions with respect to ds^2_{KE} and ds^2_Ω by $\delta_{\mathrm{KE}}(z;z')$ and $\delta_\Omega(z;z')$, respectively.

PROPOSITION 3.1. *Let Ω, Γ, r_1, r_2 be as in the Main Theorem.*

(i) *There exist constants $C_1, C_2, C_3, C_4 > 0$ such that*

$$C_1 \cdot ds^2_{\mathrm{KE}} \le ds^2_{\Omega} \le C_2 \cdot ds^2_{\mathrm{KE}} \quad \textit{and} \tag{3.2}$$

$$C_3 \cdot \delta(z; z') \le \delta_{\Omega}(z; z') \le C_4 \cdot \delta_{\mathrm{KE}}(z; z') \quad \textit{for all } z, z' \in \Omega. \tag{3.3}$$

(ii) *For any numbers $r_1' < r_1$ and $r_2' > r_2$, there exist constants $C_5 = C_5(r_1')$, $C_6 = C_6(r_2') > 0$ such that*

$$\frac{C_5}{d(z)^{r_1'}} \le V_{\mathrm{KE}}(z) \le \frac{C_6}{d(z)^{r_2'}} \quad \textit{for all } z \in \Omega. \tag{3.4}$$

(iii) *We have $r_1 \ge 2$.*

Proof. Being invariant under Γ, both ds^2_{KE} and ds^2_{Ω} descend to Kähler metrics on the compact manifold $\Gamma\backslash\Omega$; thus they are uniformly equivalent to each other on Ω, which gives (3.2). Then (3.3) is a direct consequence of (3.2), and this proves (i). For any numbers $r_1' < r_1$ and $r_2' > r_2$, it follows from (1.4) that there exist constants $C, C' > 0$ such that

$$\frac{C}{d(z)^{r_1'}} \le K(z, z) \le \frac{C'}{d(z)^{r_2'}} \quad \text{for all } z \in \Omega. \tag{3.5}$$

Since the (n, n)-form dV_{KE} is invariant under $\mathrm{Aut}(\Omega)$, $V_{\mathrm{KE}}(z)$ also satisfies the transformation rule in (2.2); that is,

$$V_{\mathrm{KE}}(z) = |\det(\partial\gamma)(z)|^2 \cdot V_{\mathrm{KE}}(\gamma(z)) \quad \text{for } z \in \Omega,\ \gamma \in \mathrm{Aut}(\Omega).$$

Together with (2.2), it follows that the ratio $V_{\mathrm{KE}}(z)/K(z, z)$ descends to a smooth positive function on $\Gamma\backslash\Omega$ and is thus bounded on Ω. This, together with (3.5), readily implies (3.4), and we have proved (ii). By [MY, Sec. 2.1], there exists a constant $C'' > 0$ such that

$$V_{\mathrm{KE}}(z) \ge \frac{C''}{d(z)^2(\log d(z))^2} \quad \text{for all } z \in \Omega.$$

This, together with the boundedness of $V_{\mathrm{KE}}(z)/K(z, z)$ on Ω, readily implies (iii), and we have finished the proof of Proposition 3.1. □

PROPOSITION 3.2. *Let Ω, Γ, r_1, r_2 be as in the Main Theorem. Then there exists a constant $C = C(\Omega) > 0$ such that, for any $\varepsilon > 0$, there exists a compact set $K = K(\varepsilon) \subset \Omega$ such that*

$$d(z') \ge d(z)^{(r_2/r_1+\varepsilon)\exp(C\cdot\delta_{\Omega}(z;z'))} \quad \textit{for all } z, z' \in \Omega\backslash K. \tag{3.6}$$

Proof. By [MY, Sec. 2.1], there exists a constant $C_1 = C_1(\Omega) > 0$ such that

$$\log(\log V_{\mathrm{KE}}(z') - c) - \log(\log V_{\mathrm{KE}}(z) - c) \le C_1 \cdot \delta_{\Omega}(z; z') \tag{3.7}$$

for all $z, z' \in \Omega$, where $c = \inf_{z\in\Omega} \log V_{\mathrm{KE}}(z)$ $(> -\infty)$. Rewriting (3.7) using (3.3), we have

$$\frac{\log V_{\mathrm{KE}}(z') - c}{\log V_{\mathrm{KE}}(z) - c} \leq \exp(C \cdot \delta_\Omega(z; z')) \tag{3.8}$$

for some constant $C > 0$ and all $z, z' \in \Omega$. Given $\varepsilon > 0$, we choose numbers $r_1', r_1'', r_2', r_2'' > 0$ such that

$$r_1'' < r_1' < r_1, \quad r_2 < r_2' < r_2'', \quad \text{and} \quad \frac{r_2''}{r_1''} < \frac{r_2}{r_1} + \varepsilon. \tag{3.9}$$

By (3.4) and (3.9), there exist constants C_2, C_3 such that

$$\frac{C_2 - r_1' \log d(z')}{C_3 - r_2' \log d(z)} \leq \exp(C \cdot \delta_\Omega(z; z')) \tag{3.10}$$

for all $z, z' \in \Omega$. Then it is easy to see that one can choose a compact set $K = K(\varepsilon) \subset \Omega$ (enlarging K if necessary) such that

$$\begin{gathered} d(\xi) < 1, \quad C_2 - r_1' \log d(\xi) > -r_1'' \log d(\xi), \\ C_3 - r_2' \log d(\xi) < -r_2'' \log d(\xi) \end{gathered} \tag{3.11}$$

for all $\xi \in \Omega \backslash K$. Then we have, for $z, z' \in \Omega \backslash K$,

$$\begin{aligned} \exp(C \cdot \delta_\Omega(z; z')) &\geq \frac{-r_1'' \log d(z')}{-r_2'' \log d(z)} \quad \text{(by (3.10), (3.11))} \\ &\geq \frac{1}{(r_2/r_1 + \varepsilon)} \cdot \frac{\log d(z')}{\log d(z)} \quad \text{(by (3.9)).} \end{aligned} \tag{3.12}$$

Notice that $d(z) < 1$. Then (3.6) follows readily from (3.12), and we have finished the proof of Proposition 3.2. □

4. Proof of the Main Theorem

Before we give the proof of the Main Theorem, we first prove several lemmas. Notation remains the same as before.

LEMMA 4.1. *Let Ω be a bounded domain in $\mathbb{C}^n$. Then there exist constants $C_i = C_i(\Omega) > 0$ $(i = 1, 2)$ such that, for any $\gamma \in \mathrm{Aut}(\Omega)$ and $z \in \Omega$,*

$$|\partial\gamma(z)| \leq \frac{C_1}{d(z)} \quad \textit{and} \tag{4.1}$$

$$|\det(\partial\gamma)(z)| \leq \frac{C_2}{d(z)^n}. \tag{4.2}$$

Here $\partial\gamma(z)$ denotes the Jacobian matrix $(\partial\gamma^i/\partial z_j(z))_{1\leq i,j\leq n}$ in Euclidean coordinates, and $|\partial\gamma(z)|$ denotes its norm $\sqrt{\sum_{1\leq i,j\leq n} |\partial\gamma^i/\partial z_j(z)|^2}$.

Proof. Inequality (4.1) follows easily from the Cauchy integral formula and the fact that the coordinate functions of all $\gamma \in \mathrm{Aut}(\Omega)$ are uniformly bounded; (4.2) is a direct consequence of (4.1). □

PROPOSITION 4.2. *Let Ω, Γ, r_2 be as in the Main Theorem. Then there exists a constant $C > 0$ such that*

$$K(z, z) \leq \frac{C}{d(z)^{2n}} \quad \text{for all } z \in \Omega. \tag{4.3}$$

In particular, we have $r_2 \leq 2n$.

Proof. Since $\Gamma\backslash\Omega$ is compact, it is well known that one can construct a fundamental domain D of $\Gamma\backslash\Omega$ in Ω; that is, $D \subset\subset \Omega$, $\pi|_D$ is one-to-one, and $\pi|_{\bar{D}}$ is onto. Here $\pi : \Omega \to \Gamma\backslash\Omega$ denotes the projection map and $\bar{D}$ denotes the closure of D in Ω. Since $\bar{D}$ is compact, there exists a constant $C_1 > 0$ such that $K(\xi, \xi) \leq C_1$ for all $\xi \in \bar{D}$. For any $z \in \Omega$, since $\pi|_{\bar{D}}$ is onto, there exists a $\gamma \in \Gamma$ such that $\gamma(z) \in \bar{D}$. Now, by (2.2) and Lemma 4.1,

$$\begin{aligned} K(z, z) &= |\det(\partial\gamma)(z)|^2 \cdot K(\gamma(z), \gamma(z)) \\ &\leq \frac{C_1 C_2}{d(z)^{2n}} \quad \text{for all } z \in \Omega, \end{aligned}$$

where C_2 is the constant in (4.2), and this proves (4.3). Then the inequality $r_2 \leq 2n$ follows readily from (1.4) and (4.3), and we have finished the proof of Proposition 4.2. □

For a point $z \in \mathbb{C}^n$ and $r > 0$, we denote the Euclidean ball by $B(z; r) := \{\xi \in \mathbb{C}^n \mid |\xi - z| < r\}$.

PROPOSITION 4.3. *Let Ω be a bounded domain in $\mathbb{C}^n$, and let K be a compact subset of Ω. Suppose r is a number such that $0 < r < d(z)$ for all $z \in K$. Then there exists a constant $C = C(\Omega, K, r) > 0$ such that, for all $\gamma \in \mathrm{Aut}(\Omega)$, $z \in K$, and $\xi \in B(z; r)$,*

$$|\det(\partial\gamma)(\xi)| \leq C|\det(\partial\gamma)(z)|^{\frac{d(z)-r}{d(z)+r}}. \tag{4.4}$$

Proof. First we observe that it follows from the bound on r and the compactness of K and $\partial\Omega$ that $\bigcup_{z\in K} B(z; r) \subset\subset \Omega$. Thus there exists a constant $d_o = d_o(\Omega, K, r) > 0$ such that $d(\xi) \geq d_o$ for all $\xi \in \bigcup_{z\in K} B(z; r)$. By Lemma 4.1, there exists a constant $C' = C'(\Omega) > 0$ such that, for any $\gamma \in \mathrm{Aut}(\Omega)$, $z \in K$, and $\xi \in B(z; r)$,

$$|\det(\partial\gamma)(\xi)| \leq \frac{C'}{d(\xi)^n} \leq \frac{C'}{d_o^n}. \tag{4.5}$$

Since $\gamma \in \mathrm{Aut}(\Omega)$, $\det(\partial\gamma)(\xi)$ is a nonvanishing holomorphic function in ξ. Together with (4.5), it follows that $\log(C'/(d_o^n \cdot |\det(\partial\gamma)(\xi)|))$ is a nonnegative pluriharmonic function. It then follows from the Harnack inequality for nonnegative harmonic functions (see e.g. [GT, p. 29]) that

$$\begin{aligned} \log\left(\frac{C'}{d_o^n \cdot |\det(\partial\gamma)(\xi)|}\right) &\geq \frac{d(z) - |\xi - z|}{d(z) + |\xi - z|} \log\left(\frac{C'}{d_o^n \cdot |\det(\partial\gamma)(z)|}\right) \\ &\geq \frac{d(z) - r}{d(z) + r} \log\left(\frac{C'}{d_o^n \cdot |\det(\partial\gamma)(z)|}\right) \end{aligned} \tag{4.6}$$

for all $z \in K$ and $\xi \in B(z; r)$. Rewriting (4.6), one has

$$|\det(\partial\gamma)(\xi)| \le \left(\frac{C'}{d_o^n}\right)^{\frac{2r}{d(z)+r}} |\det(\partial\gamma)(z)|^{\frac{d(z)-r}{d(z)+r}}$$
$$\le C \cdot |\det(\partial\gamma)(z)|^{\frac{d(z)-r}{d(z)+r}},$$

where we may let $C = \max\{1, (C'/d_o^n)^2\}$. This finishes the proof of Proposition 4.3. □

PROPOSITION 4.4. *Let Ω, Γ, r_1, r_2 be as in the Main Theorem. Then, for any number $s > r_2/r_1$, there exists a constant $C = C(\Omega, s) > 0$ such that*

$$\lambda([X_z]) \ge C \cdot d(z)^{2s} \cdot K(z, z) \tag{4.7}$$

for all $z \in \Omega$ and nonzero $X_z \in T_z\Omega$.

Proof. For any $s > r_2/r_1$, we write $s = r_2/r_1 + 3\varepsilon$, where $\varepsilon > 0$. Let D be a fundamental domain of $\Gamma\backslash\Omega$ in Ω as in Proposition 4.2. Since ds_Ω^2 is complete and $\bar{D}$ is compact, there exists a constant $r' > 0$ such that $\bigcup_{\xi\in\bar{D}} B(\xi; 2r') \subset\subset \Omega$. It follows that there exists a constant $C' > 0$ such that

$$\delta_\Omega(\xi; \xi') \le C'|\xi - \xi'| \quad \text{for all } \xi \in \bar{D} \text{ and } \xi' \in B(\xi; 2r'). \tag{4.8}$$

Let C be the constant and let $K = K(\varepsilon) \subset \Omega$ be the compact set in Proposition 3.2. Fix a sufficiently small number $r'' > 0$ such that

$$\left(\frac{r_2}{r_1} + \varepsilon\right) \exp(C \cdot C' \cdot 2r'') < \frac{r_2}{r_1} + 2\varepsilon. \tag{4.9}$$

Also let $d_o = \operatorname{dist}(\bar{D}, \partial\Omega) > 0$ be the Euclidean distance of $\bar{D}$ from $\partial\Omega$, and let $r''' > 0$ be sufficiently small that

$$\frac{4nr'''}{d_o} \le \varepsilon. \tag{4.10}$$

Finally, we let $r := \min\{r', r'', r'''\} > 0$ so that (4.8), (4.9), and (4.10) remain valid with (respectively) r', r'', r''' replaced by r. Since ds_Ω^2 is complete and $K \subset \Omega$ is compact, there exists a compact set K' (enlarging K' if necessary) such that $K \subset K' \subset \Omega$,

$$\bigcup_{\xi\in\bar{D}} B(\xi; 2r) \subset\subset K', \quad \text{and}$$
$$d(z) < 1, \ \ \delta_\Omega(z; K) > 2rC' \ \text{ for } z \in \Omega\backslash K'. \tag{4.11}$$

Next we shall show that (4.7) holds on $\Omega\backslash K'$. First, for each $\xi \in \bar{D}$, we introduce a smooth cutoff function χ_ξ (with $0 \le \chi_\xi \le 1$) on Ω such that

$$\chi_\xi(\xi') = \begin{cases} 1 & \text{if } \xi' \in B(\xi; r), \\ 0 & \text{if } \xi' \in \Omega\backslash B(\xi; 2r), \quad \text{and} \end{cases}$$
$$|d\chi_\xi(\xi')| \le \frac{2}{r} \quad \text{if } \xi' \in B(\xi; 2r)\backslash B(\xi; r). \tag{4.12}$$

Also, we define the plurisubharmonic weight functions

$$v_\xi(\xi') = 2(n+2)\log|\xi'-\xi| \quad \text{and} \quad w(\xi') = \log(1+|\xi'|^2) \tag{4.13}$$

on Ω. Since Ω is a bounded domain, there exist constants $C_1, C_2 > 0$ such that

$$C_1 \le \exp(-2w(\xi')) \le C_2 \quad \text{for all } \xi' \in \Omega. \tag{4.14}$$

Also we have, for $\xi \in \Omega$,

$$\begin{aligned} \exp(-v_\xi(\xi')) &\le \frac{1}{r^{2n+4}} \quad \text{if } \xi' \in \Omega\backslash B(\xi; r), \\ \exp(-v_\xi(\xi')) &\ge \frac{1}{(2r)^{2n+4}} \quad \text{if } \xi' \in B(\xi; 2r). \end{aligned} \tag{4.15}$$

From now on, we let z be any point in $\Omega\backslash K'$. Since D is a fundamental domain of $\Gamma\backslash\Omega$ in Ω, there exists $\gamma \in \Gamma$ such that $z' := \gamma(z) \in \bar{D}$. We denote its inverse by $\tau := \gamma^{-1} \in \Gamma$ so that $\tau(z') = z$. Then $\rho_z := \chi_{z'} \circ \gamma$ is a smooth cutoff function on Ω such that $\rho_z(\xi) = 1$ for ξ near z. Thus $X_z(\rho_z) = 0$ for any nonzero $X_z \in T_z\Omega$. Set $\beta = \bar{\partial}\rho_z$ on Ω. Then β is supported on $\tau(B(z'; 2r)\backslash B(z'; r)) \subset\subset \Omega$. The function $\exp(-v_{z'} \circ \gamma)$ is not integrable only at z. Thus we have

$$\int_\Omega |\beta(\xi)|^2 \exp(-v_{z'} \circ \gamma(\xi))\, d\mu(\xi) < \infty, \tag{4.16}$$

where $d\mu(\xi)$ denotes the Euclidean volume form. Observe that the function $v_{z'} \circ \gamma$ is also plurisubharmonic on Ω. By L^2-estimates of $\bar{\partial}$ of Hörmander ([H1, p. 94] or [H2]), there exists an $h \in \mathcal{C}^\infty(\Omega)$ such that $\bar{\partial}h = \beta$ and

$$\begin{aligned} \int_\Omega |h(\xi)|^2 \exp(-v_{z'} \circ \gamma(\xi) - 2w(\xi))\, d\mu(\xi) \\ \le \int_\Omega |\beta(\xi)|^2 \exp(-v_{z'} \circ \gamma(\xi))\, d\mu(\xi). \end{aligned} \tag{4.17}$$

Since $\beta = \bar{\partial}(\chi_{z'} \circ \gamma) = \bar{\partial}\chi_{z'} \circ \bar{\partial}\bar{\gamma}$, it follows from (4.12) that the integrand on the right-hand side of (4.17) is supported on $\tau(B(z'; 2r)\backslash B(z'; r))$. For any $\xi \in \tau(B(z'; 2r)\backslash B(z'; r))$,

$$\begin{aligned} &|\beta(\xi)|^2 \exp(-v_{z'} \circ \gamma(\xi)) \\ &\quad \le |\bar{\partial}\chi_{z'}(\gamma(\xi))|^2 |\partial\gamma(\xi)|^2 \exp(-v_{z'} \circ \gamma(\xi)) \\ &\quad \le \left(\frac{2}{r}\right)^2 \cdot \frac{C}{d(\xi)^2} \cdot \frac{1}{r^{2n+4}} \quad \text{(by (4.11), Lemma 4.1, and (4.15))}. \end{aligned} \tag{4.18}$$

Also, it follows from the invariance of ds^2_Ω under $\mathrm{Aut}(\Omega)$ and (4.8) that

$$\delta_\Omega(\xi; z) \le 2rC' \quad \text{for all } \xi \in \tau(B(z'; 2r)). \tag{4.19}$$

Since $z \in \Omega\backslash K'$, it follows from (4.19) and the last inequality of (4.11) that $\tau(B(z'; 2r)) \subset \Omega\backslash K$. Recall from (4.11) that $d(z) < 1$. Then, by Proposition 3.2, for $\xi \in \tau(B(z; 2r))$ we have

$$\begin{aligned}\frac{1}{d(\xi)} &\le \frac{1}{d(z)^{(r_2/r_1+\varepsilon)\exp(C\cdot\delta_\Omega(z;z'))}}\\ &\le \frac{1}{d(z)^{(r_2/r_1+\varepsilon)\exp(2rC'C)}} \quad \text{(by (4.19))}\\ &\le \frac{1}{d(z)^{r_2/r_1+2\varepsilon}} \quad \text{(by (4.9))}. \end{aligned} \tag{4.20}$$

Let $C_3 = C_3(\Omega, \bar{D}, 2r) > 0$ be the constant in Proposition 4.3. Recall that $z' \in \bar{D}$. By making the change of variables $\xi' = \gamma(\xi)$, or equivalently $\xi = \tau(\xi')$, we have

$$\begin{aligned}&\int_{\tau(B(z';2r))} d\mu(\xi)\\ &\quad= \int_{B(z';2r)} |\det(\partial\tau)(\xi')|^2\, d\mu(\xi')\\ &\quad\le \int_{B(z';2r)} C_3\cdot|\det(\partial\tau)(z')|^{\frac{2(d(z')-2r)}{d(z')+2r}}\, d\mu(\xi') \quad \text{(by Proposition 4.3)}\\ &\quad\le C_3C_4\cdot|\det(\partial\tau)(z')|^{\frac{2(d(z')-2r)}{d(z')+2r}}(2r)^{2n} \end{aligned} \tag{4.21}$$

for some constant $C_4 = C_4(n) > 0$. Since $\bar{D}$ is compact, there exists a constant $C_5 > 1$ such that

$$K(\xi', \xi') \le C_5 \quad \text{for all } \xi' \in \bar{D}. \tag{4.22}$$

Also, since $z' \in \bar{D}$, it follows from (4.10) that

$$\frac{8nr}{d(z')+2r} \le \frac{8nr}{d_o} \le 2\varepsilon, \tag{4.23}$$

where d_o is as in (4.10). Then

$$\begin{aligned}|\det(\partial\tau)(z')|^{\frac{2(d(z')-2r)}{d(z')+2r}} &= \left(\frac{K(z',z')}{K(z,z)}\right)^{\frac{d(z')-2r}{d(z')+2r}} \quad \text{(by (2.2))}\\ &\le \frac{C_5}{K(z,z)}\cdot K(z,z)^{\frac{4r}{d(z')+2r}} \quad \text{(by (4.22))}\\ &\le \frac{C_5}{K(z,z)}\cdot\frac{C_6}{d(z)^{\frac{2n\cdot 4r}{d(z')+2r}}} \quad \text{(by Proposition 4.2)}\\ &\le \frac{C_5\cdot C_6}{K(z,z)\cdot d(z)^{2\varepsilon}} \quad \text{(by (4.23))}, \end{aligned} \tag{4.24}$$

where C_6 is the constant in Proposition 4.2. Combining (4.21) and (4.24), we have

$$\begin{aligned}\int_{\tau(B(z';2r))} d\mu(\xi) &\le \frac{C_3C_4C_5C_6(2r)^{2n}}{K(z,z)\cdot d(z)^{2\varepsilon}}\\ &= \frac{C_7}{K(z,z)\cdot d(z)^{2\varepsilon}}, \end{aligned} \tag{4.25}$$

where $C_7 = C_7(r) > 0$ is independent of $z \in \Omega\backslash K'$. Then we have, for $z \in \Omega\backslash K'$,

$$\begin{aligned}
&\int_\Omega |\beta(\xi)|^2 \exp(-v_{z'} \circ \gamma(\xi))\, d\mu(\xi) \\
&\quad \le \int_{\tau(B(z';2r)\backslash B(z';r))} \left(\frac{2}{r}\right)^2 \cdot \frac{C}{d(\xi)^2} \cdot \frac{1}{r^{2n+4}}\, d\mu(\xi) \quad \text{(by (4.18))} \\
&\quad \le \left(\frac{2}{r}\right)^2 \cdot \frac{C}{d(z)^{2(r_2/r_1+2\varepsilon)}} \cdot \frac{1}{r^{2n+4}} \int_{\tau(B(z';2r)\backslash B(z';r))} d\mu(\xi) \quad \text{(by (4.20))} \\
&\quad \le \left(\frac{2}{r}\right)^2 \cdot \frac{C}{d(z)^{2(r_2/r_1+2\varepsilon)}} \cdot \frac{1}{r^{2n+4}} \cdot \frac{C_7}{K(z,z)\cdot d(z)^{2\varepsilon}} \quad \text{(by (4.25))} \\
&\quad = \frac{C_8}{K(z,z)\cdot d(z)^{2r_2/r_1+6\varepsilon}}, \qquad (4.26)
\end{aligned}$$

where $C_8 = C_8(r) > 0$ is independent of $z \in \Omega\backslash K'$. From (4.14) and (4.15), for $z \in \Omega\backslash K'$ we have

$$\begin{aligned}
\int_\Omega |h(\xi)|^2\, d\mu(\xi) &\le \frac{1}{C_1} \cdot (2r)^{2n+4} \int_\Omega |h(\xi)|^2 \exp(-v_{z'} \circ \gamma(\xi) - 2w(\xi))\, d\mu(\xi) \\
&\le \frac{C_8 \cdot (2r)^{2n+4}}{C_1 \cdot K(z,z)\cdot d(z)^{2r_2/r_1+6\varepsilon}} \quad \text{(by (4.17) and (4.26))} \\
&= \frac{C_9}{K(z,z)\cdot d(z)^{2r_2/r_1+6\varepsilon}}, \qquad (4.27)
\end{aligned}$$

where $C_9 = C_9(r) > 0$ is independent of $z \in \Omega\backslash K'$. Also, since γ is biholomorphic at z, it follows from (4.13) that, for an open neighborhood U such that $z \in U \subset\subset \Omega$, there exists a constant $C_{10} > 0$ such that

$$\exp(-v_{z'} \circ \gamma(\xi)) = \frac{1}{|\gamma(\xi) - \gamma(z)|^{2n+4}} \ge \frac{C_{10}}{|\xi - z|^{2n+4}} \qquad (4.28)$$

for $\xi \in U$. Then it follows from (4.14), (4.16), (4.17), and (4.28) that

$$\int_U \frac{|h(\xi)|^2}{|\xi - z|^{2n+4}}\, d\mu(\xi) < \infty,$$

which implies that $h(z) = 0$ and $X_z(h) = 0$ for any nonzero $X_z \in T_z\Omega$. For $z \in \Omega\backslash K'$, we let $f = h - \rho_z$. Then f is holomorphic on Ω, and we have $|f(z)|^2 = 1$ and $X_z(f) = 0$ for any nonzero $X_z \in T_z\Omega$. Moreover, by (4.25) and (4.27) we have

$$\begin{aligned}
\int_\Omega |f(\xi)|^2\, d\mu(\xi) &\le \frac{1}{2}\left(\int_\Omega |h(\xi)|^2\, d\mu(\xi) + \int_\Omega |\rho_z(\xi)|^2\, d\mu(\xi)\right) \\
&\le \frac{1}{2}\left(\frac{C_9}{K(z,z)\cdot d(z)^{2r_2/r_1+6\varepsilon}} + \frac{C_7}{K(z,z)\cdot d(z)^{2\varepsilon}}\right) \\
&\le \frac{C_{11}}{K(z,z)\cdot d(z)^{2s}} \quad (\text{since } s = r_2/r_1 + 3\varepsilon), \qquad (4.29)
\end{aligned}$$

where $C_{11} = C_{11}(r) > 0$ is independent of $z \in \Omega\backslash K'$. Finally, we let $\hat{f} = f\big/\big(\int_\Omega f^2\, d\mu\big)^{1/2}$. Then $\int_\Omega \hat{f}^2\, d\mu = 1$, $X_z(\hat{f}) = 0$ for any nonzero $X_z \in T_z\Omega$, and, by (4.29), $|\hat{f}(z)|^2 \geq (1/C_{11}) \cdot K(z,z) \cdot d(z)^{2s}$. Using $\hat{f}$ as a test function in (2.3), it follows easily that (4.7) holds for all $z \in \Omega\backslash K'$ and nonzero $X_z \in T_z\Omega$. By Proposition 2.1, λ is a positive continuous function on $\mathbb{P}T\Omega$. It follows easily that (4.7) actually holds for all $z \in \Omega$ and nonzero $X_z \in T_z\Omega$. Thus we have finished the proof of Proposition 4.4. □

PROPOSITION 4.5. *For any number $s > r_2/r_1$, there exists a constant $C = C(\Omega, s) > 0$ such that*

$$\|\eta(z)\| \leq \frac{C}{d(z)^s} \quad \textit{for all } z \in \Omega. \tag{4.30}$$

Proof. Proposition 4.5 can be obtained easily by combining Proposition 2.1 and Proposition 4.4. □

We also recall the following well-known result of Gaffney.

PROPOSITION 4.6 [Ga]. *Let M be an m-dimensional complete Riemannian manifold. Suppose that ν is an $(m-1)$-form on M such that the L^1 norms of both ν and $d\nu$ are finite. Then*

$$\int_M d\nu = 0.$$

As usual, the L^1-norm of a form μ is given by $\|\mu\|_{L^1} := \int_M \|\mu(x)\|\, d\,\mathrm{vol}$.

We can now give the proof of the Main Theorem.

Proof of the Main Theorem. Let Ω be as in the Main Theorem, and let s be such that $s > r_2/r_1$. First we deal with the case when $p+q < n$. For any $\phi \in \mathcal{H}^{p,q}_{2,s}(\Omega)$ with $p+q < n$, we consider the $(2n-1)$-form

$$\nu := \phi \wedge \bar{\phi} \wedge \eta \wedge \omega^{n-p-q-1} \quad \text{on } \Omega, \tag{4.31}$$

where η is as in (2.1). It is easy to see that there exists a constant $C_1 = C_1(n, p, q) > 0$ such that

$$\|\nu(z)\| \leq C_1 \|\phi(z)\|^2 \|\eta(z)\| \quad \text{for any } z \in \Omega. \tag{4.32}$$

Let $C_2 = C_2(\Omega, s) > 0$ be the constant in Proposition 4.5. Then

$$\begin{aligned} \|\nu\|_{L^1} = \int_\Omega \|\nu(z)\| \frac{\omega^n}{n!} &\leq C_1 \int_\Omega \|\phi(z)\|^2 \|\eta(z)\| \frac{\omega^n}{n!} \\ &\leq C_1 C_2 \int_\Omega \|\phi(z)\|^2 \cdot \frac{1}{d(z)^s} \frac{\omega^n}{n!} \quad \text{(by Proposition 4.5)} \\ &< \infty \quad \text{(since } \phi \in \mathcal{H}^{p,q}_{2,s}(\Omega)\text{)}. \end{aligned} \tag{4.33}$$

Since $\mathcal{H}^{p,q}_{2,s}(\Omega) \subset \mathcal{H}^{p,q}_2(\Omega)$ and ds^2_Ω is complete, it follows that ϕ and $\bar{\phi}$ are d-closed (see e.g. [Gro, 1.1.B]). Together with (2.1) and (4.31), we have $d\nu =$

$\phi \wedge \bar{\phi} \wedge \omega^{n-p-q}$. Since ϕ is of pure type (p, q) with $p+q < n$, it is easy to check that there exists a constant $C_3 = C_3(n, p, q) > 0$ such that

$$dv(z) = C_3 \|\phi(z)\|^2 \frac{\omega^n(z)}{n!} \quad \text{for all } z \in \Omega. \tag{4.34}$$

In particular, one has $\|dv(z)\| = C_3\|\phi(z)\|^2$. Then

$$\|dv\|_{L^1} = \int_\Omega \|dv(z)\| \frac{\omega^n}{n!} = C_3 \int_\Omega \|\phi(z)\|^2 \frac{\omega^n}{n!} < \infty \quad (\text{since } \mathcal{H}^{p,q}_{2,s}(\Omega) \subset \mathcal{H}^{p,q}_2(\Omega)). \tag{4.35}$$

By Proposition 4.5, it follows from (4.33) and (4.35) that $\int_\Omega dv = 0$. Together with (4.34), it follows easily that $\phi \equiv 0$. Thus we have proved (1.5) for the case when $p+q < n$. To deal with the case when $p+q > n$, we denote by L the operator of exterior multiplication by ω, that is, $L\varphi = \varphi \wedge \omega$. Then it is well known that L preserves harmonic forms, and for $p+q < n$ there exist constants $C_i = C_i(n, p, q) > 0$ $(i = 4, 5)$ such that, for $\varphi \in \mathcal{A}^{p,q}(\Omega)$,

$$C_4\|\varphi(z)\| \le \|L^{n-p-q}\varphi(z)\| \le C_5\|\varphi(z)\| \quad \text{for all } z \in \Omega \tag{4.36}$$

(see e.g. [Gro, 1.2.A′]). Then it follows easily that L^{n-p-q} induces an isomorphism between $\mathcal{H}^{p,q}_{2,s}(\Omega)$ and $\mathcal{H}^{n-q,n-p}_{2,s}(\Omega)$ for $p+q < n$. Thus (1.5) for the case when $p+q > n$ follows from that for the case when $p+q < n$. Finally it follows from Proposition 3.1 and Proposition 4.2 that $2 \le r_1 \le r_2 \le 2n$. This inequality and (1.5) readily imply (1.6), and we have completed the proof of the Main Theorem. □

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