Higher-Dimensional Analogs of Hermite's Constant

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Introduction

For integers $n > 1$, Hermite's constant is the smallest number γ_n such that, for all lattices $\Lambda \subset \mathbb{R}^n$ of rank n, there is a nonzero lattice point $\mathbf{x} \in \Lambda$ with

$$\|\mathbf{x}\| \le \gamma_n^{1/2} \det(\Lambda)^{1/n}.$$

Here $\|\mathbf{x}\|$ denotes the usual Euclidean length of $\mathbf{x}$. Hermite was the first to prove the existence of such a constant. He showed that

$$\gamma_n \le \gamma_{n-1}^{(n-1)/(n-2)} \tag{1}$$

for $n > 2$. Using (1) and a quick induction argument gives $\gamma_n \le \gamma_2^{n-1}$. After verifying that $\gamma_2 = 2/\sqrt{3}$, Hermite arrived at the upper bound $\gamma_n \le (2/\sqrt{3})^{n-1}$. Later, Minkowski used his first convex bodies theorem (see [3]) to show that

$$\gamma_n \le 4V(n)^{-2/n}, \tag{2}$$

where $V(n)$ denotes the volume of the unit ball in $\mathbb{R}^n$. Note that this upper bound for γ_n grows linearly in n as $n \to \infty$, as opposed to the exponential growth of Hermite's original upper bound.

Note that, by introducing a scaling factor, we may restrict to lattices of determinant 1 in the definition of Hermite's constant (see Lemma 4). Minkowski's work on the space of such lattices led him to state (without proof) that

$$\gamma_n \ge \left(\frac{2\zeta(n)}{V(n)}\right)^{2/n}. \tag{3}$$

This result was first proven by Hlawka (see [3, Sec. 19]); it is a special case of what is now called the Minkowski–Hlawka theorem. This, along with Minkowski's upper bound stated in (2), shows that γ_n in fact grows linearly in n as $n \to \infty$. It is not known whether γ_n/n approaches a limit as $n \to \infty$. The exact value of γ_n is known only for $n \le 8$ (see [3]).

Hermite's constant is directly related to the densest lattice packing of spheres in $\mathbb{R}^n$, and through this to many areas of mathematics and even other natural sciences (number theory, lie algebras, numerical integration, chemistry, and digital

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communication, just to name a few). The comprehensive book by Conway and Sloane [1] is an excellent source of such applications. Finding the value of γ_n, or even a better understanding of its growth as $n \to \infty$, is an important and much studied problem.

In this paper we will consider two ways of extending the notion of Hermite's constant. The first generalization we consider originated in a paper of Rankin [4], who extended Hermite's original work. For $0 < d < n$, define (a generalized) Hermite's constant to be the smallest number $\gamma_{n,d}$ such that, for any lattice $\Lambda \subset \mathbb{R}^n$ of rank n and determinant 1, there is a rank-d sublattice of determinant $\leq \gamma_{n,d}^{1/2}$. Then $\gamma_{n,1} = \gamma_n$. Extending Hermite's method, Rankin showed such a constant exists and satisfies

$$\gamma_{n,d} \leq \gamma_{m,d}(\gamma_{n,m})^{d/m} \tag{4}$$

for $0 < d < m < n$. As we will see, this reduces to Hermite's original inequality (1) when $d = 1$ and $m = n - 1$. In [4], no upper or lower bounds for general $\gamma_{n,d}$ are given.

Another way to extend Hermite's constant comes from arithmetic geometry and the notion of height. In [7] it is shown that Hermite's constant can be described as the smallest number γ_n such that, for all A in the general linear group $\mathrm{GL}_n(\mathbb{Q}_{\mathbb{A}})$ over the ring of adeles $\mathbb{Q}_{\mathbb{A}}$, there is an $\mathbf{x} \in \mathbb{P}^{n-1}(\mathbb{Q})$ with twisted height

$$H_A(\mathbf{x}) \leq \gamma_n^{1/2}|\det(A)|_{\mathbb{A}}^{1/n}.$$

(The relevant definitions will be given in Section 2.) This leads to the following generalization of Hermite's constant. For a number field K and $n > 1$, let Hermite's constant for K be the smallest number $\gamma_n(K)$ such that, for all $A \in \mathrm{GL}_n(K_{\mathbb{A}})$, there is an $\mathbf{x} \in \mathbb{P}^{n-1}(K)$ with

$$H_A(\mathbf{x}) \leq (\gamma_n(K))^{1/2}|\det(A)|_{\mathbb{A}}^{1/n[K:\mathbb{Q}]}.$$

Then $\gamma_n(\mathbb{Q}) = \gamma_n$. The "adelic" versions of Minkowski's first convex bodies theorem and the Minkowski–Hlawka theorem give generalizations to the bounds (2) and (3) for $\gamma_n(K)$. (See also Theorem 1 of [2].)

In this paper we will study the following generalization of Hermite's constant, which includes both Rankin's and the one just discussed. We use $\mathrm{Gr}_{n,d}(K)$ to denote the Grassmannian of d-planes in K^n.

DEFINITION. Let K be a number field and let $0 < d < n$. Hermite's constant is the smallest number $\gamma_{n,d}(K)$ such that, for all $A \in \mathrm{GL}_n(K_{\mathbb{A}})$, there is a $V \in \mathrm{Gr}_{n,d}(K)$ with

$$H_A(V) \leq (\gamma_{n,d}(K))^{1/2}|\det(A)|_{\mathbb{A}}^{d/n[K:\mathbb{Q}]}.$$

As explained in [7], we have $\gamma_{n,d}(\mathbb{Q}) = \gamma_{n,d}$. In particular, $\gamma_{n,1}(\mathbb{Q}) = \gamma_n$. We will prove a generalization of Rankin's result (4) and generalizations of the bounds (2) and (3). Specifically, we will prove the following.

THEOREM 1. *Let K be a number field and let $0 < d < n$. Then Hermite's constant $\gamma_{n,d}(K)$ exists and*

$$(\gamma_{n,d}(K))^{[K:\mathbb{Q}]/2d} \leq \frac{2^{r+s}|D(K)|^{1/2}}{V(n)^{r/n}V(2n)^{s/n}}. \tag{5}$$

We also have

$$\gamma_{n,d}(K) \leq \gamma_{m,d}(K)(\gamma_{n,m}(K))^{d/m} \tag{6}$$

for all $0 < d < m < n$. *Here* $D(K)$ *denotes the discriminant of* K, r *and* s *denote the number of real and complex places of* K, *respectively, and* $V(l)$ *denotes the volume of the unit ball in* $\mathbb{R}^l$ *for* $l > 1$.

Note that (5) and (6) generalize Minkowski's bound (2) and Rankin's inequality (4), respectively. The next result generalizes (3).

THEOREM 2. *Let* K *be a number field and let* $0 < d < n$. *Then*

$$(\gamma_{n,d}(K))^{n[K:\mathbb{Q}]/2} \geq \frac{w(K)n}{h(K)R(K)} \cdot \frac{\displaystyle\prod_{j=n-d+1}^{n} \frac{\zeta_K(j)|D(K)|^{j/2}}{j^{r+s}2^{js}V(j)^r V(2j)^s}}{\displaystyle\prod_{l=2}^{d} \frac{\zeta_K(l)|D(K)|^{l/2}}{l^{r+s}2^{ls}V(l)^r V(2l)^s}}.$$

Here, in addition to the notation in Theorem 1, $h(K)$, $R(K)$, *and* $w(K)$ *denote the class number, the regulator, and the number of roots of unity in* K, *respectively;* ζ_K *is the Dedekind zeta function of* K. (*As usual, the empty product is interpreted as* 1.)

Theorem 2 in the case $d = 1$ and $K = \mathbb{Q}$ yields the lower bound (3). In particular, we have the following corollary.

COROLLARY 1. *Let* $0 < d < n$. *Then Rankin's generalized Hermite's constant satisfies*

$$\frac{2^{nd}}{(V(n))^d} \geq (\gamma_{n,d})^{n/2} \geq 2n \cdot \frac{\displaystyle\prod_{j=n-d+1}^{n} \frac{\zeta(j)}{jV(j)}}{\displaystyle\prod_{l=2}^{d} \frac{\zeta(l)}{lV(l)}}.$$

Combining the upper and lower bounds in Theorems 1 and 2 gives the following estimate for the growth of $\gamma_{n,d}(K)$.

COROLLARY 2. *Let* K *be a number field and let* $0 < d$. *Then*

$$\log(\gamma_{n,d}(K)) = d \log n + O(1)$$

as $n \to \infty$, *where the implicit constant depends on* d *and* K.

The proof of Theorem 1 uses properties of the twisted height and the adelic version of Minkowski's second convex bodies theorem. The proof of Theorem 2 is more difficult, involving a mean value argument on certain homogeneous spaces and the computation of the measures of certain subsets of these spaces. In the next section we will give the definition of twisted heights along with some relevant results concerning them. Section 3 is devoted to the proof of Theorem 1. Section 4 lays out the mean value argument for proving Theorem 2, and the last section contains a measure computation that completes the proof of Theorem 2.

1. Twisted Heights

Throughout this paper K will denote a number field. In addition to the notation in the statements of Theorems 1 and 2, we let $K_{\mathbb{A}}$, $K_{\mathbb{A}}^{\times}$, and $M(K)$ denote the ring of adeles, idele group, and set of places of K, respectively. For each $v \in M(K)$, let K_v denote the completion of K at v and let $\mathfrak{O}_v$ denote the maximal compact subring of K_v whenever v is finite.

Let α_v be the Haar measure on K_v obtained by taking $\alpha_v(\mathfrak{O}_v) = 1$ if v is finite; α_v is the usual Lebesgue measure on $\mathbb{R}$ if v is real and is twice the usual measure on $\mathbb{C}$ when v is complex. We obtain a Haar measure α on $K_{\mathbb{A}}$ given by

$$\alpha = |D(K)|^{-1/2} \cdot \prod_v \alpha_v.$$

We will write α^n for the Haar measure on $(K_{\mathbb{A}})^n$ derived by taking the product measure. Then α^n is the Tamagawa measure on $(K_{\mathbb{A}})^n$, and we note that

$$\alpha^n((K_{\mathbb{A}})^n/K^n) = 1 \tag{7}$$

(see [10, Chap. 2]).

For each place v we define $|\cdot|_v$ on K_v by $\alpha_v(aM) = |a|_v \alpha_v(M)$ for any measurable set $M \subset K_v$ and $a \in K_v$. We then have the product formula [9, Chap. 4, Thm. 5]

$$\prod_{v \in M(K)} |x|_v = 1$$

for all nonzero $x \in K$. We let $|\cdot|_{\mathbb{A}}$ denote the module on $K_{\mathbb{A}}^{\times}$ as in [9]:

$$|a|_{\mathbb{A}} = \prod_{v \in M(K)} |a_v|_v.$$

Given an infinite place v, we let $\|\cdot\|_v$ denote the usual L^2 norm on $(K_v)^n$ raised to the local degree:

$$\|\mathbf{x}\|_v = \begin{cases} \left(\sum_{i=1}^n |x_i|_v^2\right)^{1/2} & \text{if } v \text{ is real,} \\ \sum_{i=1}^n |x_i|_v & \text{if } v \text{ is complex.} \end{cases}$$

If v is a finite place, we let

$$\|\mathbf{x}\|_v = \max_{1 \le i \le n} \{|x_i|_v\}.$$

For $\mathbf{x} \in K^n$ and $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ with local components A_v, define

$$\|A(\mathbf{x})\|_{\mathbb{A}} = \prod_{v \in M(K)} \|A_v(\mathbf{x})\|_v.$$

For any $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ and $\mathbf{x} \in K^n$, one sees by the product formula that $\|A(\mathbf{x})\|_{\mathbb{A}}$ is invariant under scalar multiplication of $\mathbf{x}$ by nonzero elements of K. Thus, the heights H_A for $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ defined by

$$H_A(\mathbf{x}) = \|A(\mathbf{x})\|_{\mathbb{A}}^{1/[K:\mathbb{Q}]}$$

are really heights on projective space $\mathbb{P}^{n-1}(K)$. These are the "twisted heights" in [5] and [7]. In the case where $A = I_n$ is the identity element of $\mathrm{GL}_n(K_{\mathbb{A}})$, $H_{I_n}(\mathbf{x})$ is the "usual" absolute multiplicative Weil height using L^2 norms at the infinite places. We extend H_A to $\mathrm{Gr}_{n,d}(K)$ via exterior products. Specifically, if $V \subset K^n$ is a d-dimensional subspace, then $\wedge^d V \in \mathbb{P}^{\binom{n}{d}-1}(K)$ and we define $H_A(V)$ to be $H_{\wedge^d A}(\wedge^d V)$. We also set $H_A(\{\mathbf{0}\}) = 1$ and $H_A(K^n) = |\det(A)|_{\mathbb{A}}$ for any $A \in \mathrm{GL}_n(K_{\mathbb{A}})$.

We end this section with some auxiliary results.

LEMMA 1 [5, Lemma 3.2, Cor. 4.3]. *Let $v \in M(K)$ and $A \in \mathrm{GL}_n(K_v)$. Let $0 < m \le n$ and let $\phi_v \colon K_v^m \to K_v^n$ be an injective K_v-linear map. Then there is a $B \in \mathrm{GL}_m(K_v)$ such that*

$$\|(\wedge^d B)(\mathbf{x})\|_v = \|(\wedge^d A)((\wedge^d \phi)(\mathbf{x}))\|_v$$

for all $\mathbf{x} \in \wedge^d((K_v)^m)$ and $0 < d < m$.

If $\phi \colon K^m \to K^n$ is an injective K-linear map and $A \in \mathrm{GL}_n(K_{\mathbb{A}})$, then there is a $B \in \mathrm{GL}_m(K_{\mathbb{A}})$ that satisfies

$$H_B(V) = H_A(\phi(V))$$

for all $0 < d < m$ and $V \in \mathrm{Gr}_{m,d}(K)$.

LEMMA 2 [6, Duality Theorem]. *Let $0 \le d \le n$. For a $V \in \mathrm{Gr}_{n,d}(K)$, let $V^* \in \mathrm{Gr}_{n,n-d}(K)$ denote the subspace orthogonal to V with respect to the canonical bilinear form on K^n. For $A \in \mathrm{GL}_n(K_{\mathbb{A}})$, let $A^* = (A^{-1})^{\mathrm{tr}}$, where "tr" denotes the transpose. Then, for any $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ and any $V \in \mathrm{Gr}_{n,d}(K)$,*

$$H_A(V) = H_{A^*}(V^*)|\det(A)|_{\mathbb{A}}.$$

Thus,

$$\gamma_{n,d}(K) = \gamma_{n,n-d}(K).$$

LEMMA 3 [8, Lemma 4; 6, Thm. 2]. *Let $v \in M(K)$ and $0 < d < n$. Given a $V_v \in \mathrm{Gr}_{n,d}(K_v)$, there is a linear map $P_v \colon K_v^n \to V_v$ such that, for all $\mathbf{x} \in K_v^n \setminus V_v$ and bases $\{\mathbf{x}_1, \ldots, \mathbf{x}_d\}$ of V_v,*

$$\|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d \wedge \mathbf{x}\|_v = \|\mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d\|_v \cdot \|\mathbf{x} - P_v(\mathbf{x})\|_v.$$

Let $A \in \mathrm{GL}_n(K_{\mathbb{A}})$ and $V \in \mathrm{Gr}_{n,d}(K)$. There is an isomorphism $\phi \colon K^{n-d} \to K^n/V$ (as K-vector spaces) and a $B \in \mathrm{GL}_{n-d}(K_{\mathbb{A}})$ such that

$$H_A(W + V) = H_A(V)\, H_B(W)$$

for all $W \in \mathrm{Gr}_{n-d,m}(K)$ and $m \le n - d$, where

$$W+V=\{\mathbf{x}\in K^n : \mathbf{x}+V\in\phi(W)\}\in \mathrm{Gr}_{n,d+m}(K).$$

In particular,

$$|\det(B)|_{\mathbb{A}}=\frac{|\det(A)|_{\mathbb{A}}}{H_A(V)^{[K:\mathbb{Q}]}}.$$

LEMMA 4. *Hermite's constant is the smallest number $\gamma_{n,d}(K)$ such that, for all $A\in \mathrm{GL}_n(K_{\mathbb{A}})$ with $|\det(A)|_{\mathbb{A}}=1$, there is a $V\in\mathrm{Gr}_{n,d}(K)$ with*

$$H_A(V)\le(\gamma_{n,d}(K))^{1/2}.$$

Proof. Let $A\in\mathrm{GL}_n(K_{\mathbb{A}})$. Let $a\in K_{\mathbb{A}}^{\times}$ with $|a|_{\mathbb{A}}=|\det(A)|_{\mathbb{A}}^{-1/n}$ and let $D\in\mathrm{GL}_n(K_{\mathbb{A}})$ be the diagonal element with diagonal entries all a. Then $|\det(DA)|_{\mathbb{A}}=1$ and

$$H_A(V)=H_{DA}(V)|\det(A)|_{\mathbb{A}}^{d/n[K:\mathbb{Q}]}$$

for all $0\le d\le n$ and $V\in\mathrm{Gr}_{n,d}(K)$. □

2. Proof of Theorem 1

Fix a number field K. To ease readability, we will write $\gamma_{n,d}$ for $\gamma_{n,d}(K)$ throughout this section. By the adelic version of Minkowski's second convex bodies theorem (see e.g. the corollary to [7, Thm. 3]), for any $A\in\mathrm{GL}_n(K_{\mathbb{A}})$ there are linearly independent $\mathbf{x}_1,\dots,\mathbf{x}_n\in K^n$ with

$$H_A(\mathbf{x}_1)\cdots H_A(\mathbf{x}_n)\le\frac{2^{n(r+s)}|D(K)|^{n/2}}{V(n)^rV(2n)^s}|\det(A)|_{\mathbb{A}}.$$

We may assume without loss of generality that $H_A(\mathbf{x}_1)\le\cdots\le H_A(\mathbf{x}_n)$. By [5, Lemma 4.7], for $0<d\le n$ we have

$$H_A(V)\le H_A(\mathbf{x}_1)\cdots H_A(\mathbf{x}_d),$$

where $V\in\mathrm{Gr}_{n,d}(K)$ is generated by $\mathbf{x}_1,\dots,\mathbf{x}_d$. This proves the first part of Theorem 1.

For the second part, let $0<d<m<n$ and let $A\in\mathrm{GL}_n(K_{\mathbb{A}})$ with $|\det(A)|_{\mathbb{A}}=1$. Let $V\in\mathrm{Gr}_{n,d}(K)$ with smallest height $H_A(V)$ and let $W\in\mathrm{Gr}_{n,m}(K)$ with $H_A(W)\le\gamma_{n,m}^{1/2}$. Let $\phi\colon K^m\to K^n$ be an injective linear map with image W. By Lemma 1 there is a $B\in\mathrm{GL}_m(K_{\mathbb{A}})$ that satisfies $H_B(T)=H_A(\phi(T))$ for all subspaces $T\subseteq K^m$.

There is a $T\in\mathrm{Gr}_{m,d}(K)$ with

$$H_B(T)\le\gamma_{m,d}^{1/2}|\det(B)|_{\mathbb{A}}^{d/m[K:\mathbb{Q}]}=\gamma_{m,d}^{1/2}H_A(W)^{d/m}\le\gamma_{n,d}^{1/2}\gamma_{n,m}^{d/2m}.$$

By construction,

$$H_A(V)\le H_A(\phi(T))=H_B(T)\le\gamma_{m,d}^{1/2}\gamma_{n,m}^{d/2m}.$$

This proves (6) by Lemma 4.

Another way to prove (6) is as follows. Let $0<d<m<n$ and let $A\in\mathrm{GL}_n(K_{\mathbb{A}})$ with $|\det(A)|_{\mathbb{A}}=1$. Take a $V\in\mathrm{Gr}_{n,n-m}(K)$ with $H_A(V)\le\gamma_{n,n-m}^{1/2}$.

By Lemma 3, there is a $B \in \mathrm{GL}_m(K_{\mathbb{A}})$ and a $\phi\colon K^m \to K^n/V$ with $|\det(B)|_{\mathbb{A}}^{-1} = H_A(V)^{[k:\mathbb{Q}]}$ and

$$H_A(V+W) = H_A(V)\,H_B(W)$$

for all $W \in \mathrm{Gr}_{m,m-d}(K)$. Let $W \in \mathrm{Gr}_{m,m-d}(K)$ with

$$H_B(W) \le \gamma_{m,m-d}^{1/2}|\det(B)|_{\mathbb{A}}^{(m-d)/m[K:\mathbb{Q}]}.$$

Then $V + W \in \mathrm{Gr}_{n,n-d}(K)$ satisfies

$$\begin{aligned} H_A(V+W) &\le H_A(V)\gamma_{m,m-d}^{1/2}|\det(B)|_{\mathbb{A}}^{(m-d)/m[K:\mathbb{Q}]} \\ &= H_A(V)\gamma_{m,m-d}^{1/2}H_A(V)^{(d-m)/m} \\ &= H_A(V)^{d/m}\gamma_{m,m-d}^{1/2} \\ &\le \gamma_{m,m-d}^{1/2}\cdot\gamma_{n,n-m}^{d/2m}. \end{aligned}$$

Thus, $\gamma_{n,n-d} \le \gamma_{m,m-d}\cdot\gamma_{n,n-m}^{d/m}$ by Lemma 4, and (6) follows from this and Lemma 2.

3. A Mean Value Argument

The proof of Theorem 2 uses a mean value argument and is more involved than the proof of Theorem 1. In this section we give this mean value argument; the proof of Theorem 2 will be completed in the next section, where we will carry out a certain computation.

We define

$$G_n = \{A \in \mathrm{GL}_n(K_{\mathbb{A}}) : |\det(A)|_{\mathbb{A}} = 1\}.$$

Note that $\mathrm{GL}_n(K)$ is a discrete subgroup of G_n and that $G_n/\mathrm{GL}_n(K)$ is compact. One can construct an invariant Haar measure on G_n as in [7], where by "invariant" we mean that the measure is invariant under multiplication on the left or right. Then $G_n/\mathrm{GL}_n(K)$ has finite measure; we let μ_n be the invariant Haar measure on G_n with

$$\mu_n(G_n/\mathrm{GL}_n(K)) = 1. \tag{8}$$

Define

$$G_{n,d} = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in G_n : A \in G_d,\ C \in G_{n-d} \right\},$$

so that $\mathrm{GL}_n(K)/(\mathrm{GL}_n(K) \cap G_{n,d}) = \mathrm{Gr}_{n,d}(K)$. Let $\mu_{n,d}$ be the measure on $G_{n,d}$ given by

$$d\mu_{n,d}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = d\mu_d(A) \times d\alpha^{(n-d)d}(B) \times d\mu_{n-d}(C).$$

Then

$$\mu_{n,d}\big(G_{n,d}/(G_{n,d} \cap \mathrm{GL}_n(K))\big) = 1 \tag{9}$$

by (7) and (8).

We will write $d\nu_{n,d}$ for the relatively invariant gauge form on the homogeneous space $G_n/G_{n,d}$ that satisfies $d\mu_n = d\nu_{n,d} \times d\mu_{n,d}$ in the sense of [10]. In other words, for any integrable function f on G_n,

$$\int_{G_n} f(A)\, d\mu_n(A) = \int_{G_n/G_{n,d}} \int_{G_{n,d}} f(AB)\, d\mu_{n,d}(B)\, d\nu_{n,d}(AG_{n,d}).$$

For $t > 0$ we will denote the characteristic function of $[0,t)$ by χ_t. For l a positive integer, let $S^l = \prod_v S_v^l \subset (K_{\mathbb{A}})^l$ be defined by

$$S_v^l = \begin{cases} \{\mathbf{x}_v \in (K_v)^l : \|\mathbf{x}_v\|_v < 1\} & \text{if } v \text{ is Archimedean,} \\ (\mathfrak{O}_v)^l & \text{if } v \text{ is non-Archimedean.} \end{cases}$$

Then

$$\alpha^l(S^l) = \frac{2^{lr_2} V(l)^{r_1} V(2l)^{r_2}}{|D(K)|^l}. \tag{10}$$

For an $n \times d$ matrix $X = (\mathbf{x}_1^{\mathrm{tr}} \cdots \mathbf{x}_d^{\mathrm{tr}})$ with $\mathbf{x}_i \in (K_{\mathbb{A}})^n$, define $\psi : X \to (K_{\mathbb{A}})^{\binom{n}{d}}$ by

$$\psi(X) = \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_d.$$

Define

$$\begin{aligned} f_{n,d}(X) &= \inf_{a \in K_{\mathbb{A}}^\times} \{ |a|_{\mathbb{A}} : \psi(X) \in aS^{\binom{n}{d}} \} \\ &= \inf_{A \in \mathrm{GL}_d(K_{\mathbb{A}})} \{ |\det(A)| : \psi(XA^{-1}) \in S^{\binom{n}{d}} \}. \end{aligned}$$

Certainly $f_{n,d}(X)$ is invariant under multiplication on the right by elements of G_d, so we may view $f_{n,d}$ as a function on $G_n/G_{n,d}$, for example (where it is continuous). Note that $f_{n,d}(A(V)) = H_A(V)^{[K:\mathbb{Q}]}$ for all $V \in \mathrm{Gr}_{n,d}(K)$ and $A \in \mathrm{GL}_n(K_{\mathbb{A}})$. Let

$$c(n,d) = \int_{G_n/G_{n,d}} \chi_1(f_{n,d}(AG_{n,d}))\, d\nu_{n,d}(AG_{n,d}).$$

LEMMA 5. *Let $D \in \mathrm{GL}_n(K_{\mathbb{A}})$ and $t > 0$. Then*

$$\int_{G_n/G_{n,d}} \chi_t(f_{n,d}(DAG_{n,d}))\, d\nu_{n,d}(AG_{n,d}) = |\det(D)|_{\mathbb{A}}^{-d} c(n,d) t^n.$$

For any measurable function f on $\mathbb{R}$,

$$\int_{G_n/G_{n,d}} f(f_{n,d}(AG_{n,d}))\, d\nu_{n,d}(AG_{n,d}) = nc(n,d) \int_0^\infty f(x) x^{n-1}\, dx.$$

Proof. Let

$$G'_{n,d} = \left\{ \begin{pmatrix} I_d & B \\ 0 & C \end{pmatrix} \in G_{n,d} : C \in G_{n-d} \right\}$$

and let $\mu'_{n,d}$ be the measure on $G'_{n,d}$ given by

$$d\mu'_{n,d} \begin{pmatrix} I_d & B \\ 0 & C \end{pmatrix} = d\alpha^{(n-d)d}(B) \times d\mu_{n-d}(C),$$

so that $d\mu_{n,d} = d\mu_d \times d\mu'_{n,d}$. Let $d\nu'_{n,d}$ be the relatively invariant gauge form on $G_n/G'_{n,d}$ that satisfies $d\mu_n = d\nu'_{n,d} \times d\mu'_{n,d}$. Then, by the uniqueness of Haar measure, $d\nu'_{n,d} = c \cdot d\alpha^{nd}$ for some $c > 0$.

By [9, Chap. 4, Thm. 6], $\mathrm{GL}_d(K_{\mathbb{A}}) \cong \mathbb{R}_+^\times \times G_d$, where $\mathbb{R}_+^\times$ is the multiplicative group of positive real numbers. Here $r = |\det(r \cdot A)|_{\mathbb{A}}$ for $r \in \mathbb{R}_+^\times$ and $A \in G_d$, where $r \cdot A$ denotes the image under an isomorphism from $\mathbb{R}_+^\times \times G_d$ to $\mathrm{GL}_d(K_{\mathbb{A}})$. Let Γ be a fundamental set modulo $\mathrm{GL}_d(K)$ of G_d. For $n \times d$ matrices X define

$$f'_{n,d}(X) = \inf_{r>0}\{\, r : \psi(X(r \cdot A)^{-1}) \in S^{\binom{n}{d}} \text{ for some } A \in \Gamma \,\}.$$

Since $\mu_d(\Gamma) = 1$ by (8), we have

$$\begin{aligned}
&\int_{G_n/G_{n,d}} \chi_1(f_{n,d}(DAG_{n,d}))\, d\nu_{n,d}(AG_{n,d}) \\
&\qquad = \int_{G_n/G'_{n,d}} \chi_1(f'_{n,d}(DAG'_{n,d}))\, d\nu'_{n,d}(AG'_{n,d}) \\
&\qquad = c \int_{(K_{\mathbb{A}})^{nd}} \chi_1(f'_{n,d}(DX)) \prod_{i,j} d\alpha(x_{ij}) \\
&\qquad = |\det(D)|_{\mathbb{A}}^{-d} c \int_{(K_{\mathbb{A}})^{nd}} \chi_1(f'_{n,d}(X)) \prod_{i,j} d\alpha(x_{ij}) \\
&\qquad = |\det(D)|_{\mathbb{A}}^{-d} c(n,d).
\end{aligned}$$

This proves the first part of the lemma when $t = 1$.

Now let $t > 0$ and let $a \in K_{\mathbb{A}}^\times$ with $|a|_{\mathbb{A}} = t^{-1/d}$. Let $D_t \in \mathrm{GL}_n(K_{\mathbb{A}})$ be the diagonal element with diagonal entries all equal to a. Then

$$\begin{aligned}
&\int_{G_n/G_{n,d}} \chi_t(f_{n,d}(DAG_{n,d}))\, d\nu_{n,d}(AG_{n,d}) \\
&\qquad = \int_{G_n/G_{n,d}} \chi_1(f_{n,d}(D_t DAG_{n,d}))\, d\nu_{n,d}(AG_{n,d}) \\
&\qquad = |\det(D_t D)|_{\mathbb{A}}^{-d} c(n,d) \\
&\qquad = |\det(D)|_{\mathbb{A}}^{-d} c(n,d) t^n
\end{aligned}$$

by what we have already shown. This shows that the first part of the lemma is true. In particular, the second part of the lemma is true for $f = \chi_t$, and thus for f any simple function. The case for general f follows by approximating with simple functions. □

We now give our mean value argument for proving Theorem 2.

LEMMA 6. *Let $t > 0$ satisfy $t^{-n} > c(n,d)$. Then there is an $A \in G_n$ such that, for all $V \in \mathrm{Gr}_{n,d}(K)$,*

$$H_A(V)^{[K:\mathbb{Q}]} > t.$$

In particular, by Lemma 4,

$$\gamma_{n,d}(K) \geq c(n,d)^{2/n[K:\mathbb{Q}]}.$$

This will prove Theorem 2 once we compute $c(n,d)$, which will be done in the next section.

Proof. Let $\varepsilon > 0$ satisfy $(t^n + \varepsilon)c(n,d) < 1$. Let

$$f(x) = \begin{cases} 1 & \text{if } x \leq t, \\ (x/t)^{-n-nt^n/\varepsilon} & \text{if } x > t. \end{cases}$$

Then, by Lemma 5,

$$\begin{aligned} \int_{G_n/G_{n,d}} f(f_{n,d}(AG_{n,d}))\, d\nu_{n,d}(AG_{n,d}) &= nc(n,d) \int_0^\infty x^{n-1} f(x)\, dx \\ &\leq (t^n + \varepsilon)c(n,d) \\ &< 1. \end{aligned}$$

Since $f \circ f_{n,d}$ is a positive, continuous, and integrable function, we may apply [10, Lemma 2.4.2] to yield

$$\begin{aligned} &\int_{G_n/G_{n,d}} f(f_{n,d}(AG_{n,d}))\, d\nu_{n,d}(AG_{n,d}) \\ &\qquad = \int_{G_n/\mathrm{GL}_n(K)} \left[\sum_{V \in \mathrm{Gr}_{n,d}(K)} f\big(f_{n,d}(A(V))\big) \right] d\mu_n(A), \end{aligned}$$

by (9). By (8), there is an $A \in G_n$ such that

$$1 > \sum_{V \in \mathrm{Gr}_{n,d}(K)} f\big(f_{n,d}(A(V))\big).$$

Thus, $H_A(V)^{[K:\mathbb{Q}]} = f_{n,d}(A(V)) > t$ for all $V \in \mathrm{Gr}_{n,d}(K)$. □

4. A Computation

We will not compute $c(n,d)$ directly for $d > 1$, but instead prove the following theorem.

THEOREM 3. *For* $0 < d < n$,

$$c(n,d) = \frac{\binom{n}{d}}{n} \cdot \frac{\prod_{i=0}^{d-1} c(n-i,1)}{\prod_{j=2}^{d} c(j,j-1)},$$

where the empty product is interpreted as 1.

Note that Theorem 3 implies $c(n,d) = c(n,n-d)$. In particular, $c(j,j-1) = c(j,1)$ and so Theorem 2 follows from Lemma 6, Theorem 3, (10), and the following result.

LEMMA 7 [7, Lemma 1]. *For $n > 1$,*

$$c(n,1) = \frac{\alpha^n(S^n) n^{r+s-1} h(K) R(K)}{w(K)\zeta_K(n)}.$$

Our proof of Theorem 3 requires some preliminary considerations. Define

$$\begin{aligned} g_{n,d} &= \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in G_{n,d} : A \in G_{d,d-1} \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in G_{n,d} : C \in G_{n-d+1,1} \right\} \\ &= G_{n,d} \cap G_{n,d-1}. \end{aligned}$$

For $D \in g_{n,d}$ we may write

$$D = \begin{pmatrix} A & B_1 & B_2 \\ 0 & B_3 & B_4 \\ 0 & 0 & C \end{pmatrix},$$

where $A \in G_{d-1}$, $B_3 \in G_1$, and $C \in G_{n-d}$. We let $\sigma_{n,d}$ be the measure on $g_{n,d}$ given by

$$\begin{aligned} d\sigma_{n,d}(D) = d\mu_{d-1}(A) &\times d\alpha^{d-1}(B_1) \times d\alpha^{(n-d)d}(B_2) \\ &\times d\mu_1(B_3) \times d\alpha^{n-d}(B_4) \times d\mu_{n-d}(C). \end{aligned}$$

Writing

$$A_0 = \begin{pmatrix} A & B_1 \\ 0 & B_3 \end{pmatrix} \quad \text{and} \quad C_0 = \begin{pmatrix} B_3 & B_4 \\ 0 & C \end{pmatrix},$$

we have

$$\begin{aligned} d\sigma_{n,d}(D) &= d\mu_{d,d-1}(A_0) \times d\alpha^{(n-d)d}(B_2) \times d\alpha^{n-d}(B_4) \times d\mu_{n-d}(C) \\ &= d\mu_{d-1}(A) \times d\alpha^{d-1}(B_1) \times d\alpha^{(n-d)d}(B_2) \times d\mu_{n-d+1,1}(C_0). \quad (11) \end{aligned}$$

LEMMA 8. *For $A \in G_d$ and $C \in G_{n-d+1}$, write*

$$A' = \begin{pmatrix} A & 0 \\ 0 & I_{n-d} \end{pmatrix} \in G_{n,d} \quad \text{and} \quad C' = \begin{pmatrix} I_{d-1} & 0 \\ 0 & C \end{pmatrix} \in G_{n,d-1}.$$

Let $\tau_{n,d}$ be the relatively invariant gauge form on the homogeneous space $G_n/g_{n,d}$ that satisfies $d\mu_n = d\tau_{n,d} \times d\sigma_{n,d}$. Then, for f any measurable function on $G_n/g_{n,d}$, we have

$$\begin{aligned} &\int_{G_n/g_{n,d}} f(Dg_{n,d})\, d\tau_{n,d}(Dg_{n,d}) \\ &\quad = \int_{G_n/G_{n,d}} \int_{G_d/G_{d,d-1}} f(DA'g_{n,d})\, d\nu_{d,d-1}(AG_{d,d-1})\, d\nu_{n,d}(DG_{n,d}) \\ &\quad = \int_{G_n/G_{n,d-1}} \int_{G_{n-d+1}/G_{n-d+1,1}} f(DC'g_{n,d})\, d\nu_{n-d+1,1}(CG_{n-d+1,1}) \\ &\qquad\qquad d\nu_{n,d-1}(DG_{n,d-1}). \end{aligned}$$

Proof. We have

$$G_d/G_{d,d-1} \cong G_{n,d}/g_{n,d} \quad \text{and} \quad G_{n-d+1}/G_{n-d+1,1} \cong G_{n,d-1}/g_{n,d}$$

via the maps $AG_{d,d-1} \mapsto A'g_{n,d}$ and $CG_{n-d+1,1} \mapsto C'g_{n,d}$. Further, if τ_1 and τ_2 satisfy $d\mu_{n,d} = d\tau_1 \times d\sigma_{n,d}$ and $d\mu_{n,d-1} = d\tau_2 \times d\sigma_{n,d}$, then

$$d\tau_1(A'g_{n,d}) = d\nu_{n,d}(AG_{d,d-1}) \quad \text{and}$$
$$d\tau_2(C'g_{n,d}) = d\nu_{n-d+1,1}(CG_{n-d+1,1})$$

by (11). Thus,

$$\begin{aligned}
&\int_{G_n/g_{n,d}} f(Dg_{n,d})\, d\tau_{n,d}(Dg_{n,d}) \\
&\qquad = \int_{G_n/G_{n,d}} \int_{G_{n,d}/g_{n,d}} f(DA'g_{n,d})\, d\tau_1(A'g_{n,d})\, d\nu_{n,d}(DG_{n,d}) \\
&\qquad = \int_{G_n/G_{n,d}} \int_{G_d/G_{d,d-1}} f(DA'g_{n,d})\, d\nu_{d,d-1}(AG_{d,d-1})\, d\nu_{n,d}(DG_{n,d}) \\
&\qquad = \int_{G_n/G_{n,d-1}} \int_{G_{n,d-1}/g_{n,d}} f(DC'g_{n,d})\, d\tau_2(C'g_{n,d})\, d\nu_{n,d-1}(DG_{n,d-1}) \\
&\qquad = \int_{G_n/G_{n,d-1}} \int_{G_{n-d+1}/G_{n-d+1,1}} f(DC'g_{n,d})\, d\nu_{n-d+1,1}(CG_{n-d+1,1}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\nu_{n,d-1}(DG_{n,d-1}). \qquad \square
\end{aligned}$$

Proof of Theorem 3. We will prove Theorem 3 by induction on d. The case $d = 1$ is trivially true. Now suppose $1 < d < n$. We will compute the quantity

$$C(n,d) = \int_{G_n/g_{n,d}} \chi_1(f_{n,d}(Ag_{n,d}))\, \chi_1(f_{n,d-1}(Ag_{n,d}))\, d\tau_{n,d}(Ag_{n,d})$$

two different ways.

By Lemma 1, for every $D \in G_n$ there is a $D' \in \mathrm{GL}_d(K_{\mathbb{A}})$ with

$$f_{n,d}(DG_{n,d}) = |\det(D')|_{\mathbb{A}}$$

and

$$f_{n,d-1}(DA'g_{n,d}) = f_{d,d-1}(D'AG_{d,d-1})$$

for all $A \in G_d$, where $A' \in G_{n,d}$ as in Lemma 8. By Lemma 5,

$$\begin{aligned}
\int_{G_d/G_{d,d-1}} &\chi_1(f_{d,d-1}(D'AG_{d,d-1}))\, d\nu_{d,d-1}(AG_{d,d-1}) \\
&= c(d,d-1)|\det(D')|_{\mathbb{A}}^{1-d} \\
&= c(d,d-1)(f_{n,d}(DG_{n,d}))^{1-d}.
\end{aligned}$$

Hence, by Lemmas 5 and 8,

$$
\begin{aligned}
C(n,d) &= c(d,d-1)\int_{G_n/G_{n,d}} \chi_1(f_{n,d}(DG_{n,d}))(f_{n,d}(DG_{n,d}))^{1-d}\,d\nu_{n,d}(DG_{n,d}) \\
&= c(d,d-1)c(n,d)n\int_0^1 x^{n-1+1-d}\,dx \\
&= c(d,d-1)c(n,d)\frac{n}{n-d+1}. \qquad (12)
\end{aligned}
$$

Now let $D \in G_n$ with columns $\mathbf{d}_1^{\mathrm{tr}}, \ldots, \mathbf{d}_n^{\mathrm{tr}} \in (K_{\mathbb{A}})^n$. For each place $v \in M(K)$, let $V_v \in \mathrm{Gr}_{n,d-1}(K_v)$ be generated by $(\mathbf{d}_1)_v, \ldots, (\mathbf{d}_{d-1})_v$ and take $P_v\colon K_v^n \to V_v$ as in Lemma 3. Let $\mathbf{d}'_d, \ldots, \mathbf{d}'_n \in (K_{\mathbb{A}})^n$ be given by $(\mathbf{d}'_i)_v = (\mathbf{d}_i)_v - P_v(\mathbf{d}_i)_v$ for each $v \in M(K)$ and $i = d, \ldots, n$. Then, letting X be the $n \times (n-d+1)$ matrix with columns $(\mathbf{d}'_d)^{\mathrm{tr}}, \ldots, (\mathbf{d}'_n)^{\mathrm{tr}}$ and letting Y be the $n \times (d-1)$ matrix with columns $\mathbf{d}_1^{\mathrm{tr}}, \ldots, \mathbf{d}_{d-1}^{\mathrm{tr}}$, we have

$$f_{n,n-d+1}(X)\,f_{n,d-1}(Y) = |\det(D)|_{\mathbb{A}} = 1.$$

Thus, by Lemma 1 we obtain a $D' \in \mathrm{GL}_{n-d+1}(K_{\mathbb{A}})$ with

$$
\begin{aligned}
f_{n,d-1}(DG_{n,d-1}) &= |\det(D')|_{\mathbb{A}}^{-1} \\
f_{n,d}(DC'g_{n,d}) &= f_{n,d-1}(DG_{n,d-1})\,f_{n-d+1,1}(D'CG_{n-d+1,1}) \qquad (13)
\end{aligned}
$$

for all $C \in G_{n-d+1}$, where $C' \in G_{n,d-1}$ as in Lemma 8.

By Lemma 5,

$$
\begin{aligned}
&\int_{G_{n-d+1}/G_{n-d+1,1}} \chi_t(f_{n-d+1,1}(D'CG_{n-d+1,1}))\,d\nu_{n-d+1,1}(CG_{n-d+1,1}) \\
&\qquad = c(n-d+1,1)|\det(D')|_{\mathbb{A}}^{-1}t^{n-d+1}.
\end{aligned}
$$

Setting $t = (f_{n,d-1}(DG_{n,d-1}))^{-1}$, by Lemma 8 and (13) we have

$$
\begin{aligned}
&C(n,d) \\
&= c(n-d+1,1) \\
&\quad \times \int_{G_n/G_{n,d-1}} \chi_1(f_{n,d-1}(DG_{n,d-1}))(f_{n,d-1}(DG_{n,d-1}))^{d-n}\,d\nu_{n,d-1}(DG_{n,d-1}) \\
&= c(n-d+1,1)c(n,d-1)n\int_0^1 x^{n-1+d-n}\,dx \\
&= c(n-d+1,1)c(n,d-1)\frac{n}{d}. \qquad (14)
\end{aligned}
$$

Comparing (12) and (14) gives

$$c(n,d) = \frac{n-d+1}{d}\cdot\frac{c(n-d+1,1)c(n,d-1)}{c(d,d-1)}.$$

Theorem 3 now follows by the induction hypothesis. □

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